Université Côte d'Azur **5-9 January 2026**
Winter School

Random complex curves
Exercices

Curvature reminder. Let M be a submanifold of dimension m of the Riemannian n-dimensional manifold (N, g). Let

$$\begin{array}{rcl} \sigma : T_xM \times T_xM & \to & N_xM \\ (X,Y) & \mapsto & (\nabla_X Y)^{\perp} \end{array}$$

where $NM \subset TN$ denotes the normal bundle of M in N, and ∇ the Levi–Civita connection associated to g. Recall that σ is symmetric. Then for any $x \in M$, (Gauss's equations [3, Theorem 3.6.2])

$$\begin{array}{rcl} \forall X,Y,Z,W \in T_xM,\ \langle R^M(X,Y)Z,W\rangle & = & \langle R^N(X,Y)Z,W\rangle + \\ & & \langle \sigma(Y,Z),\sigma(X,W)\rangle - \langle \sigma(X,Z),\sigma(Y,W)\rangle, \end{array}$$

where R^M and R^N denote the Riemannian curvature tensor for $(M, g_{|M})$ and (N, g) respectively. Recall that the sectional curvature along the plane $P \subset T_xM$ is

$$K(P) = \langle R^M(X,Y)Y,X\rangle,$$

where X, Y is an ONB of P.

Exercice 1

1. Let $n \geqslant 1$ be an integer and $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, vanishing transversely at 0, that is

$$\forall x \in \mathbb{R}^n,\ f(x) = 0 \Rightarrow df(x) \neq 0.$$

   Let $Z(f) := \{f = 0\}$. Express the Riemannian curvature $R(X, Y, Z, W)$ of $(Z(f), g_{0|Z(f)})$ in terms of ∇f and $d\nabla f$.

   From now on, under the hypotheses of the curvature reminder, assume that (N, g, J) is a Kähler metric, that is J is a complex structure such that $\nabla J = 0$. Assume also that $M \subset N$ is a complex submanifold.
2. [4, Proposition 9.2] For any complex line $P \subset T_xM$, express $K(P)$ in terms of $K^N(P)$ and $\|\sigma(X,X)\|^2$. Compare both.
3. Assume that $N = \mathbb{C}^2$. Show that the complex curve N has a non-positive curvature.

Exercice 2 Prove that for any integer $d \geqslant 1$,

$$N_d := \dim_{\mathbb{C}} \mathbb{C}_d^{hom}[Z_0, \cdots, Z_n] = \binom{n+d}{d}.$$

Hint : to choose a monomial is to choose a sequence of d stars $*$ and n bars $|$, where a $*$ is a cumulating exponent and a bar is separating two consecutive variables. For instance, $**||*$ denotes $Z_0^2 Z_2$.

In the sequel, we define the following Hermitian product on $\mathbb{C}_d^{hom}[Z_0, \cdots, Z_n]$:

$$\langle P, Q\rangle_d = \frac{1}{(d+n)!\pi^{n+1}} \int_{\mathbb{C}^{n+1}} P(Z)\bar{Q}(Z) e^{-\|Z\|^2} |dZ|^2,$$

with

$$|dZ|^2 = \prod_{k=0}^{n} dX_k \otimes dY_k,$$

where $Z_k = X_k + iY_k$.

Exercice 3 [2, Lemma 3.1.1] Show that the family

$$\left(\sqrt{\frac{n+d}{i_0! \cdots i_n!}} Z_0^{i_0} \cdots Z_n^{i_n}\right)_{i_0+\cdots+i_n=d}$$

is an ONB of $\left(\mathbb{C}_d^{hom}[Z_0, \cdots, Z_n], \langle\ ,\ \rangle_d\right)$.

Exercice 4 [2, Lemma 3.1.1] Let $q \in \mathbb{C}[z_1, \cdots, z_n]$ and for any $d \geqslant 1$. define

$$Q_d(Z) := Z_0^d q\left(\frac{Z_1}{Z_0}\sqrt{d}, \cdots, \frac{Z_n}{Z_0}\sqrt{d}\right).$$

1. Check that Q_d is a degree d homogeneous polynomial. Describe briefly its vanishing locus in $\mathbb{C}P^n$ for large d.
2. Show that
$$\|Q_d\|_d^2 \sim_{d\to\infty} d^n \int_{\mathbb{C}^n} |q|^2 e^{-\|z\|^2} |dz|^2.$$

Exercice 5 [2, p. 39] Let

$$P = \sum_{i_0+\cdots+i_n=d} a_{i_0\cdots i_n} \sqrt{\frac{n+d}{i_0! \cdots i_n!}} X_0^{i_0} \cdots X_n^{i_n}$$

be a random Fubini-Study complex polynomial (recall that $\Re a_I$, $\Im a_I$ are independent and follow the normal law $N(0,1)$).

1. Show that the probability measure is invariant under the action of $U(n+1)$ on $\mathbb{C}_d^{hom}[Z_0, \cdots, Z_n]$ by $P \mapsto P \circ T$.
2. For any $[Z] \in \mathbb{C}P^n$, For any $P \in \mathbb{C}_d^{hom}[Z_0, \cdots, Z_n]$, let
$$|P(x)|_d^2 = \frac{|P(Z)|^2}{\|Z\|^{2d}}.$$
show that
$$\mathbb{E}\left(|P([Z])|_d^2\right) \sim_{d\to\infty} d^n.$$
Hint : Compute at $[1 : 0 \cdots : 0]$ and use the invariance of the measure.

Exercice 6 [1, Ex 5-23] Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2]$ be a generic polynomial, and $C = Z(P)$ (C is smooth). The *flexes* of C are the points x of C where the contact between T_xC and C is of order $\geqslant 3$. Let $\operatorname{Hess} P$ be the Hessian matrix of P, that is

$$\operatorname{Hess} P = \left(\frac{\partial^2 P}{\partial Z_i \partial Z_j}\right)_{0\leqslant i,j\leqslant 2},$$

and $H = \det \operatorname{Hess} P$. We want to prove that

$$x \text{ is a flex } \Leftrightarrow H(x) = 0.$$

1. Show that both terms of the latter equivalence are invariants under the action of $PGL_3(\mathbb{C})$. In particular, it is enough to prove it for $x = [1 : 0 : 0]$ and if $T_xC = \{Y = 0\}$. We will assume this in the sequel.
2. What is the degree of H ?
3. Let $f(x, y) = P(1, x, y)$ and $h(x, y) = H(1, x, y)$. Let also

   $$g(x, y) = f_y^2 f_{xx} + f_x^2 f_{yy} - 2 f_x f_y f_{xy}.$$

   Show that

   $$(f, h)(x, y) = 0 \Leftrightarrow (f, g)(x, y) = 0.$$

   Hint : use the Euler formula to prove

   $$\forall 0 \leqslant j \leqslant 2,\ (d-1)P_j = \sum_{i=0}^{2} P_{ij}$$

   and simplify the determinant.
4. Near $(0, 0)$, write

   $$f(x, y) = y + ax^2 + bxy + cy^2 + dx^3 + \text{ other terms.}$$

   What is the condition that x is a flex ? that $g(x) = 0$? Conclude.

Références

[1] William Fulton and Richard Weiss, *Algebraic curves : an introduction to algebraic geometry*, 1969.

[2] Damien Gayet, *Topology of random algebraic and analytic hypersurfaces*, `https://www-fourier.univ-grenoble-alpes.fr/~gayetd/course-geo-random.pdf`, 2019.

[3] Jürgen Jost, *Riemannian geometry and geometric analysis*, vol. 42005, Springer, 2008.

[4] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry, volume 2*, Intersciences Publishers, 1969.