I Hyperbolic geometry on surfaces.

In the various models of the hyperbolic spaces $\rightsquigarrow$ a boundary at ∞ $\partial_\infty \mathbb{H}^2$ with the following property

(i) $\mathrm{Iso}(\mathbb{H}^2) \curvearrowright \partial_\infty \mathbb{H}^2$ and the action is conjugated to the action of $PSL_2(\mathbb{R})$ on $\mathbb{RP}^1$

(ii) $\mathcal{G}_{\mathbb{H}^2} = \{\text{pairs distinct pts of } \partial_\infty \mathbb{H}^2\} = \{\text{geodesics in } \mathbb{H}^2\}$

$$\mathcal{G}_{\mathbb{H}^2} \approx PSL_2(\mathbb{R}) / A \nearrow A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \right\}$$

In other words we have a $\mathbb{R}$-line bundle

$$\mathbb{R} \longrightarrow \underset{\substack{\wr\wr \\ PSL_2(\mathbb{R})}}{U\mathbb{H}^2} \longrightarrow \mathcal{G}_{\mathbb{H}^2}$$

The geodesic flow on $U\mathbb{H}^2$ is the identify with the $\mathbb{R}$-action on this bundle.

II Fuchsian groups and the action of $\mathbb{RP}^1$

Assume now that Γ acts on $\mathbb{H}^2$, $\Gamma \backslash \mathbb{H}^2 = S$ close surface.

(i) every non trivial element in Γ, acts on $\partial_\infty \mathbb{H}^2$ with exactly two fixed pt γ^+, γ^-

if $\gamma = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$ with $a > 1$; then $\gamma^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\gamma^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

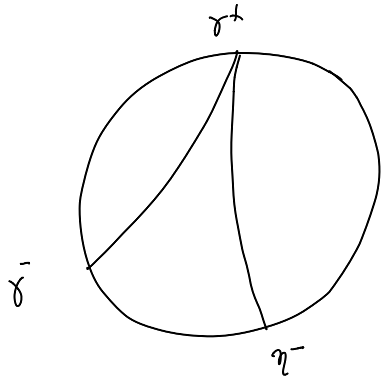
any element $\gamma \rightsquigarrow$ a geodesic $\hat{\gamma}$ so that

$$\gamma(\hat{\gamma}(t)) = \hat{\gamma}(t+L) \text{ where } L_\gamma =: \text{length of } \gamma.$$

(ii) If $\gamma^+ = \eta^+$ there $\exists n, p > 0$ $\gamma^n = \eta^p$

◀ If $\gamma^+ = \eta^+$ then there exists

a parametrisation so that $\lim_{t \to +\infty} d(\gamma(t), \eta(t)) = 0$



$\Rightarrow \forall \varepsilon$; the closed geodesic η lies in an ε-tubular neighborhood of γ

thus $\hat{\gamma}(t) = \hat{\eta}(t)$ (after a translation)

Since Γ is discrete it follows that $\mathbb{Z}L_\gamma + \mathbb{Z}L_\eta$ is discrete. Thus $\exists n, p$ so that $n L_\gamma = p L_\eta$, and then $\gamma^n = \eta^p$. ▶

(ii) every orbit of Γ on $\partial_\infty \mathbb{H}^2$ is dense

◀ let Λ be such an orbit, let C = Convex Envelope in $\mathbb{H}^2$ of $\overline{\Lambda}$

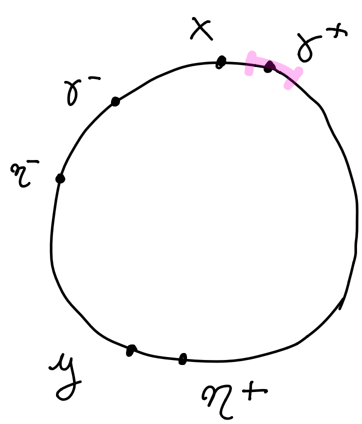
Then C is globally invariant by Γ, and bounded by geodesic

thus C/Γ is a closed surface with totally geodesic boundary Σ_1'

It follows that since Σ_1' retracts on a graph, that

$\Gamma = \pi_1(\Sigma_1')$ is free if Λ is not dense ▶

(ii) the set $\{(\gamma^+, \gamma^-) \mid \gamma \in \Gamma\} \subset \mathcal{G}$ is dense



◀

then $(\underbrace{\gamma^n \eta^{-n}}_{=\xi})(\mathcal{U}) \subset \mathcal{U}$

$(\underbrace{\gamma^n \gamma^{-n}}_{=\xi})^{-1}(V) \subset V$

It follows that $\xi_n^+ \subset \mathcal{U}$, $\xi_n^- \subset V$ ▶

III Conjugacy between ∂_∞

Proposition : given two representations $\rho_1, \rho_2 : \Gamma = \pi_1(S) \to PSL_2(\mathbb{R}) = \mathrm{Iso}(\mathbb{H}^3)$

There exists Ψ Hölder

$$\Psi : \partial_\infty \mathbb{H}^2 \longrightarrow \partial_\infty \mathbb{H}^2$$

$$\Psi(\rho_1(\gamma)) = \rho_2(\gamma)\Psi.$$

◀ Proven later ▶ but there is a topological proof below

Definition (1st version)

A boundary at ∞^{ty} is a circle $S^1 \circlearrowleft$ topological action of Γ so that this action is conjugated with the action of $\rho(\Gamma)$ on $\partial_\infty \mathbb{H}^2$

Remark (i) All boundary at ∞^{ty} are isomorphic $\leadsto \partial_\infty \pi_1(S)$

(ii) $\partial_\infty \pi(S)$ carries a Hölder structure (i.e is a Hölder manifold)

Let then $\mathcal{G}_\Gamma = \partial_\infty \Gamma \times \partial_\infty \Gamma \setminus \Delta \ \sim \ \mathcal{G}_{\mathbb{H}^2}$ using an hyperbolisation

Def : A geodesic flow for Γ is a $\mathbb{R}$- principal bundle

$$L \longrightarrow \mathcal{G}_\Gamma$$

equipped with an action of Γ so that L/Γ is compact

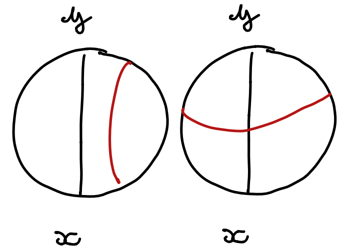
Theme : {Anosov representation of Γ} $\longrightarrow$ {geodesic flow for Γ}

spectral radius $\longrightarrow$ closed orbit

III A topological description of $\partial_\infty \Gamma$ using intersection and positivity

(iii) Given $(x,y) \in \mathcal{G}/\mathbb{Z}_2$; $(z,t) \in \mathcal{G}/\mathbb{Z}_2$; x, y, z, t all pairwise distinct

we have two possible configurations.



Proposition : all 2 possible configurations for $(x,y) = (\xi^+, \xi^-)$; $(z,t) = (\eta^+, \eta^-)$

B For $\xi, \eta \in \pi_1(S)$ are described purely topologically :

◀ Case (1) . There exists $n > 0$ so that $\Gamma = \langle \xi^n | \eta^n \rangle < \pi_1(S)$ is free and ξ, η are represented by simple closed geodesics with intersection 0

Case (2) . There exists $n > 0$ so that $\Gamma = \langle \xi^n | \eta^n \rangle < \pi_1(S)$ is free and ξ, η are represented by simple curves with exactly 1 intersection pt ▶

Proposition : given two reprentations $\rho_1, \rho_2 : \Gamma = \pi_1(S) \to PSL_2(\mathbb{R}) = \mathrm{Iso}(\mathbb{H}^3)$

There exists Ψ continuous (later we will have Hölder)

$$\Psi : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$$

$$\Psi(\rho_1(\gamma)) = \rho_2(\gamma)\Psi .$$

1) let $\Lambda^\infty(\Gamma) = \{\gamma \in \Gamma \setminus \{id\}\}/\sim$ $\quad \gamma \sim \eta$ if $\exists n, p > 0$; $\gamma^p = \eta^n$

Then we can define $\Psi : \Lambda^\infty(\Gamma) \to \partial_\infty \Gamma$ without any ambiguity

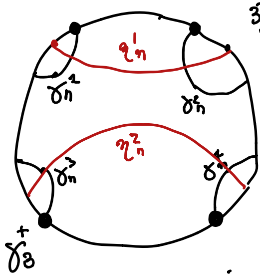
(by previous proposition)

2) Say that (i) say that $\gamma^n \to \eta^+$, if a) γ_n intersect η

b) $\forall \xi \; \exists n_0 , n > n_0 \Rightarrow \xi$ does not intersect ξ

3) say that $(x_1,x_2,x_3,x_4) \in \Lambda^\infty(\Gamma)$ is positive if there exists

$\gamma_n^1, \gamma_n^2, \gamma_n^3, \gamma_n^4, \eta_n^1, \eta_n^2$ so that $\gamma_n^i \to x_i$ and we have :



3) Show that $\Psi : \Lambda_\infty(\Gamma) \to \partial_\infty \Gamma$; $[\gamma] \longrightarrow (R(\gamma))^+$ preserves the cyclic ordering.

3) conclude by remarking that if $\Psi : \Lambda \subset \partial_\infty \mathbb{H}^2 \longrightarrow \Lambda' \subset \partial_\infty \mathbb{H}^2$ is bijective, preserves the cyclic ordering and Λ, Λ' dense. Then Ψ extends uniquely to a continuous map ▶

III Boundary at ∞ and geodesic flows.

Def : A boundary at ∞, $\partial_\infty \pi_1(S)$ is a topological circle $\Lambda \curvearrowleft \pi_1(S)$ so that (i) γ acts by $\gamma^- \circlearrowleft \gamma^+$ on Λ, $\gamma^+ = \eta^+ \Leftrightarrow \gamma \sim \eta$

(ii) Every orbit of $\pi_1(S)$ is dense

[(iii) the intersection of (γ^+, γ^-) with (η^+, η^-) agrees with the topological one]

Theorem all boundary at ∞'s are homeomorphic

⚠ The notion of ∂_∞ extends to a much more general situation, but not this proof. This proof emphasizes the order structure of $\partial_\infty \pi_1(S)$ which is a feature of dim 2, and is intimately related to positivity.