Loewner transform and Loewner energy

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Exercise 1: Computing chordal Loewner energy

Recall that the chordal Loewner energy of a simple chord γ from 0 to ∞ in the upper half-plane $\mathbb{H}$, denoted $I_{\mathbb{H},0,\infty}(\gamma)$, is the Dirichlet energy of its driving function W:

$$I_{\mathbb{H},0,\infty}(\gamma) = \frac{1}{2}\int_0^\infty \left|\frac{\mathrm{d}W_t}{\mathrm{d}t}\right|^2 \mathrm{d}t.$$

If γ is a chord in (D, a, b), then

$$I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\phi(\gamma))$$

where $\phi : D \to \mathbb{H}$ is a conformal map with $\phi(a) = 0$ and $\phi(b) = \infty$.

1. Let γ be a chord in $(\mathbb{H}, 0, \infty)$ with mapping out functions $g_t(z)$ and driving function W_t. Show that W_t has the following transformation properties. (Hint: use the expansion of $g_t(z)$ at ∞.)
   (a) (Additivity) Let $\tilde{\gamma} = g_t(\gamma) - W_t$. Show that the driving function $\tilde{W}_s$ of $\tilde{\gamma}$ is $\tilde{W}_s = W_{t+s} - W_t$.
   (b) (Scaling) Let $\lambda > 0$ and $\tilde{\gamma} = \lambda\gamma$. Show that the driving function of this curve is $\tilde{W}_s = \lambda W_{\lambda^{-2}s}$.
2. Let γ be the imaginary axis in $\mathbb{H}$.
   (a) Compute the capacity parametrization of γ and the mapping out function of $\gamma[0, t]$.
   (b) Show that $W_t = 0$ for all $t \geq 0$.
3. Consider the ray $\gamma_\alpha = \{z \in \mathbb{H} : \arg z = \alpha\}$ in $\mathbb{H}$ at angle $0 \leq \alpha \leq \pi/2$.
   (a) Show that the driving function of γ_α is $W_\alpha(t) = C\sqrt{t}$, where C is a constant depending on α with $C = 0$ if and only if $\alpha = \pi/2$. (Hint: use that γ_α is preserved under scaling.)
   (b) Calculate the chordal Loewner energy of γ_α in $(\mathbb{H}, 0, \infty)$ and observe that it is infinite when $\alpha \neq \pi/2$.
4. Consider the semicircular curve γ_t in $\mathbb{H}$ which intersects the real axis at $\gamma_0 = 0$ and $\gamma_T = 1$, for some $T > 0$ which is the half-plane capacity of the half-disk of radius 1/2. Let K_t be the compact $\mathbb{H}$-hull generated by γ_t (for $t < T$, $K_t = \gamma[0, t]$, and K_T is the half disk of radius 1/2 and center 1/2). Let $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ be their mapping out functions and $t \mapsto W_t$ be the driving function of γ.
   (a) Show that $\gamma[0, T]$ and $g_t(\gamma[t, T])$ are self-similar. Namely, show that for all $t < T$, $g_t(\gamma[t, T])$ is also a semicircle with endpoints $W_t = g_t(\gamma_t)$ and $V_t := g_t(\gamma_T)$.

(b) Let W be the driving function of γ. Take the following as fact (you can also try to show it yourself[1]):

$$\left.\frac{\mathrm{d}W_t}{\mathrm{d}t}\right|_{t=0} = 6.$$

Using the self-similarity and the Loewner equation, show that for $t < T$,

$$\frac{\mathrm{d}W_t}{\mathrm{d}t} = \frac{6}{V_t - W_t}; \qquad \frac{\mathrm{d}V_t}{\mathrm{d}t} = \frac{2}{V_t - W_t}.$$

(c) Use the differential equations above to compute T and show that the Loewner energy of γ in $(\mathbb{H}, 0, \infty)$ is infinite.

Exercise 2: Computing Loop Loewner energy

In the lecture, we saw that the Loewner energy $I^L(\cdot)$ of a Jordan curve is defined as a limit of chordal Loewner energies and can be viewed as a generalization. The set of Jordan curves γ with $I^L(\gamma) < \infty$ is exactly the set of *Weil–Petersson quasicircles.* There are equivalent expressions of the Loewner energy, one for bounded loops (i.e., ones that don't contain ∞) and one for unbounded loops.

- (Bounded case.) Suppose that γ is a bounded Jordan curve separating a bounded region Ω from an unbounded region Ω^*. Let $f : \mathbb{D} \to \Omega$ and $h : \mathbb{D}^* \to \Omega^*$ be conformal maps with h fixing ∞. Then

$$I^L(\gamma) = \frac{1}{\pi} \iint_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 \mathrm{d}A + \frac{1}{\pi} \iint_{\mathbb{D}^*} \left|\frac{h''}{h'}\right|^2 \mathrm{d}A + 4\log|f'(0)| - 4\log|h'(\infty)|$$

  where $\mathrm{d}A$ is the Euclidean area measure.

- (Unbounded case.) Suppose that γ is a Jordan curve passing through ∞ separating the regions H and H^*. Let $f : \mathbb{H} \to H$ and $h : \mathbb{H}^* \to H^*$ be conformal maps fixing ∞. Then

$$I^L(\gamma) = \frac{1}{\pi} \iint_{\mathbb{H}} \left|\frac{f''}{f'}\right|^2 \mathrm{d}A + \frac{1}{\pi} \iint_{\mathbb{H}^*} \left|\frac{h''}{h'}\right|^2 \mathrm{d}A.$$

1. Show in two ways that the loop $\gamma = \mathbb{R}_+ \cup (e^{i\alpha}\mathbb{R}_+)$ with $0 < \alpha < \pi$ satisfies $I^L(\gamma) = \infty$.

   Suggestion: Use the ray example from Exercise 1.

2. (Challenging) We show that Weil–Petersson curves may have spirals. Use Loewner loop energy to show that the spiral $t \mapsto t\exp(\mathfrak{i}\log\log|1/t|)$ can be completed into a Weil–Petersson quasicircle.

3. (Very Challenging) Since the Loewner energy is Möbius invariant, can you show directly that the expressions for I^L for bounded and unbounded loops are equivalent? (The proof I know is indirect and uses the relationship to zeta-regularized determinants of Laplacians.)

[1] See, e.g., [arxiv 2006.08574, Section 4, page 30, eq. (4.3)] or [Exact Solutions for Loewner Evolutions, Section 5].