An introduction to mixing for Anosov representations

Exercises

Exercises are ranked by difficulty:

($*$) Guided exercices.

($**$) Less guidance.

($***$) Requires a good knowledge of Lie theory.

Exercise 1. The geodesic flow of $\mathbb{H}^d$ ($*$)

Consider the bilinear form $\langle x, y\rangle = x_1 y_1 + \cdots + x_d y_d - x_{d+1} y_{d+1}$ on $\mathbb{R}^{d+1}$ and the hyperboloid model $\mathbb{H}^d \subset \mathbb{R}^{d,1}$ defined as

$$\mathbb{H}^d = \left\{ x \in \mathbb{R}^{d+1} \,\middle|\, \langle x, x\rangle = -1,\ x_{d+1} > 0 \right\},$$

with Riemannian metric g defined on the tangent spaces

$$\mathrm{T}_x\mathbb{H}^d = \left\{ v \in \mathbb{R}^{d+1} \,\middle|\, \langle x, v\rangle = 0 \right\}$$

by the formula $g_x(v, v') = \langle v, v'\rangle$.

1. Describe the unit tangent bundle $\mathrm{T}^1\mathbb{H}^d$ as a submanifold of $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$.
2. For $(x, v) \in \mathrm{T}^1\mathbb{H}^d$, describe the tangent space $\mathrm{T}_{(x,v)}\mathbb{H}^d \subset \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$.
3. Check that the geodesic flow $\varphi^t_{\mathbb{H}^d} : \mathrm{T}^1\mathbb{H}^d \to \mathrm{T}^1\mathbb{H}^d$ is given by:

$$\varphi^t_{\mathbb{H}^d}(x, v) = \Big(\cosh t x + \sinh t v, \sinh t x + \cosh t v \Big).$$

4. Give an explicit formula for the geodesic vector field $\mathcal{Z} : \mathrm{T}^1\mathbb{H}^d \to \mathrm{T}\left(\mathrm{T}^1\mathbb{H}^d\right)$ defined by

$$\mathcal{Z}(x, v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi^t_{\mathbb{H}^d}(x, v) \in \mathrm{T}_{(x,v)}\left(\mathrm{T}^1\mathbb{H}^d\right).$$

5. For $(x, v) \in \mathrm{T}^1\mathbb{H}^d$, we set:

$$\mathrm{E}^s_{(x,v)} = \left\{ (y, -y) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, \langle x, y\rangle = \langle v, y\rangle = 0 \right\}$$
$$\mathrm{E}^u_{(x,v)} = \left\{ (y, y) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, \langle x, y\rangle = \langle v, y\rangle = 0 \right\}$$

   Check that $\mathrm{T}_{(x,v)}\left(\mathrm{T}^1\mathbb{H}^d\right) = \mathrm{E}^s_{(x,v)} \oplus \mathrm{E}^u_{(x,v)} \oplus \mathbb{R}.\mathcal{Z}(x, v)$, and that this splitting is invariant under the differential of the geodesic flow.

6. For $(x,v) \in \mathrm{T}^1\mathbb{H}^d$, consider the bilinear form $\widetilde{g}_{(x,v)}$ on $\mathrm{T}_{(x,v)}\mathrm{T}^1\mathbb{H}^d \subset \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ defined by

$$\widetilde{g}_{(x,v)}\Big((y_x, y_v), (y'_x, y'_v)\Big) = \langle y_x\,,\, y'_x\rangle + \langle y_v\,,\, y'_v\rangle + \langle y_x\,,\, v\rangle\langle y'_x\,,\, v\rangle\,.$$

   Prove that this defines a Riemannian metric on $\mathrm{T}^1\mathbb{H}^d$ for which the action of $\mathrm{Isom}(\mathbb{H}^d)$ is isometric.

7. Defining norms with the Riemannian metric $\widetilde{g}$, compute the ratio

$$\frac{\left\| \mathrm{d}\varphi^t_{\mathbb{H}^d}\big|_{(x,v)}(y_x, y_v)\right\|_{\varphi^t(x,v)}}{\|(y_x, y_v)\|_{(x,v)}}$$

   for $(y_x, y_v) \in \mathrm{E}^{\mathrm{s}}_{(x,v)}$ and $(y_x, y_v) \in \mathrm{E}^{\mathrm{u}}_{(x,v)}$.

8. Prove that the geodesic flow of a closed hyperbolic manifold is Anosov.

Exercise 2. Ergodicity and mixing of the geodesic flow (∗∗)

1. Read about *Hopf's argument* to prove that the geodesic flow of a closed hyperbolic manifold is ergodic with respect to the Riemannian volume measure.

2. Read about the proof of mixing of the geodesic flow of a closed hyperbolic surface using Howe-Moore's Theorem for unitary representations.

Exercise 3. The geodesic flow of $\mathbb{H}^2_{\mathbb{R}}$ and the flow $\varphi^t_{\mathbb{L}}$ (∗)

1. Prove that there is an $\mathrm{SL}_2(\mathbb{R})$-equivariant diffeomorphism between the unit tangent bundle $\mathrm{T}^1\mathbb{H}^2_{\mathbb{R}}$ and the space

$$\mathbb{L} = \Big\{[v:\alpha] \in \mathbb{P}\Big(\mathbb{R}^2 \oplus (\mathbb{R}^2)^*\Big)\,\Big|\,\alpha(v) > 0\Big\}$$

   that conjugates the flow

$$\varphi^t_{\mathbb{L}}([v:\alpha]) = [\mathrm{e}^t v : \mathrm{e}^{-t}\alpha]$$

   to a constant speed reparametrisation of the geodesic flow.

2. Prove that for a projective Anosov representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{R})$, the domain

$$\widehat{\mathrm{M}}_\rho = \{[v:\alpha] \in \mathbb{L}\,\big|\,\forall\,\eta \in \partial_\infty\Gamma\,,\ [v] \pitchfork \xi^*(\eta) \text{ or } \xi(\eta) \pitchfork [\alpha]\}$$

   is equal to the whole space $\mathbb{L}$.

Exercise 4. Proximality (∗)

Prove the following claims:

1. If $g \in \mathrm{SL}(\mathrm{V})$ is proximal, the attracting fixed point $\ell^+(g) \in \mathbb{P}(\mathrm{V})$ of g and the repelling fixed point $\mathrm{H}^-(g) \in \mathbb{P}(\mathrm{V}^*)$ are transverse.

2. If $g \in \mathrm{SL}(\mathrm{V})$ is proximal and so is g^{-1}, then $\ell^+(g^{-1}) \subset \mathrm{H}^-(g)$.

Exercise 5. Convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^d_{\mathbb{R}})$ ($*$)

1. State/explain/prove the results mentioned in the lectures for hyperbolic groups acting on their Gromov boundary in the setting of a convex cocompact subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$ acting on its limit set $\Lambda_\Gamma \subset \partial_\infty \mathbb{H}^d$.

2. Prove that the non wandering set $\mathrm{NW}(\varphi^t_\Gamma)$ of the geodesic flow $\varphi^t_\Gamma : \mathsf{M}_\Gamma \to \mathsf{M}_\Gamma$ (where $\mathsf{M}_\Gamma = \Gamma\backslash \mathrm{T}^1\mathbb{H}^d$) has the following description:
$$\mathrm{NW}(\varphi^t_\Gamma) = \Gamma\backslash \left\{ (x,v) \in \mathrm{T}^1\mathbb{H}^d \,\middle|\, \lim_{t\to\pm\infty} \varphi^t_{\mathbb{H}^d}(x,v) \in \Lambda_\Gamma \right\}.$$

3. Using the description of the geodesic flow in Exercise 1, prove that the geodesic flow of a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ is Axiom A.

Exercise 6. Real hyperbolic groups ($**$)

Let $\Gamma < \mathrm{Isom}_\circ(\mathbb{H}^d_{\mathbb{R}})$ be a convex cocompact subgroup, and consider the inclusion $\iota : \mathrm{Isom}_\circ(\mathbb{H}^d_{\mathbb{R}}) = \mathrm{SO}_\circ(d,1) \hookrightarrow \mathrm{SL}_{d+1}(\mathbb{R})$. Using your favorite definition of Anosov representations, prove that $\iota : \Gamma \to \mathrm{SL}_{d+1}(\mathbb{R})$ is projective Anosov.

Exercise 7. The discontinuity domain for Benoist representations ($***$)

Consider a hyperbolic group Γ and a *Benoist representation* $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{V}))$. This means that ρ is projective Anosov, and preserves a properly convex domain $\Omega_\rho \subset \mathbb{P}(\mathrm{V})$ with $\mathcal{C}^1$ boundary on which Γ acts properly discontinuously and cocompactly. In this case, the limit map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathrm{V})$ is a homeomorphism onto $\partial\Omega_\rho$, and for every $\eta \in \partial_\infty\Gamma$ we have the correspondence
$$\mathrm{T}_{\xi(\eta)}\partial\Omega_\rho = \xi^*(\eta).$$

1. Explain the identification between the tangent space $\mathrm{T}_\ell\partial\Omega_\rho$ and an element of $\mathbb{P}(\mathrm{V}^*)$.

2. Describe the subsets $\widehat{\mathsf{K}}_\rho \subset \widehat{\mathsf{M}}_\rho \subset \mathbb{L}$. Draw a picture in an affine chart of $\mathbb{P}(\mathrm{V})$ when $\dim \mathrm{V} = 3$.

Exercise 8. The flow $\varphi^t_{\mathbb{L}}$ is not a geodesic flow ($**$)

1. Prove that the only $\mathrm{SL}(\mathrm{V})$-equivariant vector subbundles of $\mathrm{T}\mathbb{L}$ are E^0, E^s, E^u, $\mathrm{E}^s \oplus \mathrm{E}^0$, $\mathrm{E}^u \oplus \mathrm{E}^0$ and $\mathrm{E}^s \oplus \mathrm{E}^u$.

2. Why can this be interpreted as stating the the flow $\varphi^t_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}$ is not the geodesic flow on the unit tangent bundle of some homogeneous $\mathrm{SL}(\mathrm{V})$-space?

Exercise 9. Complex hyperbolic groups ($***$)

Let $\Gamma < \mathrm{Isom}(\mathbb{H}^d_{\mathbb{C}})$ be a convex cocompact subgroup, and consider the adjoint representation $\mathrm{Ad} : \mathrm{Isom}(\mathbb{H}^d_{\mathbb{C}}) = \mathrm{SU}(d,1) \to \mathrm{SL}(\mathfrak{su}(d,1)) = \mathrm{SL}_{d^2+2d}(\mathbb{R})$. Using your favorite definition of Anosov representations, prove that $\mathrm{Ad} : \Gamma \to \mathrm{SL}_{d^2+2d}(\mathbb{R})$ is projective Anosov. What about the real hyperbolic and quaternionic hyperbolic cases?