Exercise sheet 1

Monday 5th January 2026

Examples and properties of minimal surfaces

Exercise 1: Minimal catenoid in $\mathbb{R}^3$

1. (*Easy*) Let $\Sigma = \{(f(t)\cos(\theta), f(t)\sin(\theta), t) \,|\, t \in \mathbb{R}, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ be a surface of revolution in $\mathbb{R}^3$, for some function $f : \mathbb{R} \to (0, +\infty)$. Compute the first and second fundamental form, and the mean curvature, of Σ.

2. (*Easy*) Check that imposing $H = 0$ is equivalent to an ODE on f, and check that $f(t) = \cosh(t)$ is a solution of the ODE.

Exercise 2: The Clifford torus in $\mathbb{S}^3$

1. (*Easy*) Show that, in $\mathbb{S}^3 = \{x_1^2+x_2^2+x_3^2+x_4^2 = 1\}$ the Clifford torus $\{x_1^2+x_2^2 = x_3^2+x_4^2 = 1/2\}$ is a minimal surface.

2. (*Easy*) Show that its first fundamental form is flat and its area equals $2\pi^2$.

Comment: the smallest area closed minimal surface in $\mathbb{S}^3$ is (up to isometry) the totally geodesic sphere $\mathbb{S}^2 = \{x_4 = 0\}$. The Willmore Conjecture, proved by Marques and Neves in 2014, asserts that the second one (and the only one achieving the value $2\pi^2$, up to isometry) is the Clifford torus. Also, the Clifford torus is (up to isometry) the only minimal embedded torus in $\mathbb{S}^3$: this is the statement of the Lawson Conjecture, proved by Brendle in 2013.

Exercise 3: Minimal surface equation in $\mathbb{R}^3$

1. (*Easy*) Let $\Sigma \subset \mathbb{R}^3$ be the graph of a function $u : \Omega \to \mathbb{R}$, for Ω an open set in $\mathbb{R}^2$. Compute the first fundamental form, the area form, the second fundamental form and the mean curvature of Σ with respect to the graphical parameterization $(x, y) \mapsto (x, y, u(x, y))$.

2. (*Easy*) Show that Σ is minimal if and only if u satisfies the equation

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0 \,. \tag{1}$$

3. (*Medium*) Show that (assuming Ω and u bounded) the area of Σ equals

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |Du|^2}dxdy \,.$$

   Then prove that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(u + tv) = 0$$

   for all smooth bounded $v : \Omega \to \mathbb{R}$ if and only if the following equation (the Euler-Lagrange equation of $\mathcal{A}$) holds:

$$\mathrm{div}_{\mathbb{R}^2}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 0 \,. \tag{2}$$

4. (*Medium*) Check that (2) is actually equivalent to the minimal surface equation (1).

Comment: a consequence of the exercise is that, for (smooth!) graphs over $\mathbb{R}^2$ in $\mathbb{R}^3$ (and even for graphs in $\mathbb{R}^{n+1}$ over $\mathbb{R}^n$), being minimal (i.e. critical point of the area with respect to any variation) is actually equivalent to being a critical point of the area with respect to (a priori) only graphical variations, also called outer variations. The De Giorgi-Nash-Moser Theorem shows that this is also true for *Lipschitz* functions u that are critical points of the area functional under outer variations. The result was recently extended in any codimension by Hirsch-Mooney-Tione, who solved a conjecture of Lawson-Osserman and proved that Lipschitz weak solutions are C^2.

Exercise 4: Geodesic equation

The purpose of this exercise is to warm-up for the next one, by showing that geodesics on Riemannian manifolds are precisely the critical points of the length functional.

1. (*A bit difficult*) Show that, given a curve $\gamma : I \to (N, h)$ for $I = [a, b]$ or S^1, (N, h) a Riemannian manifold, parameterized by arclength, and a smooth variation $\Gamma : [a, b] \times (-\epsilon, \epsilon)$ of γ with fixed endpoints (i.e. $\Gamma(t, 0) = \gamma(t)$ and, if I is a closed interval, $\Gamma(a, s) = \gamma(a)$, $\Gamma(b, s) = \gamma(b)$ for all s), the following formula holds:
$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{L}(\gamma_s) = -\int h(V(\gamma(t)), \nabla^h_{\gamma'(t)}\gamma'(t))dt \ ,$$
where $\gamma_s(\cdot) = \Gamma(\cdot, s)$, $V(\gamma(t)) = d\Gamma(t, s)/ds|_{s=0}$ and $\mathcal{L}$ denotes the length of a curve.

2. (*Medium*) Deduce that γ is a geodesic (i.e. $\nabla^h_{\gamma'(t)}\gamma'(t)) = 0$) if and only if it is a critical point of the length under smooth variations of γ (with fixed endpoints, if I is an interval).

Exercise 5: Minimal surfaces are critical points of the area

The goal is now to show that minimal immersions are precisely the critical points of the area functional.

1. (*More difficult*) Show that, given an immersion $\iota : M \to (N, h)$ for (N, h) a Riemannian manifold and a smooth compactly supported variation $\Gamma : M \times (\epsilon, \epsilon) \to M$ such that $\Gamma(x, 0) = \iota$, the following formula holds:
$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{dVol}_{g_s} = -h(V, H)\mathrm{dVol}_g$$
where $V = d\Gamma(\cdot, s)/ds|_{s=0}$, H is the mean curvature of ι, g_s is the first fundamental form of $x \mapsto \Gamma(x, s)$, and recall that, in coordinates, $\mathrm{dVol}_g = \sqrt{\det g} dx_1 \wedge \ldots \wedge dx_n$. [Hint: to simplify the computation, take normal coordinates around a given point x, so that $g = g_0$ is expressed as δ_{ij} at x. First show that the derivative of dVol_{g_s} equals $\mathrm{tr}(dg_s/ds|_{s=0})$, and finally show that the latter quantity is equal to $-h(V, H)$.]

2. (*Medium*) As a consequence, if $\mathcal{A}$ denotes the area (or volume, if $\dim M > 2$) of an immersion,
$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{A}(\iota_s) = -\int_M h(V, H)\mathrm{dVol}_g \ .$$
Deduce that ι is minimal (i.e. $H = 0$) if and only if it is a critical point of $\mathcal{A}$ under compactly supported variations.

Exercise 6: Minimal surfaces are conformal harmonic maps

Recall that, given a smooth function f on a Riemannian manifold (M, g) taking values in $\mathbb{R}$, the Hessian of f is the $(0, 2)$-tensor $\nabla^2 f(X, Y) := (\nabla^g_X df)(Y)$. The Laplace-Beltrami operator is defined as $\Delta^g f := \mathrm{tr}_g \nabla^2 f$.

1. (*Easy*) Extending the definition of Hessian to a function $f : \Sigma \to \mathbb{R}^n$ coordinate by coordinate, show that the second fundamental form of an immersion $\iota : \Sigma \to \mathbb{R}^3$ is equal to the Hessian of ι with respect to the first fundamental form $g = \iota^* g_{\mathbb{R}^3}$, and that the mean curvature is equal to $\Delta^g \iota$.

2. (*Easy*) Deduce that an immersion $\iota = (\iota_1, \iota_2, \iota_3) : \Sigma \to \mathbb{R}^3$ is minimal if and only if each component ι_i is harmonic with respect to the first fundamental form $g = \iota^* g_{\mathbb{R}^3}$, that is, $\Delta^g \iota_i = 0$.

3. (*Medium*) Prove that a function $f : (M, g) \to \mathbb{R}$ (or $\mathbb{R}^n$) is harmonic if and only if it is a critical point of the energy
$$\mathcal{E}(f) := \frac{1}{2} \int_M \|df\|^2_g \mathrm{dVol_g}$$
under compactly supported variations of f, where $\|df\|^2_g = |df(e_1)|^2 + |df(e_2)|^2$ for a g-ortonormal frame (e_1, e_2). Show that if $\dim M = 2$, then $\mathcal{E}(f)$ only depends on the conformal class of g.

4. (*Medium*) Show that, for a surface Σ and an immersion $\iota : \Sigma \to \mathbb{R}^3$, $\mathcal{A}(\iota) \leq \mathcal{E}(\iota)$, with equality if and only if ι is conformal (i.e. if $\iota^* g_{\mathbb{R}^3} = e^{2\varphi} g$). Give an alternative proof of the fact that ι is minimal if and only if it is harmonic with respect to (any metric conformal to) the first fundamental form on Σ.

Comment: the above results still hold if one replaces $\mathbb{R}^3$ with any Riemannian manifold. There is a notion of harmonic maps between Riemannian manifolds (M, g) and (N, h), which is equivalent to the vanishing of the tension $\tau(f)(X, Y) = \nabla^h_X df(Y) - df(\nabla^g_X Y)$. Minimal immersions are then precisely the conformal harmonic maps.

Exercise 7: Minimal surfaces are locally area minimizing

The goal here is to show that minimal surfaces are precisely the surfaces that locally minimize area among disks (not only critical points, but actually minima, in a small neighbourhood of every point). This uses ideas from the theory of calibrations. For simplicity, we do this when the ambient space is $\mathbb{R}^3$.

- We say that a 2-form ω on (N, h) is a *calibration* if it satisfies:

  (a) $d\omega = 0$

  (b) $|\omega(E_1, E_2)| \leq 1$ for every pair of h-orthonormal vectors (E_1, E_2).

  A surface Σ in N is *calibrated* if it satisfies:

  (c) $\omega(E_1, E_2) = 1$ for every pair of h-orthonormal vectors (E_1, E_2) tangent to Σ (equivalently, $\omega|_\Sigma = \mathrm{dVol}_\Sigma$).

- Given a minimal surface $\Sigma \subset \mathbb{R}^3$ and $p \in \Sigma$, up to an isometry, we assume that in a neighbourhood U of p,

$$\Sigma_\Omega := \Sigma \cap U = \{(x, y, u(x, y)) \,|\, (x, y) \in \Omega\}$$

  for Ω a (small) convex set.

1. (*Medium*) Assume that there exists a calibration ω in $\Omega \times \mathbb{R}$ such that Σ_Ω is calibrated. Using Stokes' Theorem, prove that the area of Σ_Ω is less than or equal to the area of any other "competitor" Σ' which is topologically a disk, contained in $\Omega \times \mathbb{R}$, with $\partial\Sigma' = \partial\Sigma_\Omega$. [Hint: don't forget to use all the properties that define a calibration.]

2. (*Medium*) Now prove that the area of Σ_Ω is less than or equal to the area of any other "competitor" Σ' which is topologically a disk (not necessarily contained in $\Omega \times \mathbb{R}$) with $\partial\Sigma' = \partial\Sigma_\Omega$. [Hint: use that the nearest point projection to a convex set decreases distances.]

- Now, to apply the previous two points, we construct a calibration in $\Omega \times \mathbb{R}$ that calibrates Σ_Ω. Let ν be the unit normal vector to Σ_Ω, extended to a vector field on $\Omega \times \mathbb{R}$ in such a way that it does not depend on the z-coordinate. Define

$$\omega(X, Y) = \mathrm{dVol}_{\mathbb{R}^3}(X, Y, \nu)$$

  where $\mathrm{dVol}_{\mathbb{R}^3} = \det$.

3. (*Easy*) Prove that ω satisfies properties (b) and (c).

4. (*Medium*) Compute ω in (x, y, z)-coordinates and prove that ω satisfies property (a) if and only if Σ_Ω is minimal. [Hint: the computation is simplified if one uses the expression (2) from Exercise 3 of the minimal graph equation.]

Exercise sheet 2

Tuesday 6th January 2026

Minimal surfaces in hyperbolic manifolds

Exercise 1: Minimal graphs in the upper half-space and parabolic invariant solutions

1. (*Not too hard, if one has already done Exercise 3 of Sheet 1*) In the upper half-space
$$\mathbb{H}^3 = \left(\{(x,y,z) \,|\, z > 0\}, \frac{dx^2 + dy^2 + dz^2}{z^2} \right) ,$$
let $\Sigma \subset \mathbb{H}^3$ be the graph of a function $u : \Omega \to (0, +\infty)$ over the (x, y)-plane. Show that Σ is minimal if and only if
$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} + \frac{2(1 + u_x^2 + u_y^2)}{u} = 0 \ .$$
2. (*Easy*) Write down an ODE which is equivalent to finding a surface Σ as above that is invariant under a parabolic one-parameter group of isometries $(x, y, z) \mapsto (x, y + t, z)$.
3. (*Medium*) Show that the maximal time of existence of such ODE is finite.
4. (*Harder*) Show that there exists a unique complete minimal parabolic-invariant graph, up to isometry. What is its boundary at infinity?

Exercise 2: Geometric maximum principle

1. (*Easy*) Let $\Sigma \subset \mathbb{R}^3$ be the graph of a function $u : \Omega \to \mathbb{R}$, for Ω an open subset of $\mathbb{R}^2$. Suppose that $du_p = 0$. Prove that the second fundamental form of Σ at $p = (x, y, z)$, computed with respect to the upward normal vector field, equals the (Euclidean) Hessian of f at (x, y).
2. (*Easy*) Deduce the following statement: if two embedded surfaces Σ_- and Σ_+ are equal up to order one at p (i.e. p is in both Σ_- and Σ_+, and $T_p\Sigma_- = T_p\Sigma_+$) and Σ_+ lies above Σ_- in a neighbourhood of p, then the mean curvatures $H_\pm$ of $\Sigma_\pm$ satisfy $H_-(p) \leq H_+(p)$.
3. (*Medium*) Generalize to any ambient Riemannian manifold. [Hint: use normal coordinates centered at p.]
4. (*Optional, and intentionally vague*) Look up and apply the strong maximum principle to obtain the following additional statement: if both Σ_- and Σ_+ are minimal and connected, then $\Sigma_- = \Sigma_+$.
5. (*Medium*) Prove that there is no closed (i.e. compact without boundary) minimal surface in $\mathbb{R}^3$ or $\mathbb{H}^3$.
6. (*Medium*) Explain why the proof of the previous point fails in $\mathbb{S}^3$ (see Exercise 2 of Sheet 1 for counterexamples).

Exercise 3: Almost-Fuchsian manifolds

1. (*Medium*) Let Σ be a minimal surface in a complete hyperbolic manifold such that $\|I\!I_\Sigma\|^2 \leq 2$. Let Σ_t (for $t \in \mathbb{R}$) be the set
$$\Sigma_t = \{\exp_p(tN(p)) \,|\, p \in \Sigma\}$$
where N is the unit normal vector of Σ. Prove that Σ_t is an immersed surface and that the pull-back to Σ of its first and second fundamental forms are
$$I_t = I((\cos t)\mathrm{id} + (\sin t)B, (\cos t)\mathrm{id} + (\sin t)B) \quad \text{and} \quad I\!I_t = I((\cos t)B - (\sin t)\mathrm{id}, (\cos t)B - (\sin t)\mathrm{id})$$
[Hint: it is convenient to use the hyperboloid model of $\mathbb{H}^3$ since the exponential map is particularly simple.]

2. (*Easy*) Show that, if $\|II_\Sigma\|^2 < 2$, then Σ_t is convex for $|t| > c$, for c a constant to be computed.
3. (*Easy*) Show that the mean curvature of Σ_t is negative for $t > 0$ and positive for $t < 0$ — that is, the mean curvature vector always points towards Σ.
4. (*Medium*) Apply the geometric maximum principle to show that if a complete three-manifold M admits a foliation by closed surfaces where one leaf (say Σ) is minimal and all the other leaves have mean curvature vector pointing towards Σ, then Σ is the unique closed minimal surface in M.
5. (*Easy, at this point*) Conclude that a weakly almost-Fuchsian manifold — that is, a complete hyperbolic manifold homeomorphic to $S \times \mathbb{R}$ admitting a closed minimal surface Σ with $\|II_\Sigma\|^2 \leq 2$ — has a unique closed minimal surface.

Comment: a conjecture often attributed to Thurston asserts that every almost-Fuchsian manifold admits a foliation where all the leaves have *constant* mean curvature, where (as in point 4) one leaf is the minimal surface and the mean curvature vector of all the other leaves point towards the minimal surface.