Symmetric spaces

I: Symmetric space A symmetric space is a pseudo Riemannian manifold M, so that for every $m \in M$, there exists a (unique) isometry I_m (the symmetry at m) so that $I_m(m) = m$; $T_m I_m = -\mathrm{Id}$. It is of non compact type if it is Riemannian and the curvature $K \leq 0$; It is without Euclidian factors if one cannot write isometrically $M = N \times \mathbb{R}$ (metric product)

Examples: in dim 2, $\mathbb{R}^2$, S^2, $\mathbb{H}^2$, [$\mathbb{R}^{1,1}$, $SL_2(\mathbb{R})/\Delta = \{\text{space of geodesics}\}$]

, « symmetric space » will mean Riemannian space of non compact type (without euclidian factors), thus only $\mathbb{H}^2$ in the list.

Theorem let M be a symmetric space then

(i) M is complete and $G = \mathrm{Iso}(M)$ acts transitively on M

(ii) every element $m \in M$ defines a splitting

$$\mathfrak{g} = \mathfrak{k}_m \oplus \mathfrak{p}_m \text{ where } \mathrm{ad}(I_m)|_{\mathfrak{k}} = \mathrm{Id} \quad \mathrm{ad}(I_m)|_{\mathfrak{p}} = -p$$

(iii) Using $\mathfrak{g} \hookrightarrow \mathcal{X}(M)$ (vector fields on M)

$$\mathfrak{k}_m = \{\xi \mid \xi(m) = 0\}$$

◀ undergraduate exercise ▶

Goal: (1) Construct the symmetric space associated to semi-simple groups (ex: $SL_n(\mathbb{R})$)

(2) Describe them from a Gauge theoretic point of view.

II. Preliminaries on semi-simple groups [admitted from Humphreys]

let G be a lie group and $\mathfrak{g}$ its lie algebra

$G \curvearrowright \mathfrak{g}$ the adjoint action $g, a \mapsto Ad(g)\cdot a$

For $G \subset Gl_n(\mathbb{K})$; $\mathfrak{g} \subset M_n(\mathbb{K})$; $Ad(g)a = gag^{-1}$

$\mathfrak{g} \curvearrowright \mathfrak{g}$ the adjoint action

$a, b \mapsto [a,b] = Ad(a)\cdot b$

For a linear group $[X,Y] = XY - YX$.

The Killing metric on $\mathfrak{g}$: $K(X,Y) = Tr(Ad(X)\cdot Ad(Y))$

A group or lie algebra is semi-simple if K is non degenerate

A lie algebra is simple if it does not contain an ideal

A group G is almost simple if $\mathfrak{g}$ is simple

(⚠ confusion about $\mathbb{R}$ ...)

$\mathfrak{g}$ semi-simple $\Leftrightarrow$ $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ where $\mathfrak{g}_i$ are simple $(\neq \mathbb{R})$

A group is adjoint if Ad is injective

ex : $Sl_n(\mathbb{R})$ is not adjoint, $PSl_n(\mathbb{R})$ is

A group is linear, if it admits an injective $\rho \hookrightarrow GL_n(\mathbb{R})$

All groups are now assumed to be linear

$\rightsquigarrow$ every G-invariant quadratic form on $\mathfrak{g}$ is a multiple of Kill, if G simple $\neq \mathbb{R}$

let $aut(\mathfrak{g})$ be the Lie algebra of $Aut(\mathfrak{g})$, the group of Automorphisms of the lie algebra.

Proposition : Assume $\mathfrak{g}$ is semi-simple.

the map $u \mapsto ad(u)$ is an isomorphism of $\mathfrak{g}$ with $aut(\mathfrak{g})$

(Humphreys)

Proposition : $K \subset G$, and $kill|_K < 0$; then K is compact

Theorem (Nomizu)

the isometry group of a symmetric space is semi-simple $\times \mathbb{R}^n$

A Cartan involution is a lie algebra homomorphism ϕ such that

- ϕ is an involution
- $\mathrm{Kill}(X, \phi(X)) < 0$

Example : $\phi(A) = -{}^tA$ for $\mathfrak{sl}_n(\mathbb{R})$

Proposition (non trivial)

(i) Cartan involution exist ! see below

(ii) All Cartan involutions are conjugated. ↳ prove next time

think of $\mathfrak{sl}_n(\mathbb{R})$

(easy version of (ii), each connected component is a G-orbit of (**))

$$\phi \longrightarrow \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{with} \quad \phi|_{\mathfrak{k}} = 1, \quad \phi_{\mathfrak{p}} = -1$$

Proposition : ϕ Cartan involution $\Rightarrow [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$
and $\mathrm{Kill}|_{\mathfrak{k}} < 0$, $\mathrm{Kill}|_{\mathfrak{p}} > 0$

(the converse is true, given $\mathfrak{k}, \mathfrak{p}$ as above and ϕ defined by $\phi|_{\mathfrak{k}} = 1$, $\phi|_{\mathfrak{k}} = -1$
then ϕ is a Cartan involution)

Exercise : $\mathfrak{k}$ lie algebra of a compact group

III. Construction and properties of symmetric spaces

Let $\mathfrak{g}$ be a semi-simple lie algebra

$$\mathrm{Sym}(\mathfrak{g}) := \{\text{Cartan involutions}\} \underset{\text{submanifold}}{\overset{\phi}{\subset}} \mathrm{Aut}(\mathfrak{g})$$

$\simeq$ Maximal compacts of G
(we shall prove that later)

Exercise : show that $\mathrm{Sym}(\mathfrak{g})$ is a manifold (Hint : use the constant rank theorem).

Our goal is

(i) to show that $\mathrm{Sym}(\mathfrak{g})$ is a symmetric space whose isometry group is (up to connected component $\mathfrak{g}$), and show that it has non positive curvature

(ii) to provide a gauge theoretic description of $\mathrm{sym}(\mathfrak{g}) = S(G)$

(iii) explain NAH for adjoint groups.

Let us make the following definition (just for this lecture)

A Metric flat bundle over a manifold M is a quadruple $(\mathcal{G}, D, \rho)$ so that

(i) $\mathcal{G}$ is a $\mathfrak{g}$-bundle

(ii) D is a flat connection

(iii) ρ section of $\mathrm{Aut}(\mathcal{G})$ by Cartan involutions

Proposition given D, ρ, let $\omega = \rho D \rho \in H^1(M, \mathcal{G})$ then

- $D\rho = \frac{1}{2}[\rho, \mathrm{ad}(\omega)]$
- $d^D\omega + \frac{1}{2}[\omega \wedge \omega] = 0$
- $\rho\omega = -\omega$

ω will be called the Maurer Cartan form.

Not fearing redundancies we write the data $(\mathcal{G}, D, \rho)$ as $(\mathcal{G}, D, \rho, \omega)$

■ Construction of the Maurer-Cartan form and (a)

For any (linear) group H $\quad T_h H = h \cdot \mathfrak{h}$

Thus if $\dot{\nu} \in T_\nu S(G) \subset T_\nu \mathrm{Aut}(\mathfrak{g})$

$\dot{\nu} \in \nu \cdot \mathrm{aut}(\mathfrak{g}) = \nu \cdot \mathrm{ad}(\mathfrak{g})$

thus we have ω so that $\dot{\nu} = \nu \cdot \omega(\dot{\nu})$

Since $\nu^2 = 1$; $\nu\dot{\nu} + \dot{\nu}\nu = 0$.

ω_0 is with values in $\mathcal{P} := \{u \in \mathfrak{g} \; ; \; \rho(u) = -u\}$

It follows that $\rho . ad(\omega) + ad(\omega) . \rho = 0$.
this also means $\rho[\omega(u), v] = -[\omega(u), \rho v]$

Since ρ is an automorphism of $\mathfrak{g}$; $\rho[\omega(u), v] = [\rho\,\omega(u), \rho(v)]$
thus $\rho(\omega(u)) = -\omega(u)$.
It follows that ω is with values in $\mathcal{P}$

(b) The Maurer–Cartan equation

$$\phi : \; Sym(G) \longrightarrow End(\mathfrak{g}) \qquad d\,O(u) = \phi \,.\, ad(\omega(u))$$

thus $\quad 0 = d^2\phi = \phi\, ad\,(d\omega) + d\phi \wedge ad(\omega)$

$$0 = \phi \cdot ad(d\omega) + \phi \cdot (ad(\omega) \wedge ad\omega) = \phi\left(ad(d\omega) + \tfrac{1}{2}[ad(\omega) \wedge ad(\omega)]\right)$$

thus $0 = ad\,(d\omega + \omega \wedge \omega)$ ■ ▶

Proposition : $Sym(\mathfrak{g})$ carries a canonical $(\mathfrak{g}_0, D_0, \rho_0, \omega_0)$ for which
ω is injective.

◀ D_0 is the trivial connection on $S(G) \times \mathfrak{g}$. The section ρ_0 is the tautological one which associates to a point of $S(G)$, itself seen as an element of $Aut(\mathfrak{g})$. ▶

III another point of view : the unitary connection.

Let $(\mathfrak{g}, D, \rho, \omega)$ be a metric data on M

let $\nabla = D + \frac{1}{2}\mathrm{ad}(\omega)$; $\nabla_X \xi = D_X \xi + \frac{1}{2}[\omega(X), \xi]$

Proposition (i) $\nabla\, \mathrm{Kill} = 0$

(ii) $\nabla \rho_0 = 0$, and thus ∇ is unitary

(iii) $d^{\nabla}\omega_0 = 0$, and $\rho_0\omega_0 = -\omega_0$

(IV) $R^{\nabla} = \frac{1}{2}\mathrm{ad}[\omega\wedge\omega]$

◀ (i) $\nabla\mathrm{Kill} = 0$, because $\mathrm{ad}(\omega)$ is antisymmetric for Killing

(ii) $[\nabla_u \rho_0](v) = \nabla_u[\rho_0(v)] - \rho_0[\nabla_u v]$

$$= [D_u\rho_0](v) + \frac{1}{2}[\mathrm{ad}(\omega(u)), \rho]\, v$$

But $D_u\rho_0 = \frac{1}{2}[\rho, \mathrm{ad}(\omega)] = -\frac{1}{2}[\rho, \mathrm{ad}(\omega)]$

(iii) $d^{\nabla}\omega_0(u,v) = \nabla_u(\omega_0(v)) - \nabla_v\omega_0(u) - \omega_0[u,v]$

$$= D_u\omega_0(v) - D_v\omega_0(u) - \omega_0[u,v]$$

$$+ \frac{1}{2}[\omega_0(u), \omega(v)] - \frac{1}{2}[\omega_0(u), \omega(v)]$$

$$= d^D\omega_0 + \frac{1}{2}[\omega_0\wedge\omega_0] = 0$$

(iv) Finally, $R^{\nabla} = R^D + \frac{1}{2}\mathrm{ad}(d^D\omega) + \mathrm{ad}[\omega\wedge\omega]$

$$= \frac{1}{2}\mathrm{ad}[\omega\wedge\omega]$$ ▶

. Dictionnary

unitary $\nabla \longleftrightarrow D$ flat

$\nabla - D = \frac{1}{2}\mathrm{ad}(\omega_0)$, $D\rho_0 = \frac{1}{2}[\rho_0, \mathrm{ad}(\omega_0)]$

$$\left.\begin{array}{l} d^{\nabla}\omega_0 = 0 \\ \nabla\rho_0 = 0 \\ R^{D_0} = \frac{1}{2}\mathrm{ad}(\omega_0\wedge\omega_0) \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} d^D\omega_0 + \frac{1}{2}[\omega_0\wedge\omega_0] \quad \text{(Maurer-Cartan equations)} \\ D\rho_0 = \frac{1}{2}[\rho_0, \mathrm{ad}(\omega_0)] \\ R^{D_0} = 0 \end{array}\right.$$

Exercise: for $G = SL_2(\mathbb{R})$, recognize the Minkowski model.

IV. Application 1 : maps into symmetric spaces / NAH for general adjoint

let $f : M \longrightarrow \mathrm{Sym}(\mathfrak{g})$, then we obtain by pull back from the canonical data $(\mathcal{G}_0, D_0, \rho_0, \omega_0)$ a metric data $(f^*\mathcal{G}_0, f^*D_0, f^*\rho_0, f^*\omega_0)$ on M

Remark 1. We also have $\mathcal{G}, \nabla, \rho, \omega$ where $\nabla\rho = 0$; $\nabla\omega = 0$, $R^\nabla = \frac{1}{2}\mathrm{ad}[\omega\wedge\omega]$

Conversely :

Proposition Let $(\mathcal{G}, D, \rho, \omega)$ be a flat metric bundle on M, $\pi_1(M) = 0$

Then there exists a map $f : M \to \mathrm{Sym}(\mathfrak{g})$ unique up to the G action, so that $(f^*\mathcal{G}_0, f^*D_0, f^*\rho_0, f^*\omega_0) = (\mathcal{G}, D, \rho, \omega)$

◀ Let $D = \nabla - \frac{1}{2}\mathrm{ad}(\omega)$, then

(i) : then D is flat, thus $\mathcal{G} = \mathfrak{g} \times M$

(ii) ρ becomes : $M \longrightarrow \mathrm{Sym}(\mathfrak{g})$

(iii) and we have $D\rho = \frac{1}{2}[\rho, \mathrm{ad}(\omega)]$ $(\nabla\rho = 0)$

$\Leftrightarrow$ $D\rho = \rho\cdot\omega$ $(\mathcal{D}g = g\,\mathcal{D}g^\#)$ ▶

NAH for a general adjoint (complex) group.

A harmonic mapping on a surface X is given by

a G - Higgs bundle $(\mathcal{G}, \phi)$

$(\mathcal{G}, \nabla, \rho, \omega)$

(i) $R^\nabla + \frac{1}{2}\operatorname{ad}[\omega\wedge\omega] = 0$
(ii) $\rho\,\omega = -\omega$
(iii) $d^\nabla \omega = 0$
(iv) $\nabla \rho = 0$
(v) $\omega^{1,0}$ is holomorphic

look for ρ (Hitchin) $\Longleftarrow$

$\Longrightarrow$

(i) $\mathcal{G}, \phi, \bar\partial$
$[\bar\partial = \bar\partial^\nabla]$
$\phi = \omega^{1,0}$

look for ρ then $\omega = \rho D \rho$ $\Uparrow$

$\Downarrow$ $D = \nabla - \frac{1}{2}\operatorname{ad}(\omega)$ flat connection

V. Application 2: Sym(G) as a symmetric space

the metric on Sym(G) is defined as $\langle X | Y \rangle = \operatorname{Kill}(\omega(X), \omega(Y))$

the connection $\nabla^{LC} = \omega^* \nabla^0$ is metric ($\nabla^0 \rho_0 = 0$) and torsion free ($d^{\nabla^0} \omega_0 = 0$)

Thus the curvature of the metric is given by

$$R^\nabla(X,Y)Z = \left[[\omega(X), \omega(Y)], \omega(Z)\right]$$

$$\langle R^\nabla(X,Y)Z | W \rangle = -\langle [\omega(X), \omega(Z)] \mid [\omega(Y), \omega(W)] \rangle + \langle [\omega(X), \omega(W)] \mid [\omega(Y), \omega(Z)] \rangle$$

In particular, using Jacobi identity.

$$\boxed{\langle R^{\nabla}(X,Y)Y \mid X\rangle = -\|[\omega(X),\omega(Y)]\|^2 \leq 0}$$

Exercise : 1) $Sym(G)$ is a manifold

2) each connected component is a G-orbit.

We will now interpret

$\mathfrak{g} \subset$ lie algebra of vector fields on $M := Sym(\mathfrak{g})$

proposition

(i) $\mathfrak{k}_x = \{ u \in \mathfrak{g} \ , \ u(x) = 0 \}$

(ii) let K be the connected lie group whose lie algebra is $\mathfrak{k}$
if K fixes x, then $\mathfrak{k} \subset \mathfrak{k}_x$

◀ (i) Follows from the fact

$\mathfrak{g} \to \mathcal{X}(M)$ is given by

$g(x) = \pi_{\mathfrak{P}}(g)$ where $\pi_{\mathfrak{P}}$ is the $\perp$ projection on $\mathfrak{P}$

(ii) is a consequence of (i) : If K fixes x, then $\forall u$ in $\mathfrak{k}$ $u(x) = 0$. ▶

Proposition. Let x be a point on $Sym(G)$, let i_x be the conjugation by x $i_x : \sigma \mapsto x\sigma x^{-1}$,

(i) then i_x acts by isometries on $Sym(G)$

(ii) x is the unique fixed point of i_x on $Sym(G)$

(iii) $T_x i_x = -1$

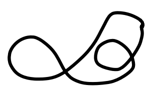
◀ Hint for (ii) $i_x(y) = y \Leftrightarrow \rho_x$ and ρ_y commutes

$\Rightarrow$ then K_x is stable by ρ_y : $k_x + k_y$ stable by ρ
but $\mathrm{Kill}(k_x + k_y) < 0 \Rightarrow$ contradiction

Consequence : Sym(g) is simply connected and connected.

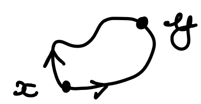
◀ let M_0 be a connected component of Sym(g)

(i) M_0 does not contain a closed geodesic γ



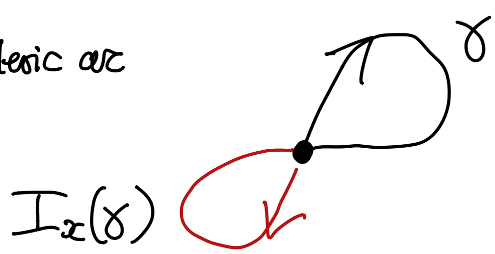
indeed (avoiding covering of your loop)
on can find $x \neq y$ in γ so that the length of two arcs
are the same (otherwise γ is a double cover)

then I_x fixes y as well



thus $x = y$ contradiction

(ii) Assume M is not simply connected then we get
a geodesic arc



then $\gamma \cup I_x(\gamma)$ is a closed geodesic

Thus M_0 is simply connected

—

let K_y associated to y and C a compact orbit of K_y, in M_0

then the center construction, produces m in K_y fixed by m

thus $m = y$ in M_0 ▶

Center construction in $K \leq 0$: Given a compact C in a connected simply connected M with $K \leq 0$; there exists x, R unique so that (i) $C \subset B(x, R)$; $R = \inf(r \mid \exists y \ \ B(y, r) \supset C)$

Corollary : all maximal compact subgroups are conjugated. (every compact subgroup is contained in the stabilizer of a Cartan involution)

VI Reductive subgroup of G.

A reductive algebra of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ so that $\mathrm{Kill}|_{\mathfrak{h}}$ is non degenerate.

Proposition

(i) If $\mathfrak{h} \subset \mathfrak{g}$ is reductive, then $\mathfrak{h} = Z(\mathfrak{h}) \times \mathfrak{l}$, where $\mathfrak{l}$ is semi-simple and $Z(\mathfrak{h})$ is the center of $\mathfrak{h}$

(ii) $\mathfrak{h}$ is reductive if and only if there is an involution i_h of $\mathfrak{h}$ so that $\mathfrak{h} = \{u \mid i(u) = u\}$

(iii) let i_h be an involution of $\mathfrak{g}$, then

$$W = \{\text{Cartan involution } i, \text{ so that } i_h \cdot i = i \cdot i_h\}$$

is a non empty, totally geodesic subspace of $\mathrm{Sym}(G)$

(iv) If $\mathfrak{h} = Z(\mathfrak{h}) \times \mathfrak{l}$, then $W_h = Z(h)/Z_K(h) \times \mathrm{Sym}(\mathfrak{l})$

reductive subalgebra $\leftrightsquigarrow$ involutions $\leftrightsquigarrow$ totally geodesic subspaces

Another caracterisation of $W = \{z \mid i_h(z) = z\}$

non empty because $z \bullet\!-\!\underset{m}{\circ}\!-\!\bullet\ i_h(z)$ m is fixed by i_h