Higher rank holomorphic bundles

$\mathrm{Rep}(\Gamma, G) = \mathrm{Hom}(\Gamma, G)/\sim \quad \rho \sim g\rho g^{-1} \quad (g \in G)$

1. The « compact » case

Given E of rank $n \rightsquigarrow$ determinant line bundle $\det(E) := \Lambda^n E$

$$\boxed{\deg(E) := \deg(\det(E))} \qquad \boxed{\deg(E) = \frac{1}{2\pi}\int_X \mathrm{tr}(iR^\nabla)}$$

$$\deg(E \oplus F) = \deg(\det(E \oplus F)) = \deg(E) + \deg(F)$$

$$\deg(L \otimes E) = n\deg(L) + \deg(E)$$

1. Analogue of Abel-Jacobi ?

$$\det \rho \in \mathrm{Rep}(\pi_1(S), U(N)) \rightsquigarrow (E_\rho, \bar\partial^\nabla) = \mathcal{E}_\rho$$

How do we characterize $\mathcal{E}_\rho$? we have some constraints:

(i) $\deg(\mathcal{E}_\rho) = 0$

(ii) lemma let $\mathcal{F}$ be a holomorphic subbundle of $\mathcal{E}_\rho$ then $\deg(\mathcal{F}) \leqslant 0$

« $\mathcal{F}$ is more negatively curved than $\mathcal{E}$ » and $\mathcal{F}$ has a holomorphic supplementary

◀ $D_X\sigma = \nabla_X\sigma + B(X)\sigma$; If σ is holomorphic $\Rightarrow B(JX)\sigma = i\,B(X)\sigma = B(X)i\sigma$

Gauss equation : $\langle R(X,Y)J\sigma | \sigma\rangle = \langle B(Y)J\sigma | B(X)\sigma\rangle - \langle B(X)J\sigma | B(Y)\sigma\rangle$

thus $\langle R(X,JX)J\sigma|\sigma\rangle < 0 \qquad c_1(E) = \frac{1}{2\pi}\mathrm{tr}(R^\nabla J)$ ▶

A degree 0 holomorphic subbundle is **stable** if for every holomorphic subbundle $\mathcal{F}$; $\deg(\mathcal{F}) < 0$, **semistable** if $\deg(\mathcal{F}) \leqslant 0$

polystable if it is the sum of stable bundles.

Theorem (Narasimhan – Seshadri 65)

let $\mathcal{E}$ be a polystable degree zero holomorphic bundle of rank n then there exists $\rho \in \mathrm{Re}\,(\pi_1(X), U(n))$ so that $\mathcal{E} = (E_\rho, \nabla^\rho)$

stable $\longleftrightarrow$ irreducible.

$\rightsquigarrow$ in particular every good holomorphic bundle admits a nice metric

2. The "split" case ($GL_n(\mathbb{R})$)

Thm (Hitchin 92) $H^0(K) \oplus \cdots \oplus H^0(K^n) \approx \mathrm{Rep}^H(\pi_1(X), GL_n(\mathbb{R}))$

$\mathrm{Rep}^H(\pi_1(X), GL_n(\mathbb{R}))$ is a specific component of $\mathrm{Rep}(\pi_1(X), GL_n(\mathbb{R}))$

a) representations of $SL_2(\mathbb{R})$,

let $V_2 = \mathbb{R}^2$; $V_n = \{$ Symmetric tensors of d° $n-1$ on $V\} = \{Q\}$

$V_n = \{$ Homogeneous polynomials of d° $n-1$ in two variables $\} = \{P\}$

$SL_2(\mathbb{R}) \circlearrowright V_n$: $Q \mapsto Q \circ A$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(X,Y) = P(aX+bY, cX+dY)$

Proposition: the representation of $SL_2(\mathbb{R})$ on V_n is the unique irreducible representation of dim n of $SL_2(\mathbb{R})$

Fuchsian representation : $\Gamma \to PSL_n(\mathbb{R})$

$$\Gamma \longrightarrow PSL_2(\mathbb{R}) \xrightarrow[\text{irr}]{} PSL_n(\mathbb{R})$$

monodromy of a hyperbolic structure.

Hitchin representation : $\rho : \Gamma \to GL_n(\mathbb{R})$ which can be deformed to a Fuchsian

$\mathrm{Rep}^H(\Gamma, PSL_n(\mathbb{R})) = \{$ Hitchin representations $\}/_{\sim}$ $\leftarrow$ conjugation

$$U(n)_{\mathbb{C}} = GL_n(\mathbb{C}) = (GL_n(\mathbb{R}))_{\mathbb{C}}$$

NS theorem $\quad$ Non Abelian Hodge ? $\quad$ Hitchin theorem

III Metrics on holomorphic vector bundle

« Is there a best metric ? »

lemma : Given a hermitian metric g on $\mathcal{E}$, there exists a unique connexion ∇ on $\mathcal{E}$, $\nabla g = 0$ and $\mathcal{E} = (E, \bar\partial^\nabla)$

◀ linear algebra ; if ∇_1 and ∇_2 are two hermitian connections then $\nabla_1 - \nabla_2 = A$; where $A(X)$ is ${}^t\bar{A} = -A$.

Similarly if $\bar\partial_1$ and $\bar\partial_2$ are two Cauchy-Riemann operators, $\bar\partial_1 - \bar\partial_2 = B$, where $B(JX) = JB(X)$; $B(X)$ $\mathbb{C}$ linear

$\Omega^1(X) \otimes \mathrm{AntiH}(E) : \{A \mid {}^t\bar{A} = -A\}$ $\qquad$ $\{B \mid B(JX) = JB(X)\} = \Omega^{1,0}(X) \otimes \mathrm{End}(E)$

$$A \longmapsto \frac{1}{2}(A(X) + i A(JX))$$

Exercise : complete the proof that is show the map above is injective ▶

∇ is called the Chern-Connexion of g. Recall that a connexion is projectively flat if $R^\nabla(X,Y) = \omega(X,Y) \otimes J$ where $\omega \in \Omega^2(S)$

(there is an extension to non degree zero bundle)

The slope of $\mathcal{E}$ is $\mathrm{slope}(\mathcal{E}) = \dfrac{\deg(E)}{\mathrm{rk}(E)}$

$\mathcal{E}$ is stable if $\forall\, \mathcal{F}$ subbundle of $\mathcal{E}$ $\mathrm{slope}(\mathcal{F}) < \mathrm{slope}(\mathcal{E})$

$\mathcal{E}$ is semi stable if $\forall\, \mathcal{F}$ subbundle of $\mathcal{E}$ $\mathrm{slope}(\mathcal{F}) \leqslant \mathrm{slope}(\mathcal{E})$

$\mathcal{E}$ is polystable if it is the direct sum of stable bundles.

Exercise, If $\deg(\mathcal{E}) = 0$, this coincide, If $\mathcal{L}$ is a line bundle and $\mathcal{E}$ stable then $\mathcal{L} \otimes E$ is stable.

(Essentially unique means, g_1, g_2 have the same Chern connections)

III Higgs bundles

A **Higgs bundle** is a pair $(\mathcal{E}, \phi)$ where

- $\mathcal{E}$ is a holomorphic bundle
- ϕ is $\in H^0(K \otimes \mathrm{End}(\mathcal{E}))$

Example: If $\mathcal{E} = \mathcal{L}$ a line; $\mathrm{End}(\mathcal{L}) = \mathbb{C}$ thus a Higgs field is $(\mathcal{L}, \alpha)$ where $\alpha \in H^0(K)$.

A **sub-Higgs bundle** is $\mathcal{F} \subset \mathcal{E}$, where $\mathcal{F}$ is holomorphic and stable by ϕ

A Higgs bundle is **semi-stable** if $\forall\, \mathcal{F}$ sub Higgs, then $\mathrm{slope}(\mathcal{F}) \leq \mathrm{slope}(\mathcal{E})$

$\mathcal{E}$ is **polystable** if it is the direct sum of stable Higgs bundles.

Goal

Non abelian Hodge correspondence

For deg 0, holomorphic bundles.

$$\{\text{poly stable Higgs bundles of rk } n\} \longleftrightarrow \{\rho : \pi_1(X) \to GL_n(\mathbb{C}) \mid \rho(\pi_1(X)) \text{ is reductive}\}$$

Hitchin – Simpson   Corlette – Donaldson

$$\{\text{harmonic metrics on } \mathcal{E}\}$$

$L \subset GL_n(\mathbb{C})$ is **reductive** if $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_k$, where V_i is stable by L and irreducible.

IV Extension to G

Let G be a Lie group, with Lie algebra $\mathfrak{g}$.

Recall that $G \circlearrowright \mathfrak{g}$ by the adjant representation, preseving

$$g, X \longrightarrow Ad(g)\cdot a$$

the induced representation of $\mathfrak{g}$ is also the adjoint representation

$$[X,Y] \to Ad(X)\cdot Y = [X,Y]$$

An adjoint bundle, has trivialisation of the form $\mathfrak{g} \times U$,

and changes of trivialisation in $Ad(G) \subset End(\mathfrak{g})$

If G is complex, one may talk of an adjoint holomorphic bundle.

ex : E is a vector bundle of rk n

$End(E)$ is an adjoint $GL_n(\mathbb{C})$-bundle

An adjoint Higgs-bundle is a pair $\mathcal{E}, \phi$ where

$\mathcal{E}$ is an adjoint holomorphic bundle and $\phi \in H^0(K \otimes \mathcal{E})$

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