The heat flow

Let f a map between Riemannian manifolds $N \to M$

Recall that

$$\Delta f := \sum_{X_i} (\nabla_{X_i} Tf)(X_i) \in \Gamma(N, f^*TM)$$

and

$$\frac{d}{dt} E(f_t) = -\int_M \langle \dot{f}, \Delta f \rangle d\mu$$

(First variation formula)

A family f_t is a solution of the heat flow if

$$\frac{\partial f}{\partial t} = \Delta f$$

If M is compact, and f an equivariant section of

then $\qquad \frac{d}{dt} E(f_t) = -\int \|\Delta f\|^2 d\mu$

Goal show

(in the good cases) that $f_t \to f_\infty$

A/ The linear situation ($M = \mathbb{R}^n$): in this situation, one can solve explicitely by taking

Theorem (Assuming N is compact).

There exists functions $p_t(x,y) : \mathbb{R}^{>0} \times N \times N$ so that

$$f_t(x) := \int_N p_t(x,y) f(y) dy$$

are a solution of the linear heat flow

$$\frac{\partial}{\partial t} f_t = \Delta f_t \; ; \text{ with } f_0 = f$$

$p_t(x,y)$ is the heat kernel.

We have more information :

Proposition : A let g_t be a family of functions positive which satify $\frac{\partial}{\partial t} g_t \leqslant \mathring{\Delta} g_t$

(i) $\|g_t\|_{C^0}$ is decreasing ,

(ii) $|g_t(x)| \leqslant K \int_N |g_{t-\varepsilon}(y)| dy$.

◀ let x the maximum of g_t, then $\Delta g_t \leqslant 0$, and thus $\frac{\partial g_t}{\partial t}(x) \leqslant 0$.

For the second part let f_t be the solution of the heat flow with $f_s = g_s$; let $h_t = g_t - f_t$ then

$$\frac{\partial}{\partial t} h_t \leqslant \Delta h_t$$

It follows that $\operatorname{Sup}(h_t)$ is non increasing. Since $h_s = 0$

It follows that for all $t > s$, $h_t(x) \leqslant 0$; thus $g_t \leqslant f_t$

then we just have to prove (ii) for f_t :

$$g_{s+\varepsilon}(x) \leqslant f_{s+\varepsilon}(x) \leqslant K \int |f_s(y)| dy = K \int |g_s(y)| dy$$

which is a result known from the heat equation using the heat kernel : $f_{s+\varepsilon}(x) = \int P_\varepsilon(x,y) f_s(y) dy$

since $P_\varepsilon(x,y)$ is bounded. $f_{s+\varepsilon}(x) \leqslant K \int f_s(y) dy$ ▶

A bis] The heat flow is a quasilinear flow

Let us choose coordinates on the target, so that the Levi-Civita connexion is written as $\nabla = D + A$ where D is trivial;

Then the equation is rewritten

as $$\frac{\partial f}{\partial t} = \mathring{\Delta} f + G(Df, f)$$

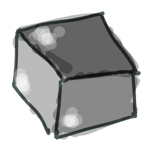
quadratic in Df

where $\mathring{\Delta}$ is the linear laplace operator on functions:

$$\mathring{\Delta} f = \sum_{u_i} D^2_{u_i u_i} f$$

B] For any f_0, M, N there exists a short time solution f_t defined on $[0, \varepsilon]$ with $\varepsilon > 0$

Black box



Idea for that:

Ideally we would have loved to consider the Banach manifold $C^k(M,N)$, and show that the heat flow is a vector field on $C^k(M,N)$.

This is not the case: if f is C^k, then Δf is C^{k-2}

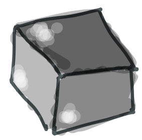
This is already a problem when $N = \mathbb{R}^N$. Nevertheless in that case we can solve the problem. The strategy is then

Using coordinates, to write the heat flow as a perturbation of the linear heat flow. Then by a contraction argument show that the perturbed linear flow has a solution.

Rk: this is a general fact about parabolic quasi linear flow.

C) (Weak Schauder estimates)

second black box



Moreover one has the Schauder estimates

$\forall n \geq 0 \quad R > 0 \ ; \forall C$ compact in M

$\exists K, \ \mathbb{R} \to \mathbb{R}$, depending on C, n, R,

so that if f_s lies in C for $t-\varepsilon < s < t$

$$\|\nabla^n T f_x\|_{C^0} \leqslant \sup_{\substack{y \in B(x,R) \\ t-\varepsilon < s < t}} K\left(\|T f_{y,s}\|^2\right)$$

◀ Some remarks;

a) we can consider the quasi linear version

$$\frac{\partial}{\partial t} f = \Delta^0 f + \underbrace{G(Df, f)}_{\leftarrow g}$$

Results from the linear theory tells you

$$\|D^2 f_x\| \leqslant K\left(\|f_0\|_{B(0,R)} + \|g\|_{B(0,R)}\right)$$

$\rightsquigarrow$ thus the result for $n = 1$ follows

b) To obtain higher derivatives you differentiate the equation above

$$\frac{\partial}{\partial t} Df = \Delta(Df) + DG(Df, f)$$

$\hookrightarrow G(D^2 f, Df, f)$

$\Rightarrow$ you use a boot strap to get the result ▶

D] When N is complete with $K \leq 0$ then the heat flow is defined on $[0, +\infty[$

This uses some information

Let $e_t = \|Tf_t\|^2$; $w_t = \|\Delta f_t\|^2$

Then explicit computations using $(K \leq 0)$ yields that

Prop : (*) $\dfrac{\partial w_t}{\partial t} \leq \mathring{\Delta} w_t$; $\dfrac{\partial e_t}{\partial t} \leq \mathring{\Delta} e_t + Re$

Where R only depends on the source

Consequence B

(i) $\|\frac{\partial f}{\partial t}\|^2$ uniformly bounded

(ii) $\|Tf_t\|$ uniformly bounded (for $t > \varepsilon \geq 0$)

(iii) Assuming $f_t(x)$ stays in a compact region $\|\nabla^m f_t(x)\|$ are uniformly bounded

◀ (i) is a direct consequence of (*) and A.(i).

(ii) let $e'_t = \exp(Rt)e_t$ then $\frac{\partial}{\partial t} e'_t \leq \Delta e'_t$ thus by A(ii)

$\varepsilon > 0$; $e'_{t+\varepsilon} \leq K_1 \int e'_t$, thus $e_{t+\varepsilon} \leq K_2 \int e_t = K_2 E(f_t) \leq K_2(f_0)$

(iii) is a consequence of Schauder estimates ▶

Then long time existence of the flow follows.

f_s ; $s \to t_0$; then

$\|f_s\|_{C^r}$ are bounded, $r \geq 1$

Thus $f_s \xrightarrow[C^\infty]{} f_{s_0}$ (using (1) + Ascoli theorem)

E] What about $t \to +\infty$?

proposition Assume $f_t(x_0) \to y_0$ in M for some subsequence, then $f_{t_i} \to f^\infty$ and f^∞ is harmonic

◀ as a consequence of our previous estimates

(i) $\|T f_{t_i}\|$ is uniformly bounded

thus $f_{t_i} \xrightarrow[t_i \to \infty]{C^0}$ to f^∞

(ii) thus using B iii) $f_{t_i} \xrightarrow{C^\infty} f_\infty$

And moreover

$$]t-1, t+1[\times M \xrightarrow{C^\infty} \text{A solution of the heat equation for } f^\infty$$

defined on $]-1,+1[$

However as $t \to \infty$ $|E(f_{t+s}) - E(f_t)| \to 0 \ \forall s$

$\Rightarrow$ implies that for the heat equation for f^∞

$E(f^\infty_t) = cst$; hence $\frac{d}{dt} E(f^\infty_t) = 0$

thus since $\frac{d}{dt} E(f^\infty_t) = -\int \|\Delta f^\infty\|^2 d\mu$

$\Rightarrow \Delta f^\infty = 0$; and f^∞ is harmonic ▶

⚠ up to here, this is exactly Eells – Sampson proof and result

1) proposition If $d(f_t(x), f_0(x)) \to \infty$ then

$\rho(\Gamma)$ preserve a Buseman function.

◀ $f_{t_i}(x) \to h$ a Buseman function

we know that $f_{t_i}(\gamma x) \to \rho(\gamma) h$

but $d(f_{t_i}(\gamma(x)), f_{t_i}(x)) \leqslant K$

[Because of our B (i)]

thus $\rho(\gamma) h = h$ ▶

Corollary If $\rho(\Gamma)$ does not preserve a Buseman function, then the heat
(L.) flow converges to a harmonic mapping

Theorem [Corlette] Assume M is a symmetric space, without euclidian factor. Assume $\rho(\Gamma) \subset \mathrm{Iso}(M)$ is reductive. Then there exists an equivariant harmonic mapping.

◀ $\overline{\rho(\Gamma)}^{z} = H$; let W be the totally geodesic subspace associated to H; then $W = \mathbb{R}^k \times W_1 \times \cdots \times W_n$

$H \to \mathrm{Iso}(W_i)$ is surjective (at the level of lie algebras)

then $\overline{\rho(\Gamma)}^{z}$ is Zariski dense in $\mathrm{Iso}(W_i)$; $\mathrm{Iso}(W_i)$ is simple.

Thus we are reduced to

a) the linear case (harmonic functions)

b) the case where $\rho(\Gamma)$ is Zariski dense in a simple group

Then $\rho(\Gamma)$ does not preserve a busemann function, thus the heat flow converges ▶

Remark: the converse also holds. Essentially this comes from the fact that in rk 1, if you preserve a Buseman function of h and φ_t is the gradient flow of h; then φ_{-t} decreases strictly the energy.