Geodesic flow of a projective representation

I. Geodesic flows of an Anosov representations

Let ρ be a projective Anosov reparametrisation

Theorem : There exists a geodesic flow $\mathbb{R} \to L \to \mathcal{G}$ so that

1) L/Γ is a Hölder reparametrisation of $U\mathbb{H}^2$
2) Periods of γ is $\ell(\gamma) = \operatorname{Log}(\lambda_1(\gamma))$
3) Any two such geodesic flows are equivalent.

We call $L \to \mathcal{G}$ with its action of Γ, the geodesic flow of ρ

and denote it $\quad U_\rho := L/\Gamma \circlearrowright \mathbb{R}$

An explicit description :

Let τ be the tautological $\mathbb{R}$-bundle over $\mathbb{P}^1(E)$.

and ξ, ξ^* be the limit maps $\partial_\infty \Gamma \to \mathbb{P}^1(E), \mathbb{P}^1(E^*)$

$\Xi = (\xi, \xi^*)$. Then $\quad U_\rho = \Xi^* \tau / \Gamma$

◀ Let $U_\rho \to \mathcal{G}$ be the geodesic flow of ρ. Let $\Psi : U_\rho \to \mathbb{P}^1(E) \times \mathbb{P}^1(E^*)$

then $L = \Psi^*(T)$, where T is the line bundle associated to τ

then L is contracting, and by construction, if γ is a periodic orbit of φ_t

then $\quad \Phi_{\ell(\gamma)} = \exp(-\ell(\gamma))$. Let then g be the metric on L so that $\Phi_t^* g = e^{-t} g$

. This follows from last lecture : Livšic theorem.

Since L has a cannonical trivialisation along the orbits of the geodesic flow

It follows that we have a Γ-equivariant map from

$$U_\rho \longrightarrow \operatorname{Met}(\Xi^* T) = \Xi^* \tau \qquad x \longmapsto g_x$$

$$\varphi_t(x) = e^{-t} g_x = \varphi_t^\circ(g_x)$$ Thus $L = \Xi^* \tau$. ▶

Theorem [Bridgeman – Canary – L – Sambarino]

"The geodesic flow U_ρ depends C^ω on ρ"

Given a C^ω map $\Psi : D \to \mathrm{Rep}(\Gamma, G)$; and $x_0 \in D$; there exists $x_0 \in D' \subset D$ so that (i) D' is open

(ii) $\exists$ a C^ω : $G : D' \to$ Hölder functions$(U_{\psi(x_0)})$

So that $G(y)$ parametrises $U_{\psi(y)}$.

II Application:

$$\frac{1}{K} \ell_{g_0}(\gamma) \leqslant \ell_\rho(\gamma) \leqslant K \ell_{g_0}(\gamma) \qquad (*)$$

Theorem (L; G.W; Delzant – G – L – Mozes)

(i) Every Anosov representation is discrete faithful

(ii) " " is a quasi isometry.

(iii) $\mathrm{Out}(\Gamma)$ acts properly on Anosov representation

(i) Observe if X_p is the symmetric space for $SL_p(\mathbb{R})$

$$\frac{1}{K} \log(\lambda_1(A)) \leqslant \inf_x (d(x, Ax)) \leqslant \log(\lambda_1(A))$$

$\forall \gamma$; $\rho(\gamma) \neq \mathrm{Id}$; moreover if $K \subset SL_p(\mathbb{R})$ is compact

then $\forall A$ in K; $|\lambda_1(A)| < M$ for some M

But according to $(*)$ there is only finitely many γ so that

$\lambda(\rho(\gamma)) < M$ Thus $\rho(\Gamma)$ is discrete

(ii) (D.G.L.M) $\Gamma \curvearrowright X$ quasi-isometry

$$\Leftrightarrow \exists K \quad \frac{1}{K}\tau(\gamma) \leqslant \inf_x d(x, \gamma(x)) \leqslant K\tau(\gamma) \qquad [\text{displacing}]$$

where $\tau(\gamma) = \inf \left(\text{word length}(\eta\gamma\eta^{-1}) \mid \eta \in \Gamma \right)$

for Γ a hyperbolic group.

(iii) exercise ▶

III Rigidity of periods

Let ρ be a projective Anosov representation and

$$\ell_\rho : \Gamma \to \mathbb{R}^+$$

Thm. [BCLS] Let $\rho_1 : \Gamma \to SL_p(\mathbb{R})$; $\rho_2 : \Gamma \to SL_q(\mathbb{R})$ be two irreducible Projective Anosov representations, then $p = q$ and ρ_1 and ρ_2 are conjugated

IV Complement on entropy, pressure intersection and metric

All that uses a black box called «Thermodynamic formalism»

$$N_\rho(A) = \{ \gamma \mid |\ell_\rho(\gamma)| < A \} \text{ is finite}$$

Thm [BCLS]

(i) $h_\rho = \text{entropy of } \rho = \lim_{A \to \infty} \left(\frac{1}{A} \log(\# N_\rho(A)) \right.$ is finite > 0

and h_ρ depends C^ω on ρ

(ii) Let ρ_1 and ρ_2 be two Anosov representations

$$I(\rho_1, \rho_2) = \lim_{A \to \infty} \left(\sum_{\gamma \in N_{\rho_1}(A)} \frac{1}{\# N_{\rho_1}(A)} \frac{\ell_{\rho_1}(A)}{\ell_{\rho_2}(A)} \right)$$

is well defined and depends C^ω on ρ_1 & ρ_2.

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I = the pressure intersection

$$J(\rho_1, \rho_2) = \frac{h(\rho_1)}{h(\rho_2)} I(\rho_1, \rho_2)$$

Theorem [BCLS] J is C^ω in ρ_1 and ρ_2, $J \geqslant 1 = J(\rho, \rho)$

$P = D^2_\rho J$ is a metric called the pressure metric

Theorem (Wolpert) For $n=2$, Hitchin representation, P is Weyl-Petersson metric

let us consider Hitchin representations as projective representations

Theorem (L–Wentworth)

$$\text{let } \phi_q \in H^0(K_q) \subset T_J(\text{Hitchin component})$$

$$P(\phi_q) = K(q,n)\left[\int_S \|\phi_q\|^2 d\mu_g\right]$$

$K(q,n) =$ explicit combinatorial coefficient.

Results by Xian Dai : $t \to t\phi_q$ is a geodesic up to second order at zero.

Remark: there are many ways to describe Hitchin representations as positive representations, and many geodesic flows associated to Hitchin representation

V Energy, Anosov representations and minimal surfaces.

let $\rho : \pi_1(S) \to G$, X_ρ the associated bundle whose fibers are $\mathrm{Sym}(G)$

For any, $f \in \Gamma(X_\rho) \mapsto e_\rho(f,J) = \frac{1}{2}\int df \wedge df \circ J$

the energy w.r to the complex structure J on S

Thus we obtain

$$E_\rho : \mathrm{Teich}(S) \longrightarrow \mathbb{R}^{>0}$$

$$J \longmapsto \mathrm{Inf}\left(e_\rho(f,J) \mid f \in \Gamma(X_\rho)\right)$$

Theorem (L.) If ρ is Anosov, then E_ρ is proper

◀ ① $e_\rho(f,J) = \frac{1}{2}\int_S \|df\|^2 dx = K_\pi \int_{US} \|df(u)\|^2 d\mu$

$$\mathrm{tr}(A^t A) = 2K_\pi \int_{S^1} \langle A(e^{i\theta}) \mid A(e^{i\theta})\rangle \, d\theta$$

② Using Cauchy Schwarz $e_\rho(f,J) \geqslant K_2 \left(\int_{US} \|df(u)\| d\mu \right)$

③ Fact [Bowen, Margulis] equirepartition of closed geodesic

$$\int f\,d\mu = \lim_{A\to\infty} \frac{1}{\#N_\gamma(A)} \sum_{\gamma\in N_{g_1}(A)} \int_\gamma f\,dt$$ where g_1 hyp. associated to J

④ In our case $\int_\gamma \|df(\dot\gamma(t))\|\,dt \geqslant d_X(f(x), \rho(\gamma)f(x)) \geqslant \tau_X(\rho(\gamma))$

where $\tau_X(A) = \inf_{x\in A} d(x, A(x))$

Exercise; $\exists K_2$ $\quad \tau_X(A) \geqslant K_2|\lambda_1(A)|$

⑤ Thus picking a reference hyperbolic metric g_0

$$\int_\gamma \|df(\dot\gamma(t))\|\,dt \geqslant K_2\, \ell_{g_0}(\gamma)$$

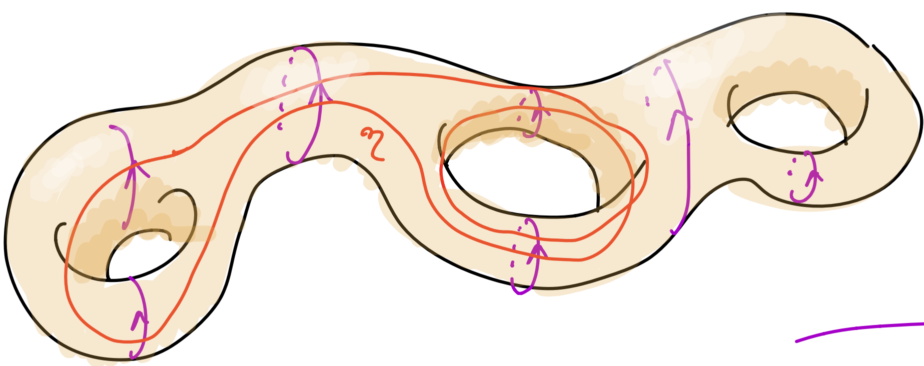
⑥ And in the end

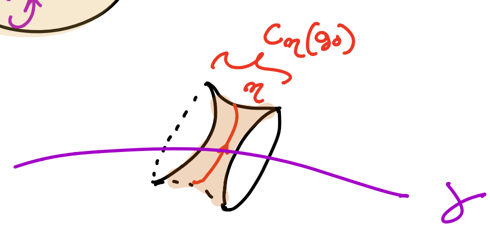
$$e_\rho(f,J) \geqslant K_3\left(\sum_{\gamma\in N_{g_1}(A)} \frac{1}{\#N_{g_1}(A)}\left[\frac{\ell_{g_0}(\gamma)}{\ell_{g_1}(\gamma)}\right]\right)^2 = K_3\left(I(g_0,g_1)\right)^2$$

Thus $E_\rho(J) \geqslant K_3\left(I(g_0,g_1)\right)^2$

Now the conclusion follows from

§ $g_1 \longmapsto I(g_0,g_1)$ is a proper map.





$\ell_{g_0}(\gamma) \geqslant C_\eta(g_0)\,\mathrm{int}(\gamma,\eta)$ if η is a simple geodesic.

Where $\mathrm{int}(\eta,\gamma) = \#(\eta\cap\gamma)$

(Fact) $\quad \ell_{g_1}(\eta) = \lim_{A\to\infty} \sum'_{\gamma\in N_{g_0}(A)} \frac{1}{\# N_{g_0}(A)} \frac{\mathrm{int}(\eta,\gamma)}{\ell_{g_1}(\gamma)}$

Then $I(g_1,g_0) = \lim_{A\to\infty} \frac{1}{\# N_{g_1}(A)} \sum'_{\gamma\in N_{g_0}(A)} \frac{\ell_{g_0}(\gamma)}{\ell_{g_1}(\gamma)}$

$$\geqslant \lim_{A\to\infty} \frac{1}{\# N_g(A)} \sum' C \frac{\mathrm{int}(\eta,\gamma)}{\ell_{g_1}(\eta)} \geqslant C\,\ell_{g_1}(\eta)$$

Finally we obtain

$$I(g_1,g_0) \geqslant C_2 \sum_i' \ell_{g_1}(\gamma_i)$$ And the result follows ▶

(generalization of an argument by Croke Fathi)

Corollary: Given an Anosov representation, there exists a conformal harmonic mapping (i.e a minimal mapping) ρ-equivariant $S \to \mathrm{Sym}(G)$

◀ Critical point of $E_\rho \Rightarrow$ conformal (Schoen-Uhlenbeck) ▶

Thm $\quad$ If G has rank 2, and the representation is « positive »

Hitchin, $\quad$ $\mathbb{R}$-split $\quad$ $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$, $\mathfrak{sl}_3(\mathbb{R})$, G_2, $Sp_4(\mathbb{R})$ $\quad$ (L, Loftin)

Maximal Hermitian symmetric $\quad$ e.g. $SO(2,n)$ $\quad$ (Collier-Tholozan-Toulisse)

Then the minimal surface is unique

Question are there other cases? $\quad SL_2(\mathbb{R})\times SL_2(\mathbb{R})\times SL_2(\mathbb{R})$