I Line bundles and their degree

Let $\mathcal{L}$ be a holomorphic line bundle over a closed surface S

$\to$ a (topological) line bundle $L \to c_1(L) \in \mathbb{N}$

$c_1(L) = \frac{1}{2\pi}\int i R^\nabla \leftarrow$ curvature of a hermitian connexion

$\deg(\mathcal{L}) := c_1(L)$; $\deg(\mathcal{L}_1 \otimes \mathcal{L}_2) = \deg(\mathcal{L}) + \deg(\mathcal{L}_2)$

$\text{Picard}(X) = \{\text{holomorphic line bundle}\}/\text{isomorphism}$

$\text{Pic}(X)$ is a group $\mathcal{L}_1, \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \otimes \mathcal{L}_2$

$\text{Picard}_0(X) = \{\mathcal{L} \mid \deg \mathcal{L} = 0\}$

Theorem (Abel–Jacobi) $\text{Pic}_0(X) \approx \text{Rep}(\pi_1(X), S^1)$

What is the map : $\text{Rep}(\pi_1(X), S^1) \longrightarrow \text{Pic}_0(X)$?

take $\rho \in \text{Rep}(\pi_1(X), S^1)$ take ∇ the associated flat connexion

on $E_\rho = (\tilde{X} \times \mathbb{C})/_{\rho(\pi_1(X))}$ $\quad \rho \longmapsto (E_\rho, \bar{\partial}^\nabla)$

converse : find the best metric on $E, \bar{\partial}$

II Abelian differentials again

Theorem $H^0(K) \approx \text{Rep}(\pi_1(X), \mathbb{R})$

$$H^0(K) \overset{?}{\longrightarrow} \text{Rep}(\pi_1(X), \mathbb{R})$$

$\alpha \in H^0(K) \longrightarrow \text{Re}(\alpha)$ is a closed 1-form. $\quad \gamma \longmapsto \int_\gamma \text{Re}(\alpha)$

converse : given $[\alpha] \in H^1(X, \mathbb{R})$ find the best representative

β which is harmonic : $d\beta = 0$, $d\beta \circ J = 0$

then $\alpha := \beta + i\beta \circ J$ is an an abelian differential.

« these two theorems can be put together »

$(\mathcal{L}, \alpha)^{\text{= abelian differential}} \longleftrightarrow \text{Rep}(\pi_1(X), GL_1(\mathbb{C}))$, where $GL_1(\mathbb{C}) = \mathbb{R} \times S^1$

$\mathcal{L}$ = line bundle

$\mathbb{R}$ = split part of $GL_1(\mathbb{C})$

S^1 = compact part of $GL_1(\mathbb{C})$