Sheet I: Hyperbolic geometry

Mini-course: Random hyperbolic 3-manifolds

Winter School Côte d'Azur 2026

1) In the upper half-space model of $\mathbb{H}^3$, every orientation-preserving isometry corresponds to an element of $\mathrm{PSL}(2,\mathbb{C})$, acting as a Möbius transformation on the boundary $\partial\mathbb{H}^3$. Let $\gamma \in \mathrm{PSL}(2,\mathbb{C})$.

(a) A parabolic element fixes exactly one point in $\partial\mathbb{H}^3$.

  i. Show that any parabolic element is conjugate to the Euclidean translation : $z \mapsto z+1$.

  ii. Deduce that γ is parabolic if and only if $|\mathrm{tr}(\gamma)| = 2$.

(b) A hyperbolic (loxodromic) element fixes two points in $\partial\mathbb{H}^3$.

  i. Show that any hyperbolic (loxodromic) element is conjugate to the translation (plus rotation) : $z \mapsto \lambda^2 z$, with $|\lambda| > 1$.

  ii. Prove that γ is hyperbolic (loxodromic) if and only if $\mathrm{tr}(\gamma) \in \mathbb{C} \setminus [-2,2]$.

(c) An elliptic element fixes a point in the interior of $\mathbb{H}^3$.

  i. Show that any elliptic element is conjugate to the rotation : $z \mapsto e^{2i\theta} z$, with $\theta \not\equiv 0 \pmod{\pi}$.

  ii. Prove that γ is elliptic if and only if $\mathrm{tr}(\gamma) \in (-2,2)$.

(d) Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ be a discrete subgroup acting freely on $\mathbb{H}^3$ (so that $M = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold). Explain why Γ cannot contain any elliptic elements.

(e) Show that two conjugate elements are of the same type.

2) Let $M = \mathbb{H}^3/\Gamma$ by a complete hyperbolic manifold, and let $\gamma \in \Gamma$ be a hyperbolic isometry. The translation length of γ, defined as the minimal displacement :

$$\ell(\gamma) = \min_{x \in \mathbb{H}^3} d(x, \gamma x),$$

is realised in its unique invariant geodesic in $\mathbb{H}^3$, its axis.

(a) Prove that γ is conjugate to

$$\begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix},$$

where $\ell = \ell(\gamma)$. *Hint : In the upper-half space, the distance between two vertical points P_1, P_2 is given by :* $d(P_1,P_2) = \int_{t_1}^{t_2} \frac{|dt|}{t}$.

Deduce that

$$l(\gamma) = \mathrm{Re}\left(\cosh^{-1}\left(\frac{\mathrm{tr}(\gamma)}{2}\right)\right).$$

(b) Suppose γ_1, γ_2 are conjugate in Γ. Show that their corresponding closed geodesics have the same length.

3) Show that in every homotopy class of non-trivial, non-parabolic closed curves in M, there exists a unique geodesic representative.

4) Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold. Given the previous exercises, show that there is a 1-to-1 correspondence between free homotopy classes of closed curves in M and conjugacy classes in Γ.

5) Let p be a point in $\partial\mathbb{H}^n$. A *horosphere* O centred at p is a connected hypersurface orthogonal to all the geodesic lines exiting from p. In the upper half-space model of $\mathbb{H}^3$, they are easily visualised : by sending p to infinity the horospheres centred at p are precisely the horizontal planes $\{t = k\}$ with $k > 0$.

Given a horosphere O centred at $p \in \partial\mathbb{H}^n$ and a domain $D \subset O$, the *cone* C of D over p is the union of all half-lines exiting from D and pointing toward p.

(a) Prove that :

$$Vol(C) = \frac{Vol_O(D)}{2},$$

where Vol_O is the area of the flat surface O. How is it for

(b) Use it to show that a finite polyhedron has finite volume.

6) Prove that every closed irreducible 3-manifold M with infinite $\pi_1(M)$ not containing $\mathbb{Z}\times\mathbb{Z}$ is hyperbolic.

Sheet II: Triangulations

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1) Let M be a closed, oriented smooth n-dimensional manifold. The *simplicial volume* (or *Gromov norm*) of M is defined as

$$\|M\| \;=\; \inf\left\{ \sum_i |\lambda_i| \;\middle|\; \big[\sum_i \lambda_i \sigma_i\big] = [M] \in H_n(M;\mathbb{R}) \right\},$$

where the infimum is taken over all real singular cycles that represent the fundamental class $[M]$.

For a closed hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, Gromov and Thurston proved that

$$\|M\| \;=\; \frac{\mathrm{Vol}(M)}{v_3},$$

where $v_3 \approx 1.01494$ is the hyperbolic volume of a regular ideal tetrahedron in $\mathbb{H}^3$. Show, using this, that the minimal number of tetrahedra in a smooth triangulation T of M is at least $\|M\|$.

2) In 1970, Andreev characterised compact hyperbolic polyhedra in $\mathbb{H}^3$ with acute dihedral angles. This was further generalized in 1996 by Rivin to ideal (non-compact) hyperbolic polyhedra as follows.

Let P be an abstract 3–dimensional polyhedron, and let P^* denote its dual polyhedron -that is, the polyhedron whose vertices correspond to the faces of P, whose edges correspond to the edges of P, and whose faces correspond to the vertices of P.

For each edge e of P, assign a number $\phi(e) \in (0, \pi)$ which will denote the exterior dihedral angle of e, defined as $\phi(e) = \pi - \psi(e)$, where $\psi(e)$ is the interior dihedral angle of e.

Theorem 1 (Rivin, 1996) *There exists a convex ideal polyhedron $Q \subset \mathbb{H}^3$ of the same combinatorial type of P and with exterior dihedral angles given by ϕ if and only if the following conditions hold :*

(a) For every vertex $v \in V$,

$$\sum_{e \in E_v} \phi(e) = 2\pi,$$

where E_v is the set of all edges containing v.

(b) For every simple closed curve γ in the 1-skeleton of the dual polyhedron P^* that bounds a disk in S^2 and does not correspond to a single vertex of P,

$$\sum_{e \in E_\gamma} \phi(e) > 2\pi,$$

where E_γ is the set of edges of P crossed by γ.

(c) $\phi(e) \in (0, \pi)$ for all $e \in E$.

Moreover, when such an ideal polyhedron exists, it is unique up to isometry of $\mathbb{H}^3$.

(a) Show, using this, that there exists a unique hyperbolic ideal right-angled regular octahedra up to isometry. Realise the polyhedron on $\mathbb{H}^3$.

(b) Can other platonic solids be realised as ideal hyperbolic polyhedra in $\mathbb{H}^3$?

3) Let $\mathcal{K}$ be a 3-dimensional polyhedral complex, in which every maximal cell is an 3-dimensional convex polyhedron. The *dual graph* $G(\mathcal{K})$ is defined as follows :

— Vertices : correspond to 3-cells of $\mathcal{P}$.

— Edges : connect two vertices if the corresponding 3-cells share a 2-dimensional face.

Let M be a tetrahedral hyperbolic 3-manifold, that is, a hyperbolic 3-manifold made out of a gluing of regular ideal hyperbolic tetrahedra.

(a) What is the dual graph of the underlying complex ?

(b) Show that the M deformation retracts onto the dual graph.

(c) Deduce that free homotopy classes of closed curves in M are in bijection with closed paths in $G(M)$.

4) A *tessellation* of $\mathbb{H}^n$ (or $\mathbb{R}^n$, $\mathbb{S}^n$) is a locally finite set of polyhedra that cover the space and may intersect only in common faces.

Let $S \subset \mathbb{H}^n$ be a discrete set. For every point $p \in S$ we define the *Voronoi cell* :

$$D(p) = \{q \in \mathbb{H}^n \mid d(q, p) \leq d(q, p') \; \forall p' \in S\}.$$

(a) Prove that :

i. The set $D(p)$ is a polyhedron.

ii. The polyhedra $D(p)$ as $p \in S$ varies form a tessellation of $\mathbb{H}^n$. This is called the Voronoi tessellation of S.

Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$. A *fundamental domain* for Γ is a polyhedron D in $\mathbb{H}^n$ such that the translates $g(D)$ as $g \in \Gamma$ varies form a tessellation of $\mathbb{H}^n$.

Let $M = \mathbb{H}^n/\Gamma$ be a hyperbolic manifold and D a fundamental domain for Γ.

(a) Prove that the projection $\pi : \mathbb{H}^n \to M$ restricts to a surjective map $D \to M$ that sends $\mathrm{int}(D)$ isometrically onto an open dense subset of M. Deduce that $\mathrm{Vol}(D) = \mathrm{Vol}(M)$.

(b) A *Dirichlet domain* for Γ centered at $p \in \mathbb{H}^n$ is

$$D_p = \{x \in \mathbb{H}^n \mid d(x, p) \leq d(x, \gamma p) \; \forall \gamma \in \Gamma\}.$$

Prove that D_p is a fundamental domain, and that $M = \mathbb{H}^n/\Gamma$ is compact if and only if D_p is compact.

5) Create other models for random 3-manifolds.