[Assume familiarity with diff geom / Lie groups)

I Riemann surfaces and holomorphic functions.

« A Riemann surface or complex curve is a 2d real manifold with a distinguished notion of holomorphic functions »

1) First definition

A Riemann surface (more generally a complex manifold) is given by the data of a (Hausdorff/σ compact) topological space X with an complex atlas $\{(U_i, \varphi_i)\}$

- U_i is a covering of X.
- z_i homeomorphisms from $U_i \to V_i \subset \mathbb{C}$
- so that $z_i \circ z_j^{-1}$ are holomorphic

Then a holomorphic function on $V \overset{\text{open}}{\subset} X$; is a map $f : V \to \mathbb{C}$ so that $\forall i$; $f \circ z_i^{-1}$ is holomorphic

$\mathcal{O}(V) = \{$space of holomorphic functions on $V\}$

A map $\Psi : X \to Y$ is holomorphic if it is continuous and $\forall V \subset Y$; $\Psi^*(\mathcal{O}(V)) \subset \mathcal{O}(\Psi^{-1}(V))$

Ψ is biholomorphic if it is a bijection and both Ψ, Ψ^{-1} are holomorphic

Example: Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$; homographies $z \longmapsto \frac{az+b}{cz+d}$

then $\operatorname{Bihol}(\mathbb{H}) = \{\text{homographies}\} = PSL_2(\mathbb{R})$

in higher dimension ⇝ complex manifold

2) Second definition, an almost complex manifold is a real manifold M equipped with a section $J \in \Gamma(\operatorname{End}(TM))$ so that $J^2 = -1$.

$$\forall x \in M; \quad J_x : T_xM \circlearrowleft, \; J_x^2 = -1$$

A holomorphic function is a C^1 function $V \subset M \to \mathbb{C}$ so that

$$df \circ J = i\, df$$

⚠ in higher dimension, holomorphic functions could be rare!

Example : ⓐ Any oriented Riemannian surface (i.e surface with a Riemanian metric g) has an almost complex structure : $J_x =$ rotation by $\frac{\pi}{2}$ in $T_x S$.

Observe two metrics give the same Riemann surface structure iff they are conformal

That is $g_1 = \lambda g_2$, $\lambda \in C^\infty(S)$

ⓑ Any Riemann surface (for (1)) is a Riemann surface (for (2)) with the same holomorphic functions

Theorem an almost complex surface is a Riemann surface with the same notion of holomorphic functions. [existence of isothermal coordinates]

◀ Prove that there exists local holomorphic functions for (2) ▶

(not true in higher dimension)

Uniformization theorem

Any connected simply connected Riemann surface is biholomorphic to $\mathbb{C}$, $\mathbb{H}$ or S^2

Corollary . Any closed Rieman surface admits a constant curvature metric, unique up to scalar multiplication.

Corollary : If S is a Riemann surface of $g \geq 2$, then $S \approx \mathbb{H}/\Gamma$ where $\pi_1(S) \approx \Gamma \subset PSL_2(\mathbb{R})$.

II Holomorphic objects

[take S "underlying surface" to a Riemann surface X]

let $T_{\mathbb{C}}S \to S$; $T_{\mathbb{C}}S = \mathrm{Hom}(TS, \mathbb{C})$ = (complex valued forms)

$T_{\mathbb{C}}^{1,0}S := \{\omega \mid \omega(Ju) = i\,\omega(u)\} \longrightarrow X$ (complex linear forms)

$T_{\mathbb{C}}^{0,1}S := \{\omega \mid \omega(Ju) = -\omega(u)\} \longrightarrow X$ (complex antilinear forms)

If $U \subset S$; $\Omega^{1,0}(U) := \{$sections over U of $T_{\mathbb{C}}^{1,0}(X)\}$

$\Omega^{0,1}(U) := \{$sections U of $T_{\mathbb{C}}^{0,1}(X)\}$

Obviously :

- $d : \mathcal{O}(U) \to \Omega^{1,0}(U)$
- $\bar{d} : f \mapsto d\bar{f}$; $\mathcal{O}(U) \to \Omega^{0,1}(U)$.

A 1-form α on $U \subset X$ is holomorphic if $\forall x \in U$, there exists $V \in \mathcal{V}(x)$; and $f_1, \dots, f_N, g_1, \dots, g_N \in \mathcal{O}(V)$ so that

$$\omega = \sum_1 f_i\, dg_i \text{ on } V$$

We also say α is an abelian differential and denote by $\omega(U)$ the set of abelian differentials over U.

Proposition

(i) Any abelian differential is closed

(ii) If z is a local complex coordinate then any abelian differential is $f(z)\,dz$ where f is holomorphic

(iii) locally any abelian differential $\omega = dh$, where h is holomorphic

Holomorphic vector fields. A vector field is holomorphic if for every holomorphic differential $\omega(\xi)$ is a holomorphic function.

Exercise. Show that ξ is holomorphic $\iff$ $L_\xi J = 0$

where $L_\xi J \in \Gamma(\mathrm{End}(TS))$ given by

$$(L_\xi J)(X) = J[\xi, X] - [\xi, JX].$$

Higher differentials

A n ic differential defined on U is a section of $\mathrm{Sym}(T^{1,0}(S))$ on U

Definition a n ic differential is holomorphic, if locally it can be written

a $\sum_i f_i (dg_i)^n$ where f_i, g_i are holomorphic

Exercise: If ω is a holomorphic differential, it can be written as

$\omega = f(z) dz^n$ in a complex chart.

III Holomorphic vector bundle

« A holomorphic bundle is a vector bundle with a notion of holomorphic sections »

1. Definition

let X be a Riemann surface, and $\{E_x\}_{x\in X}$ a family of vector spaces indexed by x; If U open $\subset X$, a trivialisation g^U is a family of isomorphisms $g^U_x : E_x \to E_U$, for all x in U

Two trivialisation g^U and g^V are holomorphically compatible if

$$U \cap V \to \mathrm{Hom}(E_U, E_V)$$
$$z \mapsto g^V_z \circ (g^U_z)^{-1}$$

are holomorphic

A holomorphic vector bundle $\mathcal{E}$ is a family of vector spaces indexed by X together with compatible trivialisation g^{U_i}, where U_i is a covering of X.

A local trivialisation of $\mathcal{E}$ is a trivialisation compatible with all the g^{U_i}

A holomorphic section of E over U, is a map $U \mapsto \bigsqcup_{x \in U} E_x$, $y \mapsto u_y$ so that for every $x_0 \in U$, there exists a local trivialisation g^V, $x_0 \in V$

so that $y \mapsto g^V(u_y)$ is holomorphic

We shall denote by $\mathcal{E}(U)$ the vector space of holomorphic sections over U,

If $\mathcal{E}, \mathcal{F}$ are holomorphic over X, Y. let $F : \mathcal{E} \to \mathcal{F}$ a bundle morphism over $f : X \to Y$, is holomorphic if given any holomorphic trivialisations $\mathcal{E}|_U \to E_u \times V$; $\mathcal{F}|_V = F_v \times V$ so that $f(U) \subset V$, then

$$\begin{array}{ccc} \mathcal{E}_U & \xrightarrow{\mathcal{F}} & F \times V \\ \updownarrow & & \updownarrow \\ E_u \times U & \xrightarrow{F_0} & F_v \times V \end{array} \qquad \text{then } F_0 : (u, x) \mapsto (g(u,x), f(x))$$

is holomorphic. In particular, f is holomorphic. Actually it is enough to check this property for trivialisation defined over a covering. Exercise: define isomorphisms

Algebraic constructions If $\mathcal{E}$ is a holomorphic vector bundle, then $\mathcal{E}^*$ is holomorphic

. If furthermore $\mathcal{F}$ is holomorphic so are $\mathcal{E} \otimes \mathcal{F}$, $\mathcal{E} \oplus F$.

2. Definition (Cauchy-Riemann operator on X)

$$\bar\partial : C^\infty(X, \mathbb{C}) \to \Omega^{0,1}(X)$$

$$f \mapsto \bar\partial f = df^{0,1} : u \mapsto \frac{1}{2}[df(u) + i\, df(Ju)]$$

- f holomorphic $\Leftrightarrow \bar\partial f = 0$
- $\bar\partial fg = \bar\partial f \cdot g + g \bar\partial f$.

let E be a complex vector bundle over X a Cauchy Riemann operator (or by a standard and confusing practice a $\bar\partial$-operator) on E is a linear operator

$$\bar\partial_{\mathcal{E}} : \Gamma(E) \longrightarrow \Omega^1(X,E) \quad \text{(complex 1-forms with values in } E)$$

so that $\bar\partial_{\mathcal{E}}(f\sigma) = f\bar\partial\sigma + \bar\partial f \cdot \sigma$

a holomorphic vector bundle is $(E, \bar\partial)$

a holomorphic section σ is so that $\bar\partial\sigma = 0$

Examples ① If ∇ is a connection on E $\bar\partial : \sigma \longmapsto \left(X \mapsto \frac{1}{2}\left(\nabla_X\sigma + i\nabla_{JX}\sigma\right)\right)$ is a $\bar\partial$-operator

② the trivial holomorphic bundle has a $\bar\partial$-operator whose holomorphic sections are the same

③ Every holomorphic bundle $\mathcal{E}$ (in the first sense) has a $\bar\partial$ operator [check the $\bar\partial$ operator on trivialisations coincide]

Theorem [Newlander–Nirenberg]

Every $(E, \bar\partial)$ has the structure of a holomorphic bundle with the same notion of holomorphic sections

◀ Show that we have a frame $(\sigma_1, \ldots, \sigma_n)$ of sections (locally) so that $\bar\partial\sigma_i = 0$ ▶

Examples of holomorphic bundles / notations

- notation if $\mathcal{E}$ is holomorphic over X
  $H^0(X,\mathcal{E}) = \{\text{holomorphic sections}\} = H^0(\mathcal{E})$
- $T^*_{\mathbb{C}}S \to S$, if S is a Riemann surface admits a holomorphic structure denoted K_S the canonical bundle
- abelian differential on X $\in H^0(K_X)$
- n-ic holomorphic differential $\in H^0(K_X^n)$