Exercises: Day 1

0.1

Why is it that $\mathcal{T}(S)/\operatorname{Mod}(S) = \mathcal{M}(S)$ (the moduli space)?

Hyperbolic geometry

0.2

Fix $0 < \Theta < \pi$. Describe a topology on the set $P(6, \Theta)$ of hyperbolic 6-gons with labeled edges $1, ..., 6$ and every interior angle equal to Θ. Prove that the map from $P(6, \Theta)$ to $\mathbb{R}^3_{>0}$ recording the lengths of sides $1, 3$, and 5 (or $2, 4, 6$) is a bijection.

How would you prove that this map is a homeomorphism?

0.3

Recall the Gauss–Bonnet Theorem that for a closed surface S equipped with a Riemannian metric h:

$$\int_S \kappa_h \; dArea_h = 2\pi\chi(S),$$

where κ_h is the Gauss curvature and $dArea_h$ is the Riemannian area form for the metric h.

Prove the following: Given a closed surface S, there is a number B such that for any hyperbolic metric h on S, there is a homotopically non-trivial simple closed curve on S of h-length at most B.

The holonomy map

0.4

Verify that the holonomy map

$$hol([f : S \to X]) = [f_* : \pi_1 S \to \mathrm{PSL}(2, \mathbb{R})]$$

is well defined, i.e., that if $f : S \to X \sim g : S \to Y$, then $[f_*] = [g_*]$.

(Hint: Give S a reference hyperbolic metric, and suppose $F : \widetilde{S} \to \mathbb{H}^2$ and $G : \widetilde{S} \to \mathbb{H}^2$ are bi-Lipschitz maps, hence have well defined boundary extensions $\partial F : \partial\widetilde{S} \to \partial\mathbb{H}^2$ and $\partial G : \partial\widetilde{S} \to \partial\mathbb{H}^2$. Prove that if $d(F, G) < \infty$, then $\partial F = \partial G$.)

0.5

Verify that the action of $\operatorname{Mod}(S)$ on $\mathcal{T}(S)$ is by real-analytic homeomorphisms.

(Hint: Every $[\varphi] \in \operatorname{Mod}(S)$ induces an *outer automorphism* $[\varphi_*]$ of $\pi_1 S$, i.e., an element of the group of $\pi_1 S$-conjugacy classes of isomorphisms of $\pi_1 S \to \pi_1 S$. Choose an automorphism φ_* representing $[\varphi_*]$ and show that the action of φ_* on representations $\operatorname{Hom}(\pi_1 S, \mathrm{PSL}(2, \mathbb{R}))$ is by polynomial maps.)

Fenchel–Nielsen coordinates

0.6

Prove that $\#\mathcal{P} = 3g - 3$ and that $S \setminus \mathcal{P}$ consists of $2g - 2$ 3-times punctures spheres.

1 Exercises: Day 2

Fenchel–Nielsen coordinates

1.1

Conclude the proof that the $\mathbb{R}^{\mathcal{P}}$-action by twists is simply transitive: That is, if $tw_{\mathcal{P}}(X, Y) = 0$ for two points X and Y in the same $\ell_{\mathcal{P}}$-fiber, then X is Teichmüller equivalent to Y.[1]

1.2

Given a section σ of $\ell_{\mathcal{P}} : \mathcal{T}(S) \to \mathbb{R}^{\mathcal{P}}_{>0}$, the rule

$$F_\sigma : X \in \mathcal{T}(S) \mapsto (\ell_{\mathcal{P}}(X), tw_{\mathcal{P}}(\sigma(\ell_{\mathcal{P}}(X)), X)) \in \mathbb{R}^{\mathcal{P}}_{>0} \times \mathbb{R}^{\mathcal{P}}$$

defines a bijection that is often called "Fenchel–Nielsen coordinates."

Sketch a strategy for showing that F_σ is homeomorphic if σ is continuous. That is, explain why F_σ and F_σ^{-1} are both continuous (proving the statement that $\ell_{\mathcal{P}}$ is a principal $\mathbb{R}^{\mathcal{P}}$-bundle).

Earthquakes

1.3

Let $X \in \mathcal{T}(S)$ with universal cover $\pi : \widetilde{X} \to X$ and deck group $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$. Let $\gamma \subset X$ be a simple closed geodesic, and identity $\widetilde{X}$ with $\mathbb{H}^2$. Let P be the collection of connected components of $\mathbb{H}^2 \setminus \pi^{-1}(\gamma)$. Given $t \in \mathbb{R}$, denote by

$$E^t_\gamma : P \times P \to \mathrm{PSL}(2, \mathbb{R})$$

the cocycle from lecture. Here is its definition again: for $p, q \in P$, let $\Lambda(p, q) \subset \pi^{-1}(\gamma)$ be the geodesics that a path from p to q crosses, in order. Given them orientations so that the positive orientation is to the left (as seen from p). Then

$$E^t_\gamma(p, q) := \prod_{g \in \Lambda(p,q)} T^t_g.$$

Prove the following

1. $E^t_\gamma(p, p) = id$ and $E^0_\gamma(p, q) = id$.
2. $E^t_\gamma(p, q) = E^t_\gamma(q, p)^{-1}$
3. $E^t_\gamma(p, q) = E^t_\gamma(p, r) E^t_\gamma(r, q)$
4. for all $\eta \in \Gamma$, $E^t_\gamma(\eta p, \eta q) = \eta E^t_\gamma(p, q) \eta^{-1}$.
5. Fix $p \in P$. For all t, the rule $\rho^t_\gamma : \eta \mapsto E^t_\gamma(p, \eta p)\eta$ is a group homomorphism.
6. The rule $t \mapsto \rho^t_\gamma \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ is continuous, i.e., for all $\eta \in \Gamma$, $t \mapsto \rho^t_\gamma(\eta) \in \mathrm{PSL}(2, \mathbb{R})$ is continuous.
7. $[\rho^t_\gamma] = hol(\mathsf{Eq}_{t\gamma}(X))$.

[1] I'm using shorthand/suppressing the equivalence relation and marking, here. By X, I really mean $[f : S \to X]$ and so on.

1.4

Recall the following from lecture:

Theorem 1.1 (First variation of length)**.** Let $\theta_{\gamma\eta} : \gamma \pitchfork \eta \to (0, \pi)$ be the function that measures the angle, measured counter clockwise, from η to γ. Then

$$\left.\frac{d}{dt}\right|_{t=0} \ell_\eta(\mathsf{Eq}_{t\gamma}(X)) = \sum_{p \in \gamma \pitchfork \eta} \cos \theta_{\nu\mu}(p)$$

Theorem 1.2 (Convextiy)**.** Then the function

$$t \mapsto \ell_\eta(\mathsf{Eq}_{t\gamma}(X))$$

is convex and strictly convex if $i(\eta, \gamma) \neq 0$.

Prove that if $\gamma \subset X$ is a simple closed geodesic, and η is another closed geodesic, then

$$\frac{\ell_\eta(\mathsf{Eq}_{t\gamma}(X))}{t} \to i(\gamma, \eta), \ t \to \infty.$$

Try proving the following statement: For every closed curve η with $i(\gamma, \eta) \neq 0$ and $\epsilon > 0$, there is a T such that for all $t \geq T$, we have

$$\ell_\eta(\mathsf{Eq}_{(T+t)\gamma}(X)) - \ell_\eta(\mathsf{Eq}_{T\gamma}(X)) \geq (1 - \epsilon) i(\gamma, \eta) t.$$

Exercises: Day 1

0.1 Hyperbolic geometry

0.1.1

Fix $0 < \Theta < \pi$ and $n \geq 3$. Describe a topology on the set $P(6, \Theta)$ of hyperbolic 6-gons with labeled edges $1, ..., 6$ and every interior angle equal to Θ. Prove that the map from $P(6, \Theta)$ to $\mathbb{R}^3_{>0}$ recording the lengths of sides $1, 3$, and 5 (or $2, 4, 6$) is a bijection.

Sketch a proof that this map is also a homeomorphism (This might involve constructing a map between two hexagons with given edge-lengths; you would want your maps to be close to an isometry if the lengths are all close).

For $n \geq 3$, describe the space $P(2n, \pi/2)$ of right angled $2n$-gons with labeled edges.

0.1.2

Recall the Gauss–Bonnet Theorem that for a closed surface S equipped with a Riemannian metric h that

$$\int_S \kappa_h \; dArea_h = 2\pi\chi(S),$$

where κ_h is the Gauss curvature and $dArea_h$ is the Riemannian area form for the metric h.

Prove the following: Given a closed surface S, there is a number B such that for any hyperbolic metric h on S, there is a homotopically non-trivial simple closed curve on S of h-length at most B.

0.2 Fenchel–Nielsen coordinates

0.2.1

Prove that $\#\mathcal{P} = 3g - 3$ and that $S \setminus \mathcal{P}$ consists of $2g - 2$ 3-times punctures spheres.

0.2.2

We saw that that the $\mathbb{R}^{\mathbb{P}}$-action on fibers of π is transitive. Now prove that the action is simply transitive. That is, if $tw(X, Y) = 0$ for two points X and Y in the same $\ell_{\mathbb{P}}$-fiber, then X is Teichmüller equivalent to Y.[1]

0.2.3

Given a section σ of $\ell_{\mathbb{P}} : \mathsf{T}(S) \to \mathbb{R}^{\mathbb{P}}_{>0}$, the rule

$$F_\sigma : X \in \mathsf{T}(S) \mapsto (\ell_{\mathbb{P}}(X), tw_{\mathbb{P}}(\sigma(\ell_{\mathbb{P}}(X)), X)) \in \mathbb{R}^{\mathbb{P}}_{>0} \times \mathbb{R}^{\mathbb{P}}$$

defines a bijection that is often called "Fenchel–Nielsen coordinates."

Sketch a strategy for showing that F_σ is homeomorphic if σ is continuous. That is, explain why F_σ and F_σ^{-1} are both continuous.

[1] I'm using shorthand/suppressing the equivalence relation and marking, here. By X, I really mean $[f : S \to X]$ and so on.

1 Exercises: Day 2

Here, you will complete the proof of The Earthquake Theorem. We have already proven continuity of the map

$$\mathsf{Eq}_X : \mathcal{ML} \to \mathsf{T}(S)$$
$$\mu \mapsto \mathsf{Eq}_X(\mu).$$

Since $\mathcal{ML} \cup \{0\}$ and $\mathsf{T}(S)$ are both cells of dimension $6g-6$, it suffices to prove that Eq_X is injective and proper. We then conclude by Invariance of Domain.

The following two results and their proofs from the lecture will be useful.

Theorem 1.1 (Variation of length)**.** Let μ and $\lambda \in \mathcal{ML}$. Let $\theta_{\nu\mu} : \mu \cap \lambda \to [0, \pi]$ be the function that measures the angle, measured counter clockwise, from leaves of μ to leaves of ν. Then

$$\frac{d}{dt}\bigg|_{t=0} \ell_\mu(\mathsf{Eq}_X(t\nu)) = \iint \cos(\theta_{\nu\mu})\ d\mu d\nu = -\frac{d}{dt}\bigg|_{t=0} \ell_\nu(\mathsf{Eq}_X(t\mu)).$$

(Note that $\theta_{\mu\nu} = \pi - \theta_{\nu\mu}$, so that $\cos(\theta_{\mu\nu}) = -\cos(\theta_{\nu\mu})$.)

Theorem 1.2 (Convexity of length functions)**.** Let $\mu \in \mathcal{ML}$ and γ be a closed curve. Then the function

$$t \mapsto \ell_\gamma(\mathsf{Eq}_X(t\mu))$$

is convex and strictly convex if $i(\mu, \gamma) \neq 0$.

1.0.1

Prove that Eq_X is proper along rays. That is, prove that if $\mu \in \mathcal{ML}$, then there is a curve γ such that

$$\ell_\gamma(\mathsf{Eq}_X(t\mu)) \to \infty,\ t \to \infty.$$

Note that this would follow from the following statement: For every closed curve γ with $i(\gamma, \mu) \neq 0$ and $\epsilon > 0$, there is a T such that for all $t \geq T$, we have

$$\ell_\gamma(\mathsf{Eq}_X((T+t)\mu)) - \ell_\gamma(\mathsf{Eq}_X(T\mu)) \geq (1-\epsilon)t.$$

1.0.2

To prove that Eq_X is proper (i.e., the preimage of a compact set is compact), consider a compact $K \subset \mathsf{T}(S)$. Suppose $\gamma_1, ..., \gamma_n$ are closed curves that fill S, so that the function

$$f : Y \in \mathsf{T}(S) \mapsto \sum_i \ell_{\gamma_i}(X) \in \mathbb{R}$$

is proper. Thus there is an $L > 0$ such that K is contained in the compact set $B_L(X) = f^{-1}([0, L])$, so it suffices to prove that $\mathsf{Eq}_X^{-1}(B_L(X))$ is compact.

Use properness of Eq_X along rays, continuity, and convexity of length functions along earthquake paths to prove this.

1.0.3

Now we prove that Eq_X is injective.

1. First along rays: Let $\mu \in \mathcal{ML}$ and consider a curve γ with $i(\gamma, \mu) \neq 0$. Consider the function $Y \mapsto \int_\gamma \cos(\theta^Y_{\mu\gamma}) d\mu$.[2] Use Theorems 1.1 and 1.2 to deduce that this function is strictly increasing along the ray $t \mapsto \mathsf{Eq}_X(t\mu)$ (hence no two points on the ray coincide).

[2] Here, I'm emphasizing that the angle is measured on the hyperbolic structure $Y \in \mathsf{T}(S)$.

2. Next we show that if μ and ν are projectively distinct, then $\mathsf{Eq}_X(\mathbb{R}_{\geq 0}\mu) \cap \mathsf{Eq}_X(\mathbb{R}_{\geq 0}\nu) = \{X\}$. For the case that $i(\nu, \mu) > 0$, use a similar argument to prove that these two rays only intersect at X.

3. Finally, assume that $i(\nu, \mu) = 0$. The difficult case is that the supports of ν and μ coincide (why?).

   Fact 1.3. If μ and ν are distinct measures with the same supports, then for any $\epsilon > 0$, there is a closed curve γ such that
   $$1 - \epsilon < \cos\theta_{\mu\gamma} = \cos\theta_{\mu\gamma} < 1$$
   and
   $$\frac{i(\mu, \gamma)}{i(\nu, \gamma)} > \frac{1}{1-\epsilon}.$$

   Given a small positive $\epsilon > 0$, let γ be as in the fact and prove that
   $$(1-\epsilon)i(\gamma, \mu) \leq \ell_\gamma(\mathsf{Eq}_X(\mu)) - \ell_\gamma(X) \leq i(\gamma, \mu)$$
   and
   $$(1-\epsilon)i(\gamma, \nu) \leq \ell_\gamma(\mathsf{Eq}_X(\nu)) - \ell_\gamma(X) \leq i(\gamma, \nu).$$

   Now assume that $\mathsf{Eq}_X(\mu) = \mathsf{Eq}_X(\nu)$ and use the fact to derive a contradiction.

This completes the proof of the Earthquake Theorem.