# Harmonic mappings

Let $f : M \to N$ be a $C^1$ map between Riemannian manifolds.

$$E(f) = \frac{1}{2} \int \|Tf\|^2 d\mu_M$$

volume on $M$

$$Tr(Tf^* \circ Tf)$$

Ex : if $N = \mathbb{R}$ ; $E(f) = \frac{1}{2} \int \|\nabla f\|^2 d\mu_{M\mathbb{R}}$

if $M = \mathbb{R}$ ; $E(f) = \frac{1}{2} \int \left\| \frac{df}{dt} \right\|^2 dt.$

A map $f : M \to N$ is a harmonic mapping if it is a critical point of the energy ; in others words if $\{f_t\}_{t \in ]-\varepsilon, \varepsilon[}$ is on on parameter family of compactly supported deformations of $f_0 = f$

Then $\frac{d}{dt} E(f_t) = 0$

## I. Forms with values in bundles

Let $E$ be a vector bundle over $M$ ; we shall consider

$\Omega^p(M, E) = \{$ p-forms with values in $E\}$ in particular

a decomposable element of $\Omega^p(M, E)$ is an element of the form $\alpha \otimes \sigma$ where $\sigma \in \Gamma(E)$ and $\omega \in \Omega^p(M)$

If $B$ is any bilinear operator on $E \times F$ with values in $H$

we define for $\alpha, \beta \in \Omega^1(M,E)$, $\Omega^q(M,F)$

$\Omega^{p+q}(M,H) \ni B(\alpha \wedge \beta)$ = antisymetrisation of $B(\alpha, \beta)$

Example if $p,q=1$ $\quad B(\alpha\wedge\beta)(u,v) = B(\alpha(u), \beta(v)) - B(\alpha(v), \beta(u))$

If $\Omega_1 = \alpha \otimes \sigma$ and $\Omega_2 = \beta \otimes \sigma_2$ where $\sigma_i \in \Gamma(E)$; are decomposable then

$$B(\Omega_1 \wedge \Omega_2) = \alpha \wedge \beta \; B(\sigma_1, \sigma_2)$$

Let $E \to M$, equipped with $\nabla$, then there exists a family of operators

$d^\nabla : \Omega^p(M,E) \to \Omega^{p+1}(M,E)$ which satisfy

(i) $\quad d^\nabla \xi = \nabla \xi \quad$ if $\xi \in \Gamma(E) \approx \Omega^0(M,E)$

(ii) $\quad d^\nabla(\alpha \wedge \Omega) = d\alpha \wedge \Omega + (-1)^{d^\circ\alpha} \alpha \wedge d\Omega$

$\alpha \in \Omega^p(M)$; $\Omega \in \Omega^q(M,E)$

Example if $\omega \in \Omega^1(M,E)$ ;

$$d^\nabla \omega = \nabla_u \omega(v) - \nabla_v(\omega(u)) - \nabla_u \omega(v)$$

(unitary connection) $\qquad \nabla$ is unitary and preserves $\langle \ \rangle$ on $E$

$$d\langle \Omega_1 \wedge \Omega_2 \rangle = \langle d^\nabla \Omega_1 \wedge \Omega_2 \rangle + (-1)^{d^\circ \Omega_1} \langle \Omega_1 \wedge d\Omega_2 \rangle$$

If furthermore $g$ is a Riemannian metric on $M$;

if $\alpha, \beta \in \Omega^p(M,E)$

$$\langle \alpha | \beta \rangle_g = \sum_{u_1, \dots u_n}' \langle \alpha(u_1, \dots u_n) | \beta(v_{1}, \dots v_{n}) \rangle$$

where $u_i$ is a orthonormal frame in $TM$

$\alpha \in \Omega^p(M,E)$; $*\alpha$ is characterised by $*\alpha \in \Omega^{\dim M - p}(M,E)$

and $\langle \beta \wedge *\alpha \rangle = \langle \alpha | \beta \rangle_g \, \mu_M \leftarrow$ vol form on $M$

In particular, if $\Omega = \alpha \otimes \sigma$ is decomposable (with $\sigma \in \Gamma(E)$)
then $*\Omega = *\alpha \otimes \sigma$

if $\alpha \in \Omega^1(E)$ ; $d(*\alpha) = \mathrm{Tr}(\nabla\alpha)\mu_M$ (divergence formula)

Where $\mathrm{Tr}(\nabla\alpha) = \sum_{u_i}^{\prime} \nabla_{u_i}(\alpha(u_i))$ for $u_i$ orthonormal frame

$\boxed{\text{For Riemann surface if } \alpha \in \Omega^1(M,E),\ *\alpha = \alpha \circ J}$

Final remark $g : M \to N$ ; $Tg \in \Omega^1(M, g^* TN)$
let's equip $g^*TN$ with the induced connection from $N$.

$$d^\nabla Tg = 0$$

◀ 1st case $N = \mathbb{R}^p$, $D$ is the trivial connexion on $\mathbb{R}^p$

$$d^D T_g = ddg = 0$$

If $\nabla$ is torsion free then locally $\nabla = D + A$ with $A(X,Y) = A(Y,X)$

$$d^\nabla\omega(u,v) = d^D\omega(u,v) + A(Tg(u), \omega(v)) - A(Tg(v), \omega(u))$$

thus $d^\nabla Tg = 0$ ▶

## II. Energy and harmonic mappings. Using our $*$ operator

$$E(f) = \frac{1}{2}\int \langle Tf \wedge *Tf \rangle$$ the energy density.

When $M$ has dimension 2, we have another definition

$$E(f) = \frac{1}{2}\int_M \langle Tf \wedge Tf \circ J \rangle$$

and this only depends on $J$ not the metric

**Proposition.** $f$ is harmonic if and only if

$$d^\nabla(*Tf) = \mathrm{Tr}(\nabla Tf)\mu_M = 0$$

some explanation: $Tf$ is a element of $\Omega^1(M, f^*TN)$
$\rightsquigarrow *Tf \in \Omega^{m-1}(M, f^*TN) \rightsquigarrow d^\nabla(*Tf) \in \Omega^m(M, f^*TN)$.

$\rightsquigarrow$ the other formulation is $\nabla^N$ the connection on $f^*TN$; $\nabla^M$ on $TM$
$\rightsquigarrow \nabla^{Hom}$ on $Hom(TM, f^*TN)$ where $(\nabla^{Hom}_u TF)(v) = \nabla^N_u(TF(v)) - TF(\nabla^M_u v)$

The case $n = 2$ is special

- $Tf : TM \to TN$ ; $Tf \in \Gamma(T^*M \otimes f^*TN) \subset$
- let $Tf^{1,0} \in \Gamma(T^*M \otimes_{\mathbb{C}} (f^*TN)_{\mathbb{C}})$, $\mathbb{C}$-linear map $TM \to (f^*TN) \otimes \mathbb{C}$.

  $Tf^{1,0}(u) = \frac{1}{2}(Tf(u) - i\, Tf(Ju))$ [complex linear part]

- let us consider $(f^*TN)_{\mathbb{C}}$ as a holomorphic bundle (using $f^*\nabla^N$)

  $Tf^{1,0} \in \Gamma(K \otimes (f^*TN)_{\mathbb{C}})$

Proposition: $f$ harmonic $\Leftrightarrow$ $Tf^{1,0}$ is holomorphic

◀ $\Delta f = 0 \Leftrightarrow df - i\, df \circ J$ is harmonic $\Leftrightarrow$ (locally $f = Re(g)$, $g$ holomorphic ▶

Proof of the proposition. This is a consequence of the proposition

Prop: Assume $f_t$ is a compactly supported deformation

$$\frac{d}{dt} E(f_t) = -\int \langle \xi \mid Tr(\nabla TF) \rangle \mu_M$$

where $\xi \in \Gamma(f^*TN)$; $\xi_x := \frac{d}{dt} f_t(x)$

◀ let consider $f_t$ as a map $F : M_0 = M \times [-\varepsilon, \varepsilon] \to N$

let $TF : \Omega^1(M_0, F^*TN)$, let $\xi = TF(\frac{\partial}{\partial t})$,

$$(*) \quad \frac{1}{2}\frac{\partial}{\partial t}\|Tf_t\|^2 \mu_M = \langle \nabla \xi \wedge *Tf \rangle$$

Let $u_i$ be an orthonormal frame of $TM$, so that $\|Tf_t\|^2 = \sum_i \langle TF(u_i) \mid TF(u_i) \rangle$

$$\frac{1}{2}\frac{\partial}{\partial t}\|Tf_t\|^2 = \sum_i \langle \nabla_{\partial/\partial t} TF(u_i) / TF(u_i) \rangle$$

$$(\text{but } d^\nabla TF = 0) \qquad = \sum_i \langle \nabla_{\partial/\partial u_i} TF(\xi) / TF(u_i) \rangle$$

Thus $\frac{1}{2}\frac{\partial}{\partial t}\|Tf_t\|^2\mu_M = \langle \nabla\xi \wedge *Tf_t\rangle$

(*) Thus $\frac{d}{dt}(E(f_t)) = \int \langle \nabla\xi \wedge *Tf\rangle = \int d(\xi\wedge *Tf)$
$- \int \langle \xi \wedge d(*Tf)\rangle$ ▶

## III · Twisted harmonic mappings.

The above notions extend in the following situation.

let $\rho : \pi_1(M) \circlearrowright \mathrm{Iso}(N)$

let $N^\rho$ be the associated bundle : $(\tilde{M}\times N)/\pi_1(M)$

with its « flat connection $D$ »

Its fiber at $x$ is isometric (though not canonically) to $N$

$\Rightarrow$ on the total space $N^\rho$ we have a natural vector bundle $TN$
= tangent space to the fiber, « vertical tangent bundle »

$\Rightarrow$ using the flat connection (local product structure)

$$TN^\rho = TN \oplus TM \quad \text{« horizontal tangent bundle »}$$

A section of $N^\rho$ is a $\rho$-twisted map : $M \to N$

that is $f : \tilde{M} \to N$ ; $f(\gamma\cdot x) = \rho(\gamma)\cdot f(x)$.

Given a section $\sigma$ on $N_\rho$ : $T^v\sigma : TM \to TN \leftarrow$ « vertical »

$\Rightarrow$ one can define the energy of a section

$$E(\sigma) = \int_M \|T^v\sigma\|^2 d\mu$$

$$= \int_\Delta \|Tf\|^2 d\mu$$

$\Delta$ a fundamental domain

As well as harmonic sections which satisfies that

$$(T^v f)^{1,0} \in \Omega^1(M, (f^* TN)_{\mathbb{C}}) \text{ is holomorphic}$$