I Preliminary on contracting maps,

λ- Contracting map :

X metric space

$\phi : X \to X$ is contracting if $\exists\, \lambda < 1$ so that

$$d(\phi(x), \phi(y)) < \lambda d(x,y)$$

Thm (Banach 1922)

If X is complete non empty every contracting map has a unique fixed point

◀ Sketch : $\{\phi^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to a fixed point ▶

Some improvement :

If $(x,z) \overset{\Psi}{\mapsto} \phi_z(x)$ is C^0, $X\times Z \to X$ and $\exists\, \lambda < 1$ s.t that for all $z \in Z$

$$d(\phi_z(x), \phi_z(y)) \leqslant \lambda d(x,y)$$

then the fixed point $g(y)$ of ϕ_y depends continuously on y.

◀ Exercise ! various proof ▶

« the fixed points of a continuous family of contracting maps depend continuously on the parameter »

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Assume furthermore, X, Y are Banach vector spaces and Ψ is C^1 then g is C^1

◀ ① ϕ_y C^1 + λ contracting $\Rightarrow \| D_{g(z)}\phi \| \leqslant \lambda < 1$

thus $1 - D_{g(z)}\phi$ is invertible

② You can apply the implicit function theorem to the equation

$$x - \phi_z(x) = 0$$

⇝ the result follows ▶

Remarks :

1) By the chain rule applied to $\Phi_z(g(z)) = g(z)$

You obtain an actual equation $D_z g = F(\Psi, g(z))$ (for some F)

2) Then, we can promote by bootstrapping if Ψ is C^k, C^ω

that g is C^k, C^ω

3) Finally this extends when X, Z are (Banach) manifolds.

II. Bundle contracting mapping

Let M be a metric space ; a Lipschitz bundle with fiber X over M is a metric space N, together with a map

$$\pi : N \longrightarrow M$$

admitting local lipschitz trivialisation.

$$\begin{tikzcd} N \supset \pi^{-1}(U) \arrow[r, leftrightarrow] \arrow[d] & X \times U \\ M \supset U \arrow[ur, leftrightarrow] & \end{tikzcd}$$

$\leftarrow$ with the sup metric

The fiber at $m \in M$ is $N_m := \pi^{-1}(m)$

π is the projection or fiber map N is the total space, M is the base.

Given $\varphi : M \to M$ a bundle map over φ is a map $\phi : N \to N$ such that $\phi(N_m) \subset N_{\varphi(m)}$

A section is a map σ from M to N such that $\pi \circ \sigma = \mathrm{Id}$

Example

1) the trivial bundle $N = M \times X$, then bundle maps are just family of maps from X to X depending on a parameter in M.

then sections are maps from M to X.

2) Vector Bundles, Grassmannian Bundles.

A contracting bundle map is a map over φ is a bundle map ϕ over φ so that there exists $\lambda < 1$ so that

$$\forall m, \forall x, y \in N_m,\ d(\phi(x), \phi(y)) \leqslant \lambda d(x,y)$$

Given a bundle map ϕ a fixed section is a section σ so that

$$\phi(\sigma(m)) = \sigma \varphi(m)$$

Another interpretation:

Let $\Gamma = \{\text{section : of } \pi : N \to M\}$ A bundle map defines

$$(\phi_* \sigma)(m) = \phi(\sigma(\varphi^{-1}(m))) \quad : \quad \phi_* \ \Gamma \circlearrowleft$$

(Assuming φ is invertible)

a fixed section is a fixed point in Γ of ϕ_*

Theorem [Existence of a fixed section]

Assume ϕ over φ is contracting, assume M is compact and N is complete, and the fiber is contractible.
Then ϕ admits a unique fixed section

◀ Let Γ^0 be the space of continuous sections equipped with

$$d_\infty(\sigma, \eta) = \sup_{x \in M} d(\sigma(x), \eta(x))$$

Then Γ^0 is complete, ϕ_* is contracting thus we can apply Banach theorem ▶

Application: centralizers.

By uniqueness if $\Psi : M \circlearrowleft M$ over ψ commuts with ϕ
Then by uniqueness $\Psi_* \sigma = \sigma$, if σ is fixed by ϕ

Say Φ is eventually contracting if $\exists\, N>0$ so that Φ is contracting. Then the theorem holds for eventually contracting map. Observe the notion of eventually contracting is independent of the choice of metrics bilipschitz equivalent

Corollary: if $\{\Phi_t\}$ is a flow over $\{\varphi_t\}$, if Φ_T is contracting for $T>0$, if $\{\varphi_t\}$ is a flow by bilipschitz map. Then $\exists$ a Hölder invariant section for all $\{\Phi_t\}$.

III The regularity of a fixed section

Assuming we are in the C^1-category: Φ, φ, M, N are C^1. Is σ C^1?

- In general $\underline{\text{NO}}$ (no easy counterexample at this stage)
- In special cases: $\varphi = \text{Identity}$, yes: this is the «parameter theorem» above.

However we have an improvement in the Hölder category.

$f : X \circlearrowleft$ is α-Hölder, if there exists K so that

$$d(f(x), f(y)) < K\, d(x,y)^\alpha$$

Thm [Hölder regularity] (Hirsch-Pugh-Shub).

Assume that (i) Φ is λ-contracting, and lipschitz

(ii) φ is invertible and φ^{-1} is k lipschitz.

let α so that $\lambda k^\alpha < 1$, $\alpha < 1$

Then the fixed section is α-Hölder

($\underline{\text{Rk}}$: such an α always exists since $\lambda < 1$)

◀ Let us consider for every σ, and $\alpha < 1$, $\varepsilon > 0$

$$V_\alpha^\varepsilon(\sigma) = \sup_{0<d(x,y)<\varepsilon} \left(\frac{d(\sigma(x), \sigma(y))}{d(x,y)^\alpha} \right)$$

let us consider the space of « Hölder sections »

$$\Gamma_\varepsilon(H,\alpha) = \{ \sigma \in \Gamma^0 , V_\alpha^\varepsilon(\sigma) \leqslant H \}$$

Then we have the following steps

① $\Gamma_\varepsilon(H,\alpha) \neq \emptyset$ [Comes from the local lipschitz structure]

② Using the compactness of M; If $\sigma \in \Gamma_\varepsilon(H,\alpha)$ then σ is α-Hölder, Moreover $\Gamma_\varepsilon(H,\alpha)$ is closed in Γ^0

③ let us understand the action of ϕ_* on $\Gamma_\varepsilon(H,\alpha)$. To conclude

we just have to show that $\Phi_* (\Gamma_\varepsilon(H,\alpha)) \subset \Gamma_\varepsilon(H,\alpha))$.

* Assume first that N is the trivial bundle $N = M \times X$

with the metric $d((x,m),(x',m')) = \sup(d(x,x'), d(m,m'))$

↳ then the bundle map is

$$\phi(m,u) = (\varphi(m), \Psi_m(u))$$

↳ a section is a map $f : M \to X$: $\sigma(m) = (m, f(m))$

let $\sigma \in \Gamma(H,\alpha)$

$$d(\phi(\sigma(x)), \phi(\sigma(y))) = \sup \left(d(\varphi(x), \varphi(y)) , d(\Psi_x(f(x)), \Psi_y(f(y))\right)$$

■ The crucial term is

$$A(x,y) = d(\Psi_x(f(x)), \Psi_y(f(y)))$$

$$\leqslant d(\Psi_x(f(x)), \Psi_x(f(y))) + d(\Psi_x(f(y)), \Psi_y(f(y)))$$

$$\leqslant \lambda H d(x,y)^\alpha + \Lambda d(x,y)$$ ↶ using the fact that $f \in \Gamma(H,\alpha)$

Where Λ is the lipschitz constant of Ψ

$$d(\phi^* \sigma(x), \phi^* \sigma(y)) = d(\phi(\sigma(\varphi^{-1}(x)), \phi(\sigma(\varphi^{-1}(y)))$$
$$\leqslant d(x,y) + \lambda H d(\varphi^{-1}(x), \varphi^{-1}(y))^\alpha + \Lambda\, d(\varphi^{-1}(x), \varphi^{-1}(y))$$
$$\leqslant d(x,y)(1+\Lambda k) + (\lambda k^\alpha) H d(x,y)^\alpha$$

thus $V_\alpha(\phi^*(\sigma)) \leqslant H$ ■

Thus $\phi^* : \Gamma(H,\alpha) \circlearrowleft$

* Actually the computation is local in x, y : we are only interested in the situation when $d(x,y) \to 0$.

Thus we can assume $x, y \in$ a trivializing open set by choosing ε small enough.

Then we can resume the computation ▶

Exercise : what happens when $\lambda k < 1$; i.e $\alpha = 1$

$$H \lambda k + 1 + k \leqslant H \quad ?$$

III The parameter case

Assume now that we are in the parameter case : that is we have a family φ_z of mappings $M \circlearrowleft M$; $z \in Z$ covered by contracting bundle maps ϕ_z. What can we say about the fixed section σ_z

Theorem [with parameters]

Assume Z, M, N $(z,n) \to \phi_z(n)$ are respectively $C^0, C^1, \dots C^\omega$

then $z \mapsto \sigma_z$ is a C^0, C^1, C^ω map with values in Γ^α (Hölder sections)

Dominated splittings
Anosov bundles.

II Splitting, bounded splitting.

Let E be a vector bundle, a splitting : $E = U \oplus V$ is the data of a decomposition $E_x = U_x \oplus V_x$ at every point, so that $\dim(U_x)$ and $\dim(V_x)$ are constant.

A splitting is bounded if $\bar{U} \cap \bar{V} \cap E_x = \{0\}$ for all x; where $\bar{U}$ and $\bar{V}$ are the closure of U and V as subsets of E

I Dominated splitting.

Let us assume that $E \to M$ is a vector bundle

(ii) Assume that ϕ is a linear bundle map over φ, a subbundle $U \subset E$ is invariant if $\phi(U) \subset U$.

(iii) Given an invariant bounded decomposition $E = U \oplus V$ then we say that U dominates V $(U > V)$ if $\exists \lambda > 0$ $\exists n$ so that $\forall u \in U, \forall v \in V$ $\frac{\|\phi^n(v)\|}{\|\phi^n(u)\|} < \lambda \frac{\|v\|}{\|u\|}$

Remark (i) $\|\phi^n_x\|_U \cdot \|\phi^{-n}_{\varphi(x)}\|_V \leqslant \lambda < 1$

Corollary | If $E = U + V$ and $V < U$ for ϕ
| then $E^* = U^\perp \oplus V^\perp$ and $U^\perp < V^\perp$ for ϕ^*

(ii) (Assuming compactness of M) $\exists n_0$ so that

$$U < V \text{ for } \phi^{n_0} \Leftrightarrow \lim_{m\to\infty} \frac{\|\phi^m(u)\|}{\|\phi^m(v)\|} = 0$$

Proposition : every dominated splitting is continuous

◀ let $x_n \to x$; let $U'_x = \lim_{x_n\to x} U_x$; $V'_x = \lim_{x_n \to x} V_x$, after possibly extracting a subsequence.

} Our goal is to prove that $U'_x = U_x$; $V'_x = V_x$

if $u' \in U'$; $v' \in V'$ we have

$$\frac{\|\phi(v')\|}{\|\phi(u')\|} \leqslant \lambda \frac{\|v'\|}{\|u'\|} \quad \lambda < 1, \text{ in particular } E = U'_x \oplus V'_x$$

Step 1 : Let us show that $V' = (V' \cap U) \oplus (V' \cap V)$

■ let $v' \in V'$, then $v' = \underset{\underset{U}{\cap}}{u_0} + \underset{\underset{V}{\cup}}{v_0}$; let us write $u_0 = \underset{\underset{U'}{\cap}}{u'_1} + \underset{\underset{V'}{\in}}{v'_1}$

Observe that $u'_1 = 0$. Otherwise $u_0 \neq 0$ and $\|\phi^n(v')\| \sim \|\phi^n(u_0)\| \sim \|\phi^n(u'_1)\|$ and the contradiction. Thus $u_0 \in U \cap V'$, and then $v_0 \in V \cap V'$ ■

Step 2 : Assume now that $V \neq V'$, then $U \cap V' \neq \{0\}$, let $w_0 \in U \cap V' \setminus \{0\}$
It follows that $U \neq U'$, and thus $U' \cap V \neq \{0\}$ let $w_1 \in U' \cap V \setminus \{0\}$.

Then : $\dfrac{\|\phi^n(w_0)\|}{\|\phi^n(w_1)\|} \to 0$ because $w_0 \in V'$, $w_1 \in U'$;

and $\dfrac{\|\phi^n(w_0)\|}{\|\phi^n(w_1)\|} \to \infty$; because $w_0 \in U$, $w_1 \in V$

This is the contradiction ▶

Corollary (of the proof) ; if $E = U \oplus V$; $U > V$; $E' = U' \oplus V'$; $U' > V'$

and $\mathrm{rk}(U) = \mathrm{rk}(U')$; $\mathrm{rk}(V) = \mathrm{rk}(V')$ then $U = U'$; $V = V'$

Corollary | [Assuming M is compact]
domination is independent on the metric

(iii) ϕ contracts the bundle $V \otimes U^*$ $\Leftrightarrow$ $V > U$

(iv) $E = U + V$ is a dominated splitting if $U > V$ in the future and $V > U$ in the past.

Theorem [Assuming M is compact]
(i) A dominating decomposition is Hölder
(ii) If φ_z, ϕ_z depends C^k on some parameter z, and ϕ_{z_0} admit a dominating splitting, then ϕ_z admits a dominating splitting and more over the splitting depends C^k on z.

◀ let $E = U \oplus V$ the dominated splitting.

Assume V has rank k, let N the bundle over M whose fiber at m is $\mathrm{Grassm}_k(E_m)$, observe ϕ acts on N

$\exists$ a (ball) subbundle $B \subset N$ such that V is a section of B (fixed by ϕ)

More over since for all $W \in \overline{B}_x$, $W \pitchfork U_x$, we have a map $B_x \longrightarrow \mathrm{Hom}(V_x, U_x)$, whose image is a compact ball

Since the action of ϕ on $\mathrm{Hom}(V_x, U_x)$ is contracting. ϕ is also contracting on B. Thus

(i) V is Hölder

(ii) For a small perturbation of ϕ, ϕ is still contracting; hence you also have a dominated splitting ▶

Extensions (i) $E = E_1 \oplus \cdots \oplus E_n$ with $E_i > E_j$ if $i > j$

($\Leftrightarrow E_n + \cdots + E_{k+1} > E_k + \cdots + E_1$)

the same holds

IV Flows and connections, central leaves

We will mainly be interested in the following case:

$\{\varphi_t\} \circlearrowleft M$, and ∇ is a flat connection on E;

the flot $\{\phi_t\}$ on E is the parallel transport along $\{\varphi_t\}$.

Thus $F \subset E$ is ϕ_t invariant $\Leftrightarrow$ F is $//$ along φ_t

Let φ_t be a flow on M; A set $\mathcal{L}$ in M is centrally stable, if $\forall \varepsilon > 0$, $\forall x, y \in \mathcal{L}$; $\exists z_0, \ldots, z_n$, with $z_0 = x$, $z_n = y$; $\forall t > 0$; $d(\varphi_t(z_i), \varphi_t(z_{i+1})) < \varepsilon$

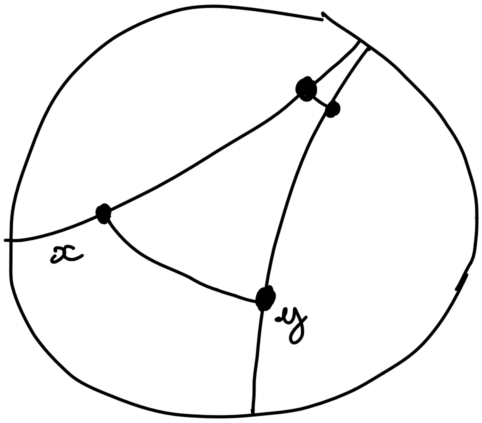
Example: Horospheres / horocycles w.r to the geodesic flow

Let $\mathcal{L}$ be a set; $V \subset E$ is $//$ along $\mathcal{L}$ if $\forall x, y$, $\forall \varepsilon$ there exists $z_0, \ldots, z_n$; $z_0 = x$, $z_n = y$; with $z_i \in \mathcal{L}$. $B(z_i, \varepsilon)$ is trivializing for ∇; and

E is constant along $B(z_i, \varepsilon)$; X is constant along the trivialization.

Proposition Let $E = F_1 \oplus F_2$ with $F_1 > F_2$ w.r to φ_t then F_2 is $//$ along stable sets.

◀ Using the compactness of M let $\alpha > 0$ so that $B(x,\alpha)$ is a trivializing open set of E for all x in M. Then if $y \in B_x(\alpha)$
$H_{xy} : E_x \to E_y$ the parallel transport (well defined)



Thus for t large enough

$$d(\varphi_t(x), \varphi_t(y)) \longrightarrow 0$$

Thus

$$d\left(F^2_{\varphi_t(x)}, H_{\varphi_t(y)\varphi_t(x)} F^2_t(y)\right) \to 0$$

But since F^2 is an attractive fixed in the past
We have that

$$d\Big(F^2_x, \underbrace{\Phi_{-t}\big(H_{\varphi_t(y)\varphi_t(x)}\big)\Phi_t}_{= H_{yx}} F^2_y\Big) \to 0$$

thus F^2_x is parallel along stable leaves ▶

III Two important examples.

Application to Fuchsian groups.

Let Γ be a Fuchsian group; $U\mathbb{H}^2/\Gamma = M$ equipped with the geodesic flow. Let ρ_0 be a lift $\pi_1(S) \to SL_2(E)$, let $\mathbb{H}^2 = \mathrm{Sym}(SL_2(E))$
Let us consider

$$\begin{array}{ccc} & U\mathbb{H}^2 \times E & \circlearrowleft \Gamma \\ \underset{\text{geodesic flow}}{\mathbb{R} \circlearrowright} & \downarrow & \\ & U\mathbb{H}^2 & \end{array}$$

we have $U\mathbb{H}^2/\mathbb{R} = \{(x,y) \in \mathbb{P}^1(E) \times \mathbb{P}^1(E) \mid x \neq y\}$

⇝ We have a splitting of $\mathcal{E}$ as $\mathcal{E} = L^+ \oplus L^-$ where

$L^+_z = x$; $L^-_z = y$ where $z \mapsto (x,y) \in \mathbb{RP}^1 \times \mathbb{RP}^1$

Moreover every points on $\mathbb{H}^2$ is a metric on E, thus we have a euclidian metric g on $\overline{E}$, so that $g_z = G_{\pi(z)}$; where
$\pi : U\mathbb{H}^2 \to \mathbb{H}^2$

Proposition: (i) g and $E = L^+ \oplus L^-$ are invariant by Γ
(ii) L^+, L^- are invariant by ϕ_t
(iii) the action of $\mathbb{R}$ dilates L_x^+, and contracts L_y^-

$\|\phi_t(u)\| = e^t \|u\|$ if $u \in L_x$; $\|\phi_t(u)\| = e^{-t}\|u\|$ if $u \in L_-$

◀ $UH^2 = \{(u_1, u_2) \in E;\ \omega(u_1, u_2) = 1\}$

geodesic flow is given by $t(u_1, u_2) = (tu_1, t^{-1}u_2)$

$$g_{u_1 u_2}(u_i, u_j) = \delta_{ij}$$

$L^+ = \langle u_1 \rangle$; $L^- = \langle u_2 \rangle$ ▶

Corollary let E_ρ be the flat bundle over $UH^2/\rho(\Gamma) \circlearrowleft \varphi_t$ geodesic flow. let ϕ_t the flow lifting the geodesic flow by parallel transport then $E = L^+ \oplus L^-$ is a dominated splitting.

Corollary. let ρ' be a representation close to ρ then there exists a Hölder splitting : $E_{\rho'} = L'_+ \oplus L'_-$ for ϕ'_t over φ_t on $UH^2/\rho(\Gamma)$

Corollary 2 : there exists a Hölder map : $\mathbb{P}(E) \to \mathbb{P}(E)$ so that $\Psi(\rho(\gamma)) = \rho'(\gamma)$.

◀ lifting everything to the universal cover the new splitting gives a ρ, ρ' equivariant map $UH^2 \xrightarrow{\Psi} \mathbb{P}(E)$, $\Psi(z) = L_z^-$

Moreover Ψ is constant along the flow ; and Ψ is constant along stable leaves (cf proposition above).

It follows that $\Psi : \mathbb{P}(E) \longrightarrow \mathbb{P}(E)$

because $\mathbb{P}(E) = \{$central stable leaves of $UH^2\}$ ▶

Higher rank situation

Let $B_v = \{$space of frames in V, $(e_1,\dots,e_n) \mid \det(e_1,\dots,e_n)=1\}$

$SL_n(\mathbb{R}) \circlearrowright B_v \circlearrowleft SL(E)$

Let as above $V \times B_v = \mathcal{V}$

Then $\mathcal{V}$ has

(i) a cannonical frame (tautological) $\rightsquigarrow V = L_1 \oplus \cdots \oplus L_n$
(ii) a metric $g(e_i,e_j) = \delta_{ij}$, invariant by the right action of Γ
(iii) is equipped by a left action of $SL_n(\mathbb{R})$, right action of $SL(V)$
(iv) a flat connection whose parallel sections are given by the orbit of $SL(V)$

Let Γ be a discrete subgroup of $SL(E)$, $\Delta = \text{diago}(\mathbb{R}^n)$

(i) the action of Δ on B_v/Γ is the Weyl Chamber flow

(ii) the action of Δ on B_v lifts to an action on $(V \times B_v)/\Gamma = V_\Gamma$ preserving the flat connection

(iii) the metric g descend to a metric on V_Γ

Proposition. Let $\delta = \begin{pmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_1 \end{pmatrix}$ be a trace zero matrix. Let $v \in L^i_{x_0} = \langle e_i \rangle$

Let v_t be the parallel transport along $e^{\delta t} \cdot x_0$ of v

then $\|v_t\| = e^{-\lambda_i t} \|v_0\|$

Corollary [Weyl chamber flow]

Let $\delta = \left(\begin{array}{c|c} \lambda_1 & \\ \hline & \boxed{\lambda_2} \end{array} \right)$ with $\lambda_1 > \lambda_2 > \lambda_p$

then the decomposition

$V = V_1 \oplus \cdots \oplus V_p$, where V_i is the eigenspace of λ_i

(i) is invariant under the $/\!/$ lift on $e^{\delta t}$

(ii) is dominating for the action of $e^{\delta t}$.