EXERCISES
Measured laminations and train tracks

1. (*) Recall that the *geometric self-intersection number* of a (homotopy class of a) closed curve is the minimum number of times a representative of the curve intersects itself transversely (where we require it to be in general position, i.e. not have a triple points). We say that a curve is *simple* if its geometric self-intersection number is 0. By drawing pictures, do the following:

   (a) Convince yourself that every closed curve on the (flat) torus is simple.

   (b) Let T_1 be the torus with one puncture (i.e. one point removed). Show that there are non-simple curves on T_1. In fact, that it has a curve with self-intersection number k for every $k \in \mathbb{Z}_{\geq 0}$. Draw some with $k = 1, 2, 3$.

   (c) Let S_2 be the closed genus 2 surface. Repeat part (b) for S_2.

   (d) In both the case of T_1 and S_2 above, convince yourself that there are *infinitely many* distinct simple curves.

   (e) Convince yourself that for any surface of negative Euler characteristics there are infinitely many distinct curves that are simple, as well as infinitely many distinct curves with self-intersection k for any natural number k.

2. (*) *(Simple curves can be complicated!)* In Figure 1 a train track, with weights, on the 4-holed sphere is shown. Draw the corresponding simple closed curve. (This is "Thurston's simple closed curve" once painted by Thurston and Sullivan on a wall in Berkeley.)

   Find other (bigger) integer weights that satisfy the switch equations and draw the corresponding (multi)-curve.

3. (**) Show, by providing an explicit example, that not every lamination on a surface is the support of a measured lamination.

4. *(Train tracks)*

   (a) (**) Let τ be a trivalent train track on a closed surface of genus $g \geq 2$ with V vertices and E edges. Show that $E - V \leq 6g - 6$ with equality if and only if τ is maximal (every complementary region is an ideal triangle). If you want, just consider the case $g = 2$. Here you can use as a fact that the area of any closed hyperbolic surface of genus g is $2\pi|\chi(S)| = 2\pi(2g - 2)$

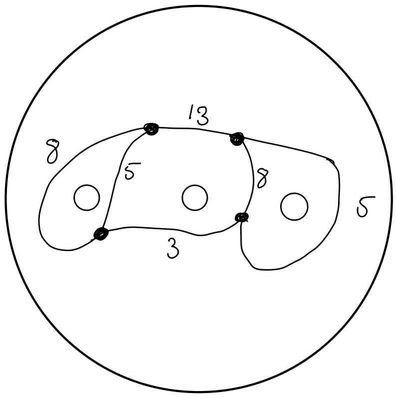

Figure 1: A train track on the 4-holed sphere.

(b) (*) Convince yourself that every simple closed curve on the punctured torus is carried by one of the two standard train tracks shown in Figure 2.

(c) (**) We say a train track is *recurrent* if it carries at least one multi-curve (or measured lamination) which traverses every edge (that is, there is at least one solution to the switch equations that assigns positive weight to all edges). Let τ be one of the two standard train tracks on the punctured torus (Figure 2). Show that, even though there are train tracks τ' for which τ is a proper sub-train track of, no such τ' is recurrent.

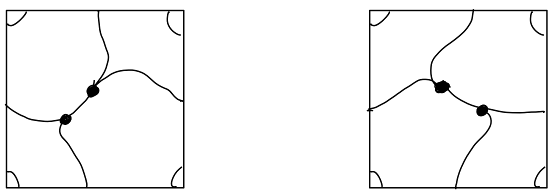

Figure 2: The standard train tracks on the punctured torus. (The curves in the corners are supposed to represent the puncture—not part of the train track)

(d) (**) Let S be the closed genus 2 surface. Draw a non-recurrent train track on S. Draw a recurrent one. Draw a recurrent maximal train track, that is, one for which all the complementary regions are triangles.

5. (**) Recall that we say a sequence of measures (m_n) on some space X converges to m in the weak-* topology if $\int f dm_n \to \int f dm$ for all compactly supported continuous functions $f : X \to \mathbb{R}$. An equivalent definition (by the Portmanteau theorem) says that (m_n) converges to m if $m_n(A) \to m(A)$ for all measurable sets A for which $m(\partial A) = 0$.

   Show that the condition on the boundary is necessary, that is: Find a sequence (m_n) of measures (on some space) which converge to some m and a measurable subset B for which $m_n(B)$ does not converge to $m(B)$.

6. (**) For $L > 0$, consider the measure on $\mathbb{R}^2$ defined by

$$m^L = \frac{1}{L^2} \sum_{p \in \mathbb{Z}^2} \delta_{\frac{1}{L}p}$$

   where, as usual, δ_x denotes the Dirac measure defined by $\delta_x(U) = 1$ if $x \in U$ and $\delta_x(U) = 0$ if $x \notin U$ and for $p = (a, b) \in \mathbb{Z}^2$ we define $\frac{1}{L}p = (a/L, b/L) \in \mathbb{R}^2$.

   (a) Let S_n be a (closed or open) square of side length n. Show that

$$m^L(S_n) \to \text{Leb}(S_n)$$

   as $L \to \infty$ for every $n > 0$. Here Leb is the usual Lebesgue measure (area) on $\mathbb{R}^2$.

   (b) Convince yourself that this implies that m^L converges to the Lebesgue measure in the weak-* topology and that the same argument (where m^L is appropriately scaled) works more generally for $\mathbb{R}^n$.

7. (***) Let S be a closed surface of genus $g \geq 2$. Recall that Thurston measure on the space of measured laminations $\mathcal{ML}(S)$ is defined as

$$m_{\text{Thu}} = \lim_{L \to \infty} \frac{1}{L^{6g-6}} \sum_{\lambda \in \mathcal{ML}_{\mathbb{Z}}} \delta_{\frac{1}{L}\lambda}$$

   where $\mathcal{ML}_{\mathbb{Z}}$ denotes the set of all integrally weighted simple multicurves and the convergence is with respect to weak-* convergence.

   (a) Show that it satisfies the following scaling property:

$$m_{\text{Thu}}(t \cdot U) = t^{6g-6} m_{\text{Thu}}(U)$$

   for any measurable set U and $t > 0$. Here $t \cdot U = \{t\lambda \mid \lambda \in U\}$.

(b) Conclude that $m_{\text{Thu}}(\partial B) = 0$ for any measurable set B satisfying $B \cap t \cdot B = \emptyset$ for all $t > 0$.

(c) Let $F : \mathcal{ML} \to \mathbb{R}_{\geq 0}$ be continuous, positive (i.e. $F(\lambda) = 0$ if and only if $\lambda = 0$) and homogeneous (i.e. $F(t\lambda) = tF(\lambda)$) function and let $B(F) = \{\lambda \in \mathcal{ML}(S) \,|\, F(\lambda) \leq 1\}$. Show that $m_{\text{Thu}}(\partial B(F)) = 0$.

8. (**) *(Counting all simple multicurves)* Let S be a closed surface of genus $g \geq 2$ equipped with a hyperbolic metric X. You can take as fact that hyperbolic length of curves extends continuously (and positively and homogenously) to $\mathcal{ML}(S)$.

(a) Show that the existence of the Thurston measure (defined as the limit in the previous exercise) implies that we can count all simple integral multicurves on S, that is, that we know that the limit

$$\lim_{L \to \infty} \frac{\#\{\lambda \text{ simple integral multicurve on } S \;|\, \ell_X(\lambda) \leq L\}}{L^{6g-6}}$$

exists and find the (an expression of) the limit.

Hint: Use the equivalent definition of weak-* convergence given in Exercise 5 together with Exercise 7(b).

(b) Do the same as in part (a) but where $\ell_X(\cdot)$ is replaced by any continuous, positive, and homogenous $F : \mathcal{ML}(S) \to \mathbb{R}_{\geq 0}$.

EXERCISES
Currents

Throughout let S denote a closed surface with $\chi(S) < 0$ and $\mathcal{C}(S)$ its space of geodesic currents.

1. (*) (i) Let $F_1, F_2 : \mathcal{C}(S) \to \mathbb{R}_{\geq 0}$ be two continuous, positive, and homogenous functions (meaning $F_i(\lambda) > 0$ for all $\lambda \neq 0$ and $F_i(c\lambda) = cF_i(\lambda)$ for all $c > 0$). Show that there exists $A \geq 1$ such that

$$\frac{1}{A} F_1(\lambda) \leq F_2(\lambda) \leq A F_2(\lambda)$$

   for all $\lambda \in \mathcal{C}(S)$.

   (*) (ii) Let A be a compact subset of $\mathcal{C}_{\text{fill}}(S)$ (the set of filling currents). Show that there exists $A \geq 1$ such that

$$\frac{1}{A} \iota(\sigma_1, \lambda) \leq \iota(\sigma_2, \lambda) \leq A \iota(\sigma_1, \lambda)$$

   for all $\lambda \in \mathcal{C}(S)$ and all $\sigma_1, \sigma_2 \in A$.

2. (*) Prove that the set of filling currents, $\mathcal{C}_{\text{fill}}(S)$, is an open and dense subset of $\mathcal{C}(S)$.

3. (**) Argue that the intersection form $\iota(\cdot, \cdot)$ does indeed define a function $\mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}_{\geq 0}$, that is, that $\iota(\lambda, \mu) < \infty$ for all $\lambda, \mu \in \mathcal{C}(S)$.

4. (**) Let X be a hyperbolic surface homeomorphic to S and equip the universal cover with the pull-back metric. Given a (unit speed) geodesic segment $\sigma : (-a, a) \to \tilde{X}$ in the universal cover, let $G(\sigma)$ be the set of (unoriented) geodesics in $\tilde{X}$ that transversely intersect σ. Note that $G(\sigma)$ can be parametrized by (t, θ) where $\sigma(t)$ and θ are the point and angle of intersection of the geodesic with the segment. The *Liouville current for X*, denoted L_X can be (locally) defined by the measure

$$L_X|_{G(\sigma)} = \frac{1}{2} \sin(\theta) d\theta dt.$$

   Note that this measure is locally finite and invariant under the full isometry group, and hence in particular by $\pi_1(S)$, so does indeed define a current.

   (i) Using this expression, show that

$$L_X(G(\sigma)) = \ell_{\tilde{X}}(\sigma)$$

   where $\ell_{\tilde{X}}(\sigma)$ is the $\tilde{X}$-length of the segment σ.

(ii) Conclude that for any closed geodesic γ on X we have

$$\iota(L_X, \gamma) = \ell_X(\gamma)$$

where $\ell_X(\gamma)$ is the X-length of γ.

Remark: Otal proved that currents are separated by their intersection with closed curves, that is, if $\iota(\mu, \gamma) = \iota(\lambda, \gamma)$ for all closed curves γ, then $\mu = \lambda$. Hence the Liouville current for X is the *unique* current with the property in (ii).

(iii) Combining part (ii) with Exercises 1, conclude that for any hyperbolic surface X, the induced hyperbolic length function on closed curves extends to a length function $\ell_X : \mathcal{C}(S) \to \mathbb{R}_{\geq 0}$ that is continuous, positive, and homogeneous and which is then comparable to any other such function (in particular, to the length function given by intersection with a fixed filling curve or current).

5. (***) (*This exercise requires some basic knowledge of Teichmüller space*) Let $\mathcal{T}(S)$ denote the Teichmüller space of S. The purpose of this exercise is to outline a proof, due to Bonahon, of Thurston's compactification of Teichmüller space which says that the "boundary" of $\mathcal{T}(S)$ is $\mathbb{P}\mathcal{ML}(S)$. Here $\mathbb{P}\mathcal{ML}(S) = \mathcal{ML}(S)/\mathbb{R}_+$ is the space of projective measured laminations. With L_X denoting the Liouville current corresponding to X as in the exercise above, you can take the following as facts:

- The map $\mathcal{T}(S) \to \mathcal{C}(S)$, $X \mapsto L_X$ is injective (in fact an embedding). This follows from the fact that X is determined by the length it assigns to (finitely many, simple) closed curves.
- $\iota(L_X, L_X) = C$ where $C > 0$ is a constant independent of X (in fact, $C = \pi^2|\chi(S)|$).

(a) Show that $\mathcal{T}(S) \to \mathbb{P}\mathcal{C}(S)$, $X \mapsto [L_X]$ is still injective.

We will from now on identify $\mathcal{T}(S)$ with its image in $\mathcal{C}(S)$ or in $\mathbb{P}\mathcal{C}(S)$.

(b) Show that $\mathcal{T}(S)$ and $\mathcal{ML}(S)$ are disjoint in $\mathcal{C}(S)$ (and $\mathcal{T}(S)$ and $\mathbb{P}\mathcal{ML}(S)$ are disjoint in $\mathbb{P}\mathcal{C}(S)$).

(c) Let (X_n) be a sequence in $\mathcal{T}(S)$. Argue that, up to passing to a subsequence, there exists $\mu \in \mathcal{C}(S)$ and a sequence of positive numbers (ϵ_n) such that $\epsilon_n X_n \to \mu$. (That is, that X_n converges projectively to μ.)

(d) Conclude that $\ell_{X_n}(\gamma)$ converges to $\iota(\mu, \gamma)$ projectively for all closed curves γ.

(e) If (X_n) does not have a limit point in $\mathcal{T}(S)$, show that the sequence (ϵ_n) from part (c) converges to 0. *Hint:* Note that if (X_n) diverges, there is a curve α in S with $\ell_{X_n}(\alpha) \to \infty$.

(f) If (X_n) does not have a limit point in $\mathcal{T}(S)$, show that μ from part (c) is in fact a measured lamination and conclude that $\overline{\mathcal{T}(S)} \subset \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$ where $\overline{\mathcal{T}(S)}$ denotes the closure of $\mathcal{T}(S)$ inside $\mathbb{P}\mathcal{C}(S)$.

(g) Convince yourself that we actually have equality above, that is,

$$\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$$

by arguing as follows: since closed curves are dense in $\mathbb{P}\mathcal{ML}(S)$ it is enough to show that any such α is the projective limit of a sequence (X_n) in $\mathcal{T}(S)$. Create such a sequence by "pinching" alpha (i.e. one where the lengths $\ell_{X_n}(\alpha)$ goes to 0) in such a way that $\iota(L_{X_n}, \beta)$ projectively converge to $\iota(\alpha, \beta)$ for all closed curves β (some version of the collar lemma will be helpful).