HARMONIC MAPS FROM RIEMANN SURFACES – EXERCISE SESSION 1

Exercise 1: A bit of linear algebra

Let $\mathrm{Sym}(n)$ denote the space of symmetric $n \times n$ matrices. Let Φ be a linear form on $\mathrm{Sym}(n)$ such that

$$\Phi(\mathrm{U}SU^{-1}) = \Phi(S)$$

for every orthogonal matrix U.

1) Prove that there exists $\lambda \in \mathbb{R}$ such that $\Phi(S) = \lambda \mathrm{Tr}(S)$ for all S.

Let (E, g_E) be a Euclidean space of dimension n, F a vector space and $B : E \times E \to E \to F$ a symmetric bilinear map. Define

$$\mathrm{Tr}_{g_E} B = \sum_{i=1}^{n} B(e_i, e_i) \ ,$$

where $(e_1, \ldots, e_n)$ is an orthonormal basis for g.

2) Prove that $\mathrm{Tr}_{g_E} B$ is well-defined, i.e. independent of the choice of the orthonormal basis.

3) Prove that

$$\mathrm{Tr}_{g_E} B = \frac{1}{\mathbf{Vol}(S(0,r)} \int_{S(0,r)} B(u,u)\mathrm{d}u \ ,$$

where $S(0,r)$ is the sphere of center 0 and radius r in (E, g_E) and the integral is with respect to the spherical measure.

Assume now that F is also equipped with a scalar product g_F. Given $L : E \to F$ a linear map, define:

$$\|L\| = \sqrt{\mathrm{Tr}_{g_E}\left(g_F(L\,\cdot, L\,\cdot)\right)} \ .$$

Prove that $\|\cdot\|$ is a euclidean norm on $\mathrm{Hom}(E,F)$.

Exercise 2: Harmonic maps to products of Riemannian manifolds

Let S be a Riemann surface, and (N_1, g_1), (N_2, g_2) and $f_1 : S \to N_1$ and $f_2 : S \to N_2$ smooth maps.

Date: January 13, 2025.

1) Using your favourite definition of a harmonic map, show that the map

$$(f_1, f_2) : S \to (N_1 \times N_2, g_1 \oplus g_2)$$

is harmonic if and only if f_1 and f_2 are harmonic.

2) Now, do it again with the other definitions.

Exercise 3: Harmonic maps and convex functions

Let (M_1, g_1), (M_2, g_2) and (M_3, g_3) be three Riemannian manifolds, and $f : M_1 \to M_2$ and $g : M_2 \to M_3$ smooth maps.

1) Show that

$$\mathrm{Hess}(g \circ f)_x(u,u) = \mathrm{d}g_{f(x)}(\mathrm{Hess}\, f_x(u,u)) + \mathrm{Hess}\, g_{f(x)}(\mathrm{d}f(u), \mathrm{d}f(u)) ,$$

for every $x \in M_1$ and $u \in T_x M_1$.

A smooth function $h : M_2 \to \mathbb{R}$ is *convex* if $\mathrm{Hess}\, h$ is non-negative.

2) Let $f : M_1 \to M_2$ be a harmonic map and $h : M_2 \to \mathbb{R}$ a convex function. Prove that $h \circ f$ is *subharmonic*, i.e.

$$\Delta_{M_1}(h \circ f) \geq 0 .$$

3) Conversely, assume that for every $x \in M_1$, every neighborhood U of $f(x)$ in M_2 and every convex function $h : U \to \mathbb{R}$, the function $h \circ f$ is subharmonic. Prove that f is harmonic.

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HARMONIC MAPS FROM RIEMANN SURFACES – EXERCISE SESSION 2

Exercise 1: Embeddings of symmetric spaces

Let G be a semisimple Lie group with finite center and finitely many connected components, and H a semisimple subgroup. Denote by X_G and X_H their respective symmetric spaces, which we see as spaces of Cartan involutions of X_G and X_H.

1) Let θ be a Cartan involution of G preserving H. Prove that the H-orbit of θ in X_G is totally geodesic and isomorphic to X_H.

2) Conversely, let $X' \subset X_G$ be a connected totally geodesic subspace and $\theta \in X'$. Prove the existence of a Lie subgroup $H \subset G$ preserved by θ such that $H \cdot \theta = X'$.
Hint: you may start by assuming that $G = \mathrm{Isom}(X_G)$.

3) Let σ be an involution of G. Prove the existence of a Cartan involution of G which commutes with σ, and remark that this Cartan involution preserves the subgroup fixed by σ.
Hint: again, it may be useful to assume that $G = \mathrm{Isom}(X_G)$.

Remark 0.1. More generally, for any semsimple subgroup H of G, there exists a Cartan involution of G preserving H. The proof, however, is more involved.

4) For each of the pairs $H \subset G$ below, describe an involution of G fixing H, and a Cartan involution of G preserving H:

- $\mathrm{SL}(n, \mathbb{R}) \subset \mathrm{SL}(n, \mathbb{C})$
- $\mathrm{SU}(p, n-p) \subset \mathrm{SL}(n, \mathbb{C})$
- $\mathrm{SO}(p, n-p) \subset \mathrm{SL}(n, \mathbb{R})$
- $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$.

Exercise 2: Non-positively curved geometry

Let (M, g_M) *be a smooth Riemannian manifold and* ∇ *its Levi–Civita connection. Let*

$$F : \begin{array}{ccc} [0,1] \times [0,1] & \to & M \\ (s,t) & \mapsto & \gamma_s(t) \end{array}$$

be a one parameter family of geodesics parametrized at constant speed. We set

$$\dot\gamma = \frac{\partial F}{\partial t} \quad \textit{and} \quad J = \frac{\partial F}{\partial s} \ ,$$

Date: January 8, 2025.

seen as sections of the bundle F^*T_M equipped with the pull-back metric and connection, which are still abusively denoted g_M and ∇ respectively.
Recall that the vector field $\dot\gamma$, being geodesic, satisfies

$$\nabla_{\partial_t}\dot\gamma = 0 \ ,$$

while the Jacobi field J satisfies the equation

$$\nabla_{\partial_t}\nabla_{\partial_t} J = R(\dot\gamma, J)\dot\gamma \ ,$$

where R is the curvature tensor of ∇. Finally, since ∂_s and ∂_t commute and ∇ is torsion-free, we have

$$\nabla_{\partial_t} J = \nabla_{\partial_s}\dot\gamma \ .$$

We now assume that M is complete, simply connected and non positively curved, meaning that

$$g_M(R(u,v)u,v) \geq 0$$

for every pair of tangent vectors at a point.

1) Prove that the function $\sqrt{g_M(J,J)}$ is convex in the direction t.

Assume that $s \mapsto \gamma_s(0)$ and $s \mapsto \gamma_s(1)$ are geodesics parametrized at constant speed. Define $l^2(s) = \int_0^1 g_M(\dot\gamma(s,t)\dot\gamma(s,t))\mathrm{d}t$ the square length of the geodesic segment γ_s.

2) Prove that

$$\frac{\partial^2}{\partial s^2} l^2(s_0) = \int_0^1 \frac{\partial^2}{\partial t^2} g_M(J(s_0,t), J(s_0,t))\mathrm{d}t \ .$$

Assume moreover that $s \mapsto \gamma_s(0)$ is constant and $s \mapsto \gamma_s(1)$ is parametrized at speed 1 (i.e. $g_M(J(s,1), J(s,1)) = 1$).

3) Prove that

$$\frac{\partial^2}{\partial s^2} l^2(s_0) \geq 2 \ .$$

4) Let x be a point in M. Prove that the exponential map at x is a global diffeomorphism from T_xM to M.

5) Let d_x^2 denote the squared to the point x. Prove that d_x^2 is is smooth and satisfies $\mathrm{Hess} d_x^2 \geq g_M$ at every point.

6) Let N be a complete totally geodesic submanifold of M. For every point $x \in M$, prove that there exists a unique point $\pi(x) \in N$ minimizing the distance to x.

7) Prove that the map $\pi : M \to N$ is smooth and 1-Lipschitz.

Let S be a closed Riemann surface, $\rho : \pi_1(S) \to \mathrm{GL}(n,\mathbb{C})$ a reductive representation and $G \subset \mathrm{GL}(n,\mathbb{C})$ the Zariski closure of the image of ρ. We admit that there exists a G-equivariant totally geodesic embedding of the symmetric space X_G into $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$ (see Remark 0.1).

8) Let $f : \tilde{S} \to \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$ be a ρ-equivariant harmonic map. Show that $f = \iota \circ \hat{f}$, where $\hat{f} : \tilde{S} \to X_G$ is a ρ-equivariant harmonic map and $\iota : X_G \to \mathrm{SL}(n, \mathbb{C})$ is a ρ-equivariant totally geodesic embedding.

Exercise 3: G-Higgs bundles

Given a real linear semisimple algebraic group G, we denote by K its maximal compact subgroup, by $\mathfrak{p}$ the orthogonal of $\mathrm{Lie}(K)$ *in* $\mathrm{Lie}(G)$ *(with respect to the Killing form), and by $K_\mathbb{C}$ and $\mathfrak{p}_\mathbb{C}$ their complexifications.*

1) For each of the following real algebraic subgroups G of $\mathrm{GL}(n, \mathbb{C})$, describe K, $\mathfrak{p}$, $K_\mathbb{C}$ and $\mathfrak{p}_\mathbb{C}$:

- $G = \mathrm{SO}(n, \mathbb{C})$
- $G = \mathrm{SL}(n, \mathbb{R})$
- $G = \mathrm{SU}(p, n-p)$
- $G = \mathrm{SO}(p, n-p)$
- $G = \mathrm{Sp}(n, \mathbb{R})$ (n even).

2) Let S be a Riemann surface, $\rho : \pi_1(S) \to G \subset \mathrm{GL}(n, \mathbb{C})$ a reductive representation and $f : \tilde{S} \to G/K \subset \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$ a ρ-equivariant harmonic map. For each of the above groups G, describe the Higgs bundle encoding the algebraic properties of $\mathrm{d}^{1,0} f$.

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