The Hitchin fibration and its fibers

let us give examples of Higgs bundles / NAH Correspondence

(i) : $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle of degree 0

$\alpha \in H^0(K) \simeq H^0(K, \mathrm{End}(\mathcal{L}))$

What is the harmonic metric ? What is

i) we have a parallel metric g_0 for ∇, where $\bar{\partial}^{\nabla} = \partial_{\mathcal{L}}$

$\rightsquigarrow \rho_K : \pi_1(S) \longrightarrow S^1 \subset \mathbb{C}$; $\theta \mapsto e^{i\theta}$

ii) $\alpha = \mathrm{Re}(df)$ f is ρ_S equivariant

$\pi_1(S) \longrightarrow \mathbb{R} \subset \mathbb{C}$; $\lambda \mapsto e^{\lambda}$

Then $\rho : \rho_K \cdot \rho_S : \pi_1(S) \longrightarrow \mathbb{C}^*$

and the harmonic metric is

$$g_0 = f \cdot g$$

(ii) let q_2 be a quadratic differential

Let S be a spin structure, that is $S^2 = K$

let $\mathcal{E} = S \oplus S^*$ $\quad (\bar{\mathcal{L}}^{i} = (\mathcal{L}^*)^{-1})$

then $\mathrm{End}(\mathcal{E}) = (S \oplus S^*)^* \otimes (S \oplus S^*)$

$$= (S^* \otimes S) + (S^* \otimes S^*) + (S \otimes S^*) + S \otimes S$$

Thus $\mathcal{E}nd(\mathcal{E}) = \begin{pmatrix} \mathbb{C} & K^* \\ K & \mathbb{C} \end{pmatrix}$

even though $\mathcal{E}$ does depend on the choice of S, $\mathcal{E}nd(\mathcal{E})$ does not

Then
$$K \otimes \mathcal{E}nd(\mathcal{E}) = \begin{pmatrix} K & \mathbb{C} \\ K^2 & K \end{pmatrix}$$

Thus $\phi_{q_2} = \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix}$ is a Higgs Field.

⚠ we shall check later that the associated representations (by NAH) actually takes values in $PSL_2(\mathbb{R})$ and corresponds to monodromy of hyperbolic structures.

Thus we have the following picture, for X_0 a Riemann surface

$$\begin{tikzcd}
 & \{\text{harmonic diffeos}\} \arrow[dl, "\text{taking the Hopf differential}"'] \arrow[dr, leftrightarrow] & \\
H^0(K^2) \arrow[rr, bend left, leftrightarrow, "\text{M. Wolf}"] \arrow[rr, bend right, leftrightarrow, "\text{N. Hitchin}"'] \arrow[dr] & & \{\text{Hyperbolic structures}\} \\
 & \text{Higgs bundle} \arrow[ur, "\text{Hitchin theorem}"'] &
\end{tikzcd}$$

Schoen-Yau : there exists a unique diffeomorphism between 2 hyperbolic surface (in a given homotopy class)

a) the Hopf differential.

let $f : X \to N$ be harmonic, X Riemann surface, N Riemannian manifold with metric g, the Hopf differential is $H_f = g_{\mathbb{C}}(Tf^{1,0}, Tf^{1,0})$

$$H_f(X,Y) = g_{\mathbb{C}}(Tf^{1,0}(X), Tf^{1,0}(Y))$$
$$= \frac{1}{4} g_{\mathbb{C}}(Tf(X) - iTf(JX), Tf(Y) - iTf(JY))$$

$$= \frac{1}{4}\left[h(X,Y) - h(JX,JY) \right] - \frac{1}{4} i \left(h(JX,Y) + h(X,JY) \right)$$

Thus $H_f = 0 \iff Tf$ is conformal

I The Hitchin fibration

Let $\mathcal{M}_H$ be the space of polystable Higgs bundle $(\mathcal{E}, \phi)$.

where $\phi \in H^0(K, \mathrm{End}(\mathcal{E}))$. We can now produce objects as in the context of characteristic classes [starting from $R^\nabla \in \Omega^2(M, \mathrm{End}(E) \ldots]$

Namely let us consider

$$\mathrm{Tr}(\phi^n) : (u_1, \ldots, u_n) \longmapsto \mathrm{Tr}(\phi(u_1) \cdot \ldots \cdot \phi(u_n))$$

Then (i) $\mathrm{Tr}(\phi^n) \in \Omega^{1,0}(K^n)$

Moreover (ii) $\mathrm{Tr}(\phi^n)$ is holomorphic

: it is enough to check that in a (local) holomorphic trivialisation.

Exercise : $\mathrm{Tr}(\phi^2) \propto$ Hopf differential (by factors of 2) for $f : X \to SL(n,\mathbb{C})/SU(n)$

We could have replaced $\mathrm{Tr}(\phi^n)$, by $\sigma^n(\phi)$ which the coefficients of the characteristic polynomials $\det(u - \phi)$

Thus we have constructed the Hitchin fibration

$$\boxed{\begin{aligned} \mathcal{M}_H &\longrightarrow \underbrace{H^0(K) \oplus \cdots \oplus H^0(K^n)}_{\text{the Hitchin base.}} \\ \phi &\longmapsto (\sigma^1(\phi), \ldots, \sigma^n(\phi)) \end{aligned}}$$

Where $\det(u \, Id - \phi) = \sum_{i=1}^{n} u^i \sigma^{n-i}(\phi)$

(for $u \in K$)

II] Normalizations : Let π the Hitchin fibration

(i) If $\mathcal{L} \in \mathrm{Pic}_0(X)$, $(\mathcal{L}\otimes\mathcal{E}, \phi)$ is still a Higgs bundle

and $\pi(\mathcal{L}\otimes\mathcal{E}, \phi) = \pi(\mathcal{E}, \phi)$.

We normalize $\mathcal{E}$ so that $\det(\mathcal{E}) = \mathbb{C}$, trivial

(ii) we also take $\mathrm{tr}(\phi) = 0 \Leftrightarrow \rho \hookrightarrow SL_n(\mathbb{C})$ (and not $GL_n(\mathbb{C})$)

III Spectral and cameral covers .

A] The spectral curve

Let $Q = (q_1, \dots, q_n)$ be an element of the Hitchin base

Let $\Sigma_{1,Q} = \{ u \in K \text{, such that } \sum_{i=0}^{n-1} u^i \otimes q_{n-i}(u) = 0 \}$

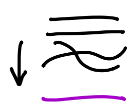
$P_Q : K : u \mapsto \sum_{i=0}^{n-1} u_i \, q_{n-i}(u) \in K^n$ ($P_Q \in (K^{-n+1})$)

Def : $\Sigma_{1,Q}$ is the spectral curve of Q, for Q in the Hitchin base

Q is generic if Σ_1 is generic

(i) $\Sigma_{1,Q}$ is smooth

(ii) $\pi : \Sigma_{1,Q} \subset K \to X$ is ramified, and has only double points



This can be explicitely described using Q

Here is the condition on Q that we shall use

every pt has a neighborhood and local coordinates so that

$$P_Q(u) = \prod_{i=1}^{n-2} (u - u_i(z))\left((u - u_0(z))^2 - z\right), \text{ all } u_i \text{ pairwise distinct.}$$

- In the case $n = 2$, $Q = (q_2)$ then this equivalent to the fact that q_2 has simple
- We can use a translation $u \to u - u_0(z)$ to get rid of u_0.

Exercise : this is equivalent to the fact that

$\hat{Q} := \mathrm{Res}(P_Q, P_Q') \in H^0(K^{n(n-1)})$ has only simple zeros

However it remains to show that this condition is indeed generic

Example $n=2$; $Q = (q_2) \in H^0(K^2)$, then we are looking at the double cover.

the genericity condition is that q_2 has simple zeros.

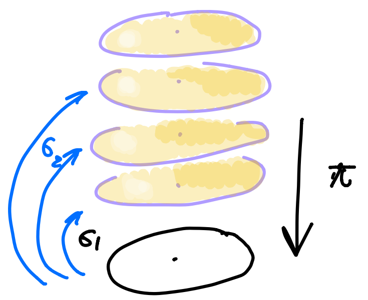
Observe that K carries a natural holomorphic form called the Liouville form

$$\lambda_u(v) = \langle u | T\pi(v) \rangle$$

Exercise : understand λ_u in coordinates and check λ_u is holomorphic

In particular we can restrict λ_u to Σ_Q'

In a covering patch (outside the ramification points)



$$P_Q(u) := \sum_{i=1}^{n} u^i \otimes q^i = \prod_{i=1}^{n} (u - (\sigma_i)_* \lambda)$$

$$\{Q \in \bigoplus_i H^0(K^i) \text{ generic}\} \underset{\text{using the Liouville form}}{\rightleftarrows} \{\text{generic spectral curve}\}$$

B) The Cameral cover

The spectral curve Σ_Q' is pretty nice, but this is not a Galois covering : Here is a smarter construction. To turn a cover into a Galois covering it is enough to label the fibers

Given a spectral curve Σ_Q', the cameral cover is the

$$\Sigma_Q^C = \{(u_1, \dots, u_n) \in K \oplus K \oplus \cdots \oplus K \mid \pi(u - \lambda_{u_i}(u)) = P_Q(u)\}$$

$$= \{(u_1, \dots, u_n) \mid u_i \in \Sigma_Q' \, ; \, \{u_1, \dots, u_n\} = \pi^{-1}(x)\}$$

Thus $\Sigma^{\prime C}_{Q}$ and $\Sigma^{\prime}_{Q}$ define the same information

Fact $\{\ \Sigma^{\prime C}_{Q} / \mathfrak{S}(n) = X$; where $\mathfrak{S}_n$ = symmetric group $\}$

A Cameral cover is $\Sigma' \subset K^n$ invariant by the action of $\mathfrak{S}_n$

Rq secretly $\mathfrak{S}_n = \{$Weyl group of $Sl_n\}$ and cameral $\leftrightarrow$ chamber

C | Liouville forms

Finally remark that $\Sigma^{\prime C}_{Q}$ carries n holomorphic forms λ_i, (corresponding to the different factors) and that

$$P_{\pi^* Q}(v) = \prod (v - \lambda_i) \in K^{-n+1}_{\Sigma^{\prime C}_{Q}}$$

$$\sigma^* \lambda_i = \lambda_{\sigma(i)}$$

$$\boxed{\text{« The characteristic polynomial factorizes using the liouville forms on the cameral cover »}}$$

III The fiber of Hitchin fibration, Beauville–Narasimhan–Ramanan (BNR) Correspondence.

or Line bundles over the cameral cover (always assume genericity)

Let $(\mathcal{E}, \phi)$ be a Higgs bundle associated to the point Q in the Hitchin base.
Let $\Sigma^{\prime}_{Q}$ be the spectral cover $\Sigma^{\prime}_{Q} \xrightarrow{\pi} X$ $\Sigma^{\prime C}_{Q} \xrightarrow{\pi_i} \Sigma^{\prime}_{Q}$ be the cameral cover, $\hat{\pi} : \Sigma^{\prime C}_{Q} \to X$ the cameral cover: .

Let now consider $(\pi \mathcal{E}, \pi^* \phi)$. Now

« outside the ramification points. $\pi^*\phi$ is diagonalisable : $\mathcal{E} = \bigoplus^{n} \mathcal{L}_i$
where $\mathcal{L}_i = \pi_i^* \mathcal{L}$, $\mathcal{L}$ defined on $\Sigma^{\prime}_{Q}$ outside the ramification point. »

Theorem (BNR). Assume Q in the Hitchin base is generic, let $\mathcal{H}_Q$ be the fiber of the Hitchin fibration, then we have a bijection;

$$\mathcal{H}_Q \ni (\mathcal{E}, \phi) \longleftrightarrow \mathcal{L} \in \mathrm{Pic}_0(\Sigma^{\prime}_{Q})$$

where $\mathcal{L} \subset \pi^* \mathcal{E}$, $\pi^* \phi|_{\mathcal{L}} = \lambda \cdot \mathrm{Id}$ and $H^0(E, U) = H^0(L, \pi^{-1}(U))$ as $\mathcal{O}(U)$-module.

(λ = Liouville form)

A] From $(\mathcal{E},\phi)$ to $\mathcal{L}$

Proposition : There exists a (uniquely defined) extension of $\mathcal{L}$

◀ Let us first consider the case $Q=(0,q_2)$ in rank 2 so that q_2 only has simple zeros ; at a zero we can choose coordinate so that $q(z)=z\,dz^2$. Then the spectral curve is defined as $\Sigma_1'=((z,u)\mid u^2=z)$ (Indeed $P(u)=u^2-z$)

We may choose a trivialization of $\mathcal{E}$. Then we have $\phi(z)\in \mathrm{End}(\mathcal{E})$, Let us define for $u\neq 0$ in Σ_{1Q}'

$$\mathcal{L}_u^{\pm}=\{v\in\mathcal{E}\mid \phi(u^2)v=\pm uv\}$$

Observe that $\phi(0)\neq 0$, otherwise $\phi(z)=zA(z)$, $q_2=\det(\phi(z))=z^2\det(A(z))$ has a zero of order 2.

Let us choose a non zero section u_1 so that $\phi\cdot u_1\neq 0$ in particular $u_1,\phi(u_1)$ are independent at zero hence everywhere

In the corresponding base $\phi=\begin{pmatrix}0&u^2\\1&0\end{pmatrix}$

Then $\mathcal{L}_u^+=\{(x,y)\mid +u^2y=ux\,;\ x=uy\}=\langle(u,1)\rangle$

$\mathcal{L}_u^-=\{(x,y)\mid u^2y=-ux,\ x=-uy\}=\langle(-u,1)\rangle$

Thus $\mathcal{L}_u^+$ and $\mathcal{L}^-$ extends as subbundle of $\pi^*\mathcal{E}$ on Σ_{1C}'.

The general reduces at each ramification to this study ▶

⚠ however $\pi^*\mathcal{E}$ does not split everywhere as $\mathcal{L}^+\oplus\mathcal{L}^-$

But we have the following

Proposition : for every U small enough, if $\pi:\Sigma_{1Q}'$ (spectral curve) $\to X$

$$H^0(E,U)=H^0(L,\pi^{-1}(U))\quad \text{as } \mathcal{O}(U)\text{ module.}$$

proof for $n=2$

◀ (i) outside ramification points we have $\pi^{-1}(U)=U_1\cup U_2$

$$H^0(E,U)=H^0(E,U_1)=H^0(L,U_1)\oplus H^0(\sigma^*L,U_1)=H^0(L,U_1)\oplus H^0(L,U_2)=H^0(L,U_1\cup U_2)$$

(ii) at a ramification point : let $\zeta \in H^0(\mathcal{L}_u, \pi^{-1}(u))$

Observe that $\zeta - \sigma^*\zeta = u\eta$ since $\zeta - \sigma^*\zeta$ vanishes at 0 [Check for $S_1 = (u,1)$) . Observe that $\sigma^*(\eta) = \eta$, and thus $\eta \in H^0(E, u)$

$\Psi : \sigma \mapsto \eta$ is clearly injective. Finally, Ψ is surjective

Indeed $u \cdot \eta = \frac{1}{2}\underbrace{(\phi + u)\cdot\eta}_{\in \mathcal{L}^-} - \frac{1}{2}\underbrace{(\phi - u)\eta}_{\eta_{\mathcal{L}^+}}$, because $\phi^2 = u$ ▶

σ_*

$$\boxed{\{(\mathcal{E},\phi) \text{ generic }\} \longrightarrow \{\Sigma_1' \subset K, \mathcal{L} \in \mathrm{Pic}_0(\Sigma_1')\}}$$

If we impose our condition $\det(\mathcal{E}) = \mathbb{C}$, the image is in

$\mathrm{Prym}(\Sigma_1') = \{\mathcal{L} \mid \bigotimes_i \pi_i^* \mathcal{L} = \mathbb{C}\}$ $\underline{\alpha}$ $n=2$: $\sigma^*\mathcal{L} = \mathcal{L}^*$ }

Prym variety is a compact connected abelian group hence a torus

D] From $\mathcal{L}$ to $(\mathcal{E},\phi)$.

let $\Sigma_1' \xrightarrow{\pi} X$ be a ramified covering of degree n, let $\mathcal{L}$ be a line bundle over Σ_1'

Proposition There exists a bundle of rank n, denoted $\pi_* L$ for every open set $U \subset X$, we have an isomorphism

$$H^0(\pi_* L, U) \xrightarrow{\Psi} H^0(L, U)$$

so that $\Psi(f\cdot\sigma) = f\circ\pi \cdot \sigma$

Moreover this characterizes $\pi_* L$.

This is a general result, we will only prove it when $\Sigma_1' \to X$ is generic

lemma 1 : Every $x \in X$ has a neighborhood U so that

$H^0(L, \pi^{-1}(U)) \simeq \underbrace{\mathcal{O}_X(U) \oplus \cdots \oplus \mathcal{O}_X(U)}_{n}$; $\mathcal{O}(U)$ = holomorphic function on U

(as $\mathcal{O}(U)$-module)

◀ (i) If x is not a critical point of π then there exists $U \in \mathcal{V}(x)$ so that $\pi^{-1}(U) = U \sqcup \cdots \sqcup U_n$ with L trivialized on U_i

thus $H^0(L, \pi^{-1}(\mathcal{U})) \simeq \bigoplus_i H^0(L, U_i) \simeq \bigoplus_i \mathcal{O}_{\Sigma^1}(\mathcal{U}_i)$

and thus the result since $\mathcal{O}_{\Sigma^1}(\mathcal{U}_i) \simeq \mathcal{O}_X(\mathcal{U})$.

(ii) at a ramification point. We only have to consider double ramification point: it is enough to consider the situation after a suitable choice of coordinates: $\Sigma^1 \to X$: $u \to u^2 = z$

We pick a neighborhood $\mathcal{U}$ of 0 in Σ^1 invariant by $u \mapsto -u$;

so that $\mathcal{L}$ is trivialized over $\mathcal{U}$, : $\mathcal{L}|_{\mathcal{U}} = \mathcal{U} \times \mathbb{C}$

Let $\sigma_1 \in H^0(\mathcal{U}, L)$ $u \mapsto (u, 1)$
$\sigma_2 \in H^0(U, L)$ $u \mapsto (u, u)$

Observe that, if $\sigma \in H^0(U, L)$ then

$$\sigma = f(u)\sigma_1 = \frac{1}{2}\underbrace{(f(u) - f(-u))}_{u\, g(u)}\sigma_1 + \frac{1}{2}\underbrace{(f(u) + f(-u))}_{h(u)}\sigma_1$$

Thus $\sigma = g(u)\sigma_2 + h(u)\sigma_1$

Thus $H^0(L, \mathcal{U}) \simeq \mathcal{O}(\mathcal{U})\cdot\sigma_1 \oplus \mathcal{O}(\mathcal{U})\cdot\sigma_2$. This concludes the proof ▶

◀ We can conclude the construction: Let $\mathcal{U}_i$ be a covering of X, so that $F(\mathcal{U}_i) = H^0(L, \pi^{-1}(\mathcal{U}_i)) \simeq \mathcal{O}(\mathcal{U}) \oplus \cdots \oplus \mathcal{O}(\mathcal{U})$,

let $\sigma_1, \dots, \sigma_n$ be elements of $F(\mathcal{U}_i)$ so that

$$F(\mathcal{U}_i) = \bigoplus_j \mathcal{O}(\mathcal{U}_i)\cdot\sigma_i^j$$

We then have $g^{ik} : \mathcal{U}_i \cap \mathcal{U}_j \to GL_n(\mathbb{C})$

Whose coefficients are $(a^{ik})_{\ell m} \in \mathcal{O}(\mathcal{U}_i \cap \mathcal{U}_j)$

where $\sigma_\ell^i = \sum_m a^{ik} \sigma_m^j$

We consider these g^{ik} as change of trivialisations.

⤳ a bundle $\mathcal{E}$ whose holomorphic sections are

$$H^0(\mathcal{E}, \mathcal{U}_i) = \mathcal{F}(\mathcal{U}_i) ▶$$

Proposition let $(\Sigma_1', \mathcal{L})$ as above, assume furthermore that $\Sigma_1' \subset K$ then, there exists $\phi \in H^0(K \otimes \pi^* \mathcal{L})$ so that

$$\mathcal{L}_u = \{ v \in \pi^*(\pi_* \mathcal{L}) \mid \phi|_{\mathcal{L}} = u \cdot Id \}$$

(i.e. we can reconstruct $(\mathcal{E}, \phi)$ from $(\Sigma_1, \mathcal{L})$)

◀ It is actually enough to construct (for every U small enough)

$$\phi : H^0(\pi_* L, U_i) \to H^0(\pi_* L, U_i) \otimes H^0(K, U_i) \text{ as } \mathcal{O}(U_i)$$

modules.

(i) If x is not a ramification pt:

$$H^0(\mathcal{E}, U) = \bigoplus_i H^0(L, U_i) \; ; \text{ where } \pi^{-1}(U) = U_1 \sqcup \cdots \sqcup U_n$$

then we define ϕ as

$$\phi = \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix}$$ where u_i is the enclusion of $U_i \hookrightarrow K$

(ii) At a ramification pt (as usual we only consider $u \to u^2 = z$)

$$\sigma_1 = (u, 1) \; ; \; \sigma_2 = (u, -u)$$

$$\phi(\sigma_1) = \sigma_2 \; ; \; \phi(\sigma_2) = -u^2 \sigma_1$$

$$\phi = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} dz$$

Observe that in $\pi^* \mathcal{E}$: $\phi_u \sigma_1^* = \sigma_2^* = u \cdot \sigma_1^*$

⇝ Exercice show that indeed this is the inverse. ▶

Combining the constuction we have obtained $\pi^{-1}(Q) \longleftrightarrow \mathrm{Pic}_0(\Sigma_{1Q}')$