Simplicial volume and bounded cohomology — January 2026

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Exercises

Simplicial volume.

(1) (Easy) We have seen that if $f : M^n \to N^n$ is a continuous map between n-dimensional oriented, connected, closed manifolds then

$$|\deg(f)| \cdot \|N\| \leq \|M\|.$$

Show that this inequality is an equality if f is a covering.

(2) (Unknown (to me)) Find a topological proof, or a proof not relying on hyperbolic geometry of the inequality $4g - 4 \leq \|\Sigma_g\|$ for surfaces of genus $g \geq 2$.

(3) (Medium) Let $0 \neq \beta \in H^n(M, \mathbb{R}) \cong \mathbb{R}$. Use Hahn-Banach to prove the inequality

$$\frac{|\langle \beta, [M] \rangle|}{\|\beta\|_\infty} \geq \|M\|.$$

(4) (Medium) Let M and N be m and n-dimensional closed oriented connected manifolds. Prove that

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{m+n}{n} \cdot \|M\| \cdot \|N\|.$$

Hint: For the second inequality: A product of simplices $\Delta^m \times \Delta^n$ can be triangulated in $\binom{m+n}{n}$ simplices in a canonical way. For the first inequality: use cup products in cohomology.

(5) (Very hard and unknown) Direct proof of the inequality

$$\|\Sigma_g \times \Sigma_h\| \leq 6\chi(\Sigma_g \times \Sigma_h).$$

Find explicite cycles z_k representing the fundamental class of a product $\Sigma_g \times \Sigma_h$ of surfaces of genus $g, h \geq 2$ with

$$\|z_k\|_1 \to \|\Sigma_g \times \Sigma_h\| = 6\chi(\Sigma_g \times \Sigma_h) = 24(g-1)(h-1).$$

The only thing I know is that such cycles will not come from triangulating a fundamental domains of the form a product of fundamental domains of each surface.

(Bounded) group cohomology.

(1) (Easy if you don't mess up the signs) Let G be a group. Recall that we endowed

$$C^k(G) := \{f : G^{k+1} \to \mathbb{R} \mid f \text{ is } G-\text{invariant}\}$$

with the homogenous coboundary operator

$$\delta : C^k(G) \longrightarrow C^{k+1}(G)$$

given by

$$\delta(f)(g_0, \dots, g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i f(g_0, \dots, \widehat{g_i}, \dots, g_{k+1}),$$

for $f \in C^k(G)$ and $g_i \in G$. Show that

$$\delta \circ \delta = 0.$$

(2) (Easy) Further define

$$\overline{C^k}(G) := \{f : G^k \to \mathbb{R}\}.$$

(a) Verify that the following maps φ and ψ are isomorphisms between $C^k(G)$ and $\overline{C^k}(G)$:

$$\begin{array}{rcl} C^k(G) & \longleftrightarrow & \overline{C^k}(G) \\ \varphi : f & \longrightarrow & \{(h_1, \dots, h_k) \mapsto f(e, h_1, h_1h_2, \dots, h_1h_2 \dots h_k)\} \\ \{(g_0, \dots, g_k) \mapsto \overline{f}(g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{k-1}^{-1}g_k)\} & \longleftarrow & \overline{f} : \psi \end{array}$$

(b) Show that the corresponding inhomogoneous coboundary operator

$$d = \varphi \circ \delta \circ \psi : \overline{C^k}(G) \longrightarrow \overline{C^{k+1}}(G)$$

is given by

$$dh(g_1, \dots, g_k) = h(g_2, \dots, g_k) + \sum_{i=1}^{k-1} (-1)^i h(g_1, \dots g_{i-1}, g_ig_{i+1}, g_{i+2}, \dots g_k) + (-1)^k h(g_1, \dots, g_{k-1}).$$

(c) The group cohomology is defined as $H^n(G, \mathbb{R}) := \mathrm{Ker}(\delta)/\mathrm{Im}(\delta) \cong \mathrm{Ker}(d)/\mathrm{Im}(d)$ and analogously for the bounded group cohomology $H_b^n(G, \mathbb{R})$. Show that

$$H^0(G, \mathbb{R}) = H_b^0(G, \mathbb{R}) = \mathbb{R},$$
$$H^1(G, \mathbb{R}) = \mathrm{Hom}(G, \mathbb{R}), \quad H_b^1(G, \mathbb{R}) = 0.$$

(3) (Medium) Define the set of quasimorphisms on the group G as

$$QM(G) := \{f : G \to \mathbb{R} \mid \exists\, C,\ |df(g,h)| = |f(g) - f(gh) + f(h)| < C,\ \forall g, h \in G\},$$

and denote the set of bounded functions on G by $B(G)$. Show that the kernel of the comparison map

$$H_b^2(G, \mathbb{R}) \longrightarrow H^2(G, \mathbb{R})$$

induced by the natural inclusion of cocomplexes is isomorphic to

$$QM(G)/(\mathrm{Hom}(G, \mathbb{R}) \oplus B(G)).$$

(4) (Medium-hard. Fun but peripheral to the content of this course) Likewise one can define homogeneous quasimorphisms with $\mathbb{Z}$-coefficients

$$QM(G, \mathbb{Z}) := \{f : G \to \mathbb{Z} \mid \exists\, C,\ |df(g,h)| = |f(g) - f(gh) + f(h)| < C,\ \forall g, h \in G\},$$

and denote by $B(G, \mathbb{Z})$ the bounded $\mathbb{Z}$-valued functions. Show that

$$\mathbb{R} \cong QM(\mathbb{Z}, \mathbb{Z})/B(\mathbb{Z}, \mathbb{Z}).$$

This can be taken as a construction of the real numbers. Somehow much more elementary than the two standard constructions with Cauchy sequences or Dedekind cuts. See the article of A'Campo "A natural construction for the real numbers" for more details.