I Non positively curved manifolds

Proposition: Assume M is complete with $K \leq 0$, then $\exp : T_m M \to M$ is a local diffeomorphism.

A **Hadamard manifold** is a complete Riemannian manifold, simply connected, with $K \leq 0$. Then $\exp_m : T_m M \to M$ is a diffeomorphism.

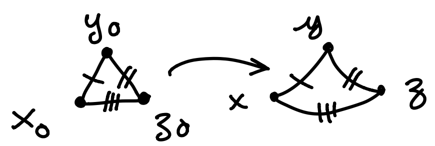
In particular, given x and y there exists a unique geodesic between x and

Let M be a metric space and $x, y, z \in M$, the comparison triangle is the triangle $V_0 = (x_0, y_0, z_0)$ in $\mathbb{R}^2$ so that

$$d(x,y) = \|x_0 - y_0\| \ , \ d(y,z) = \|y_0 - z_0\| \ , \ d(x,z) = \|x_0 - z_0\|$$

The comparison map is

$$\partial T_0 = [x_0, y_0] \cup [z_0, x_0] \cup [y_0, z_0] \longrightarrow \underbrace{[x,y] \cup [z,x] \cup [y,z]}_{\text{geodesic segments}}$$



Proposition [CAT(0)]

For any (x,y,z) in M, the comparison map is contracting

Moreover, if the comparison map preserves the metric.

The comparison map extends to a totally geodesic isometric embedding from T_0 to M

Corollary [flat strip]: If γ_1 and γ_2 are two geodesics within Bounded distance: $\exists A$; $d(\gamma_1(t), \gamma_2(t)) < A$. then There exists a totally geodesic isometric embedding

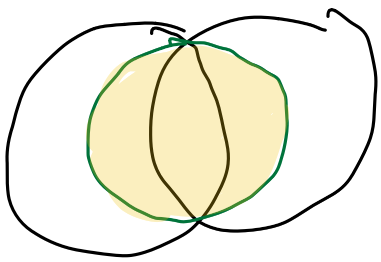
$$\gamma : [0,b] \times \mathbb{R} \longrightarrow M \text{ so that}$$

$$\gamma|_{\{0\} \times \mathbb{R}} = \gamma_1 \ ; \ \gamma|_{\{b\} \times \mathbb{R}} = \gamma_2$$

Proposition [Convexity of the metric]

Let γ_1 and $\gamma_2 : \mathbb{R} \to M$ be two geodesics in some Hadamard manifold; then

$$t \mapsto d(\gamma_1(t), \gamma_2(t)) \text{ is a convex function}$$



Corollary let m be the middle of $x \neq y$

then $\forall R, \exists r < R \quad B(m,r) \supset B(x,R) \cap B(y,R)$

Corollary : if C is a compact set; then exists a unique x, R so that $C \subset B(x,R)$; and if $C \subset B(y,R')$ then $R' > R$

x is the center of C.

[Infinitesimal version of the convexity property]

let γ be a geodesic, if γ_t is a family of geodesics then

$J = \frac{\partial}{\partial t}\gamma_t$, vector field along γ is a

Jacobi field ; it satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma} J = R(\dot\gamma, J)\dot\gamma$ (with a sign)

Proposition

If J is a Jacobi field $\|J_{\gamma(t)}\|^2$ is a convex function of t

Moreover if $\|J_{\gamma(t)}\|$ is constant then $R(\dot\gamma, J)\dot\gamma = 0$ and J is $/\!/$

Consequence

Proposition . The energy is a convex function of the set of mappings

$M \to N$ (if $N \leq 0$) :

If f_t is a family of maps so that $\forall x \; f_t(x)$ is a geodesic

then (i) $t \to \|T_x f_t\|^2$ is convex

(ii) $E(f_t)$ is convex.

◀ Indeed $\forall u$; $T_x f_t(u)$ is a Jacobi field by definition ▶

II Buseman / horofunction compactification

let M be a metric space. The Busemann embedding the map
$M \longrightarrow C^\infty(M)/C^{st}$ function , $x \longmapsto d_x : (y \mapsto d(x,y))$

lemma let M be metric so that balls are compact
then the image of M by i is relatively compact.

Def: The Busemann compactification is $\overline{i(M)}$;
A Busemann function (or horofunction) is an element of $\overline{i(M)} \setminus i(M)$
by a slight abuse of language a representative of h in $C^k(M)$ will also be called a Busemann function.

Fixing a point m_0 in M, one can identify $C^0(M)/_{cst}$ with
$\{ f \in C^0(M) / \ f(m_0) = 0 \}$
Then h is an horofunction if there exists a sequence x_i so that

$$h(m) = \lim_{x_i \to \infty} \left(d(x_i, m) - d(x_i, m_0) \right)$$

Proposition if $x_i \in M \longrightarrow$ Busemann function h,
then if $d(y_i, x_i) \leq K$; then $y_i \longrightarrow h$

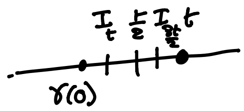
Assume $K(M) \leq 0$; let $\partial_\infty M = \{$space of horofunctions$\}$

Proposition given any $x \in M$; there exists a unique geodesic γ so that $h(\gamma(t)) = t + c$
[gradient line of γ]
Moreover $\lim_{t \to \infty} \gamma(t) = [h]$

◀ Exercise on the triangular inequality ▶

Corollary Assume that $g \in \mathrm{Isom}(M)$ satisfies $g(\gamma(t)) = \gamma(t+b)$ where γ is a gradient line of h; Then $g[h] = h$

Corollary: Assume M is a symmetric space; let h be a Busemann function. Then $\underline{\mathrm{Stab}_G}[h]$ is a non trivial sub algebra of $\mathfrak{g}$.

◀ 

$$\underbrace{I_{\frac{3t}{2}} \cdot I_{\frac{t}{2}}}_{g(t)} \longrightarrow \mathrm{Id}.$$

; $g_t(\gamma(0)) = \gamma(t)$, g_t preserves γ globally

$$\text{thus } g_t(\gamma(s)) = \gamma(s+t)$$

let $H = \mathrm{Stab}[h]$; this is a closed subgroup of G, hence a lie subgroup, and since it contains element arbitrarily close to the identity it follows that $\mathfrak{h} \neq \{0\}$

- Exercise (If M has no euclidian factor) $\mathfrak{h} \neq \mathfrak{g}$

Theorem let Γ be a subgroup of $G = \mathrm{Isom}(\mathrm{Sym}(\mathfrak{g}))$, If $\mathfrak{g}$ is simple. Γ preserves a Busemann function, then Γ is not Zariski dense

◀ If Γ preserves $[h]$, then $\Gamma \subset \mathrm{Stab}[h]$; and in particular $\Gamma \subset \mathrm{Normalizer}(\mathfrak{h})$ = algebraic subgroup of $\mathrm{Ad}(\mathfrak{g})$; non trivial if $\mathfrak{g}$ is semisimple (otherwise $\mathfrak{h}$ is an ideal)

III Harmonic mappings.

1. Some general propeties

Proposition 2: let $f: M \to N$ be a harmonic mapping and g be a convex function on N; then

$$\Delta f \circ g \geqslant 0$$

Proposition [Maximum principle]

Let $f : N \to \mathbb{R}$; N Riemannian, let f be a function so that $\Delta f \geqslant 0$; assume that f has a maximum at m_0, then f is constant.

3. Uniqueness of harmonic mappings

Given $f, g : M \to N$ which are ρ-equivariant

$\Rightarrow F : [0,1] \times M \to N$, $F(0,x) = f(x)$; $F(1,x) = g(x)$, $t \mapsto F(t,x)$ is a constant speed geodesic.

$\Rightarrow u(x) = TF\left(\frac{\partial}{\partial t}\right)$

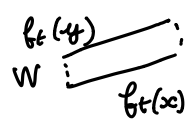
Def: f and g are *parallel*, if u is parallel along F: $\nabla_X u = 0$ and $\forall v \in TM$; $\langle R(TF(v), u) TF(v) | u \rangle = 0$

Theorem if f and g harmonic mappings : $M \to N$ then f and g are parallel.

◀ Let f_t as before, then $E(f_t)$ is convex, since it is critical at $t = 0$ and $t = 1$, it is constant on $[0,1] \Rightarrow e(f_t)$ is also constant $\Rightarrow \forall u$; $\|Tf_t(u)\| = cst$.

Trick: If $g_1 + g_2 = cst$, g_1 and g_2 convex, then g_1, g_2 are cst.

Since $Tf_t(W)$ is a jacobi field along $f_t(x)$



$\Rightarrow \nabla_u W = 0 \Rightarrow \nabla_W u = 0$, parallel ▶

Corollary: If ϕ: harmonic map $T^n \to N$ then ϕ is totally geodesic;

◀ Compare ϕ and $\phi \circ T_g$ = translation ▶