Complex projective geometry

- $\mathbb{CP}^1 = \mathbb{C} \cup \{0\}$, • $[x,y,z,t] := \frac{x-z}{x-t} \frac{y-t}{y-z}$ crossratio
- $PSL(2,\mathbb{C})$ acting by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$ preserves the crossratio
- circles in $\mathbb{C}$ passing through x,y,z is $\{t \mid \mathrm{Im}[x,y,z,t]=0\}$

Corollary : $PSL(2,\mathbb{C})$ preserves circles

Hypobolic plane and its geodesics

1. First definitions

The upper half plane model $H^2 = \{z \mid \mathrm{Im}(z) > 0\}$, boundary at ∞ $\partial H^2 = \mathbb{R} \cup \{\infty\}$

the length of a curve $c(t) = (x(t), y(t))$: $\ell(c) = \int \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{y} dt$

$PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$ preserves H^2, acts on $\partial_\infty H^2$

2. Geodesics

A geodesic in H^2 is an arc length parametrized curve whose image is a circle (or line) $\perp$ to $\partial_\infty H^2$

prop : $\exists$ unique geodesic between two points

◀ Sketch : let $a,b \in H^2$; then $\mathrm{Im}[(a,\bar{a},b,\bar{b}] = 0$

thus let C be the circle passing through $a,b,\bar{b}$. It passes through $\bar{a}$. hence is invariant under $x \to \bar{x}$, hence $\perp$ to $\partial_\infty H^2$.

Thus $C = \{z \mid \mathrm{Im}[a,b,\bar{b},x] = 0\}$. Conversely if $C \perp$ to $\partial_\infty H^2$, passes through a,b It passes through $\bar{a},\bar{b}$. If now $a \in \partial_\infty H^2$, $b \in H^2$ a similar argument works

the case $a,b \in \partial_\infty H^2$ is trivial ▶

$d_{H^2}(x,y) := \inf(\ell(c) \mid c(0) = x, c(1) = y \}$

proposition 1) $PSL(2,\mathbb{R})$ preserves geodesics

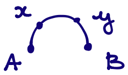
2) If γ is a geodesic then $d(\gamma(s),\gamma(t)) = |s-t|$

2) If c is a curve $]a,b[\to H^2$ is $\ell(c) = d(c(a),c(b))$ then c is a geodesic

3) If $\gamma :]a,b[\to H^2$; if $\forall t \in]a,b[$

$|a-b| = d(\gamma(a), c(t)) + d(c(t), \gamma(b))$ then γ is a geodesic

4) $d_H(x,y) = \log [A, B, y, x]$



3. Geodesics and the boundary at ∞ty

Let $\gamma :]-\infty, +\infty[$ be an oriented geodesic. Then we define

$$\partial H^2 \ni \gamma(-\infty) = \lim_{t \to -\infty} \gamma(t) \; ; \; \lim_{t \to +\infty} \gamma(t) = \gamma(+\infty) \in \partial H^2$$

We say that γ and η are asymptotic if $\exists K$ s.t $\forall s > 0$; $d(\gamma(s), \eta(s)) < K$ and we write $\gamma \sim_{+\infty} \eta$

rks. If $\gamma \sim_{+\infty} \eta$ then $\gamma \sim \eta_\alpha$ where $\eta_\alpha(s) = \eta(s+\alpha)$

being asymptot is an equivalence relation

Theorem : $\gamma(+\infty) = \eta(+\infty)$

$\Leftrightarrow \gamma \sim \eta$

Moreover, $\exists \alpha$ so that $d(\gamma(s), \eta_\alpha(s)) \leq K e^{-s}$ for $K > 0$

◀ Assume $\gamma(+\infty) = \eta(+\infty)$ we may as well assume that $\gamma(+\infty) = \eta(+\infty) = \infty$

and $\operatorname{Im}(\gamma(0)) = \operatorname{Im}(\eta(0))$. then $\operatorname{Im}(\gamma(s)) = \operatorname{Im}(\eta(s)) \; \forall s$.

and $d(\gamma(s), \eta(s)) \leq \dfrac{K}{\operatorname{Im}(\gamma(s))} = K e^{-s}$

(see exercise) for $\overline{\gamma(+\infty) \neq \eta(+\infty)} \Rightarrow d(\gamma(s), \eta(s)) \to +\infty$ ▶

4. Angle between geodesics

the angle $\measuredangle(\gamma_1, \gamma_2)$ between two intersecting geodesics is the angle between the two corresponding circles in the upper half plane model

rk : the angle between two geodesics is invariant under homographies

$\measuredangle(\gamma_1, \gamma_2) = \measuredangle(f(\gamma_1), f(\gamma_2))$ if $f \in PSL(2, \mathbb{R})$

lemma: If $\theta = \measuredangle(\gamma_1, \gamma_2)$ with γ_i geodesics and $\gamma_1(0) = \gamma_2(0)$

then $1 - 2\cos\theta = \lim_{t \to 0} \frac{1}{t^2} d^2(\gamma_1(t), \gamma_2(t))$

◀ proof in exercise ▶

A local isometry is a map $\varphi : B(x, R) \subset \mathbb{H}^2 \to \mathbb{H}^2$, so that $d(z, w) = d(\varphi(z), \varphi(w))$

rks: a local isometry sends geodesic arcs to geodesic arcs, and preserves the angles between geodesics.

A local isometry f preserves the orientation if $(\dot\gamma_1(0), \dot\gamma_2(0))$ has the same orientation as $(f(\dot\gamma_1(0)), f(\dot\gamma_2(0)))$ for γ_i geodesic arcs in $B(x, R)$, with $\gamma_1(0) = \gamma_2(0)$.

Thm | let $\varphi : B(x, R) \to \mathbb{H}^2$ be a local isometry preserving the orientation; then $\exists \varphi \in PSL(2, \mathbb{R})$ s.t $\varphi|_{B(x,R)} = f$

Corollary | $PSL(2, \mathbb{R})$ is the group of isometry presering orientation of $\mathbb{H}^2$.

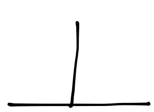
Polygons and convexity

1. Convexity

A set $A \subset H^2$ is convex if given $x, y \in A$ the geodesic segment $[x, y]$ is included in A.

Examples

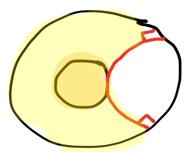
1) Hyperbolic halfplane: by definition on of the connected component of $H^2 \setminus \gamma$ when γ is a geodesic. proof of convexity: using homographies, one may assume the γ is vertical



since for any geodesic arc $\gamma(t) = (x(t), y(t))$ the function $t \to x(t)$ is montone (or constant) the result follows

2) Convex polygons : by definition, those are intersection of geodesic half planes proof the intersection of convex sets is convex.

3) Metric balls : $B(x,R) = \{ y \mid d(x,y) < R \}$ proof of convexity : use that we may choose a Poincaré disk model so that $x = 0$, the we see that $B(x,R)$ is a ball and thus $B(x,R) = \bigcap$ half planes associated to ∂ pts



2. Area

Let C be a (Borel measurable) set in H^2. Let then

$$\text{Area}(C) = \int_C \frac{dx\,dy}{y^2} \quad (\text{in the hyperbolic upper half plan})$$

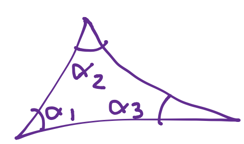
Exercise : show that if $f \in PSL(2,\mathbb{R})$ then $\text{Area}(f(C)) = \text{Area}(C)$

1) compute $J(f)(z)$ and show that $J(f) = \frac{|\text{Im} f(z)|^2}{\text{Im}(z)^2}$

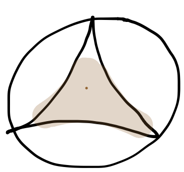
2) use the change of variables formula

Theorem (Gauss-Bonnet)

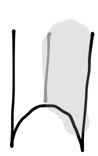
Let T be a triangle, then $\text{Area}(T) = \pi - \alpha_1 - \alpha_2 - \alpha_3$



◀ ⓐ calculons l'aire du « triangle idéal »



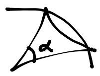
on se ramène au cas de $(-1, 1, \infty)$



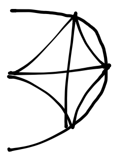
$$\text{Aire}(\;) = \int_0^1 \left[\int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} \right] dx$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{d(\sin(u))}{\cos u} = \int_0^{\frac{\pi}{2}} du = \frac{\pi}{2}$$

thus $\text{Area}(T) = \pi$

(b)  let T_α as in the picture

$$\text{Area}(T_\alpha) + \text{Area}(T_\beta) = \text{Area}(T(\alpha+\beta)) + \pi$$

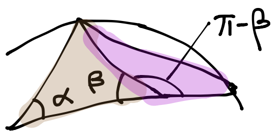


Thus: $f(\alpha) := \pi - \text{Area}(T_\alpha)$ satisfies

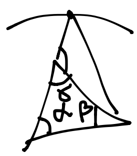
$$f(\alpha) + f(\beta) = f(\beta+\alpha)$$

It follows since f is continuous that $f(\alpha) = \lambda\alpha$, since $f(\pi) = \pi$

$\lambda = 1$

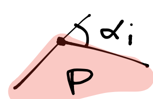
(c) 

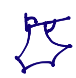
$$\text{Area}(T_{\alpha\beta}) = \text{Area}(T_\alpha) - \text{Area}(T_{\pi-\beta}) = \pi - \alpha - (\pi - (\pi - \beta))$$
$$= \pi - (\alpha + \beta)$$

(d) 

$$\text{Area}(T_{\alpha\beta\gamma}) = \text{Area}(T_{\alpha\beta}) - \text{Area}(T_{\pi-\gamma}) = \pi - (\alpha - \beta - \gamma)$$

Corollary [exercise] If P is a convex polygon then $\text{Area}(P) = \sum_i^n \alpha_i - 2\pi$



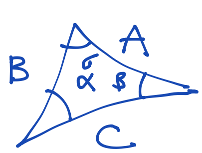
$3\pi - 2\pi = \pi$ 

Example: the area of a right hexagon is π

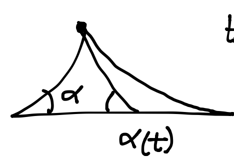
◀ $6\left(\frac{\pi}{2}\right) - \pi = 3\pi - 2\pi = \pi$ ▶

3. Classification of triangles

Theorem let $\alpha, \beta, \gamma \in [0, \pi]$ so that $\alpha + \beta + \gamma < \pi$. Then there exists a unique triangle (up to isometries) with internal angles α, β, γ



◀ Construction:

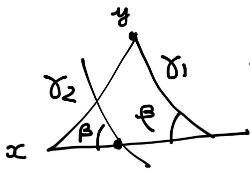


the fonction $t \to \alpha(t)$ is injective from

$$[0, \infty[\;\to\;]0, \pi - \alpha]$$

It follows that there exists a unique t_0 so that $\alpha(t_0)=\beta$

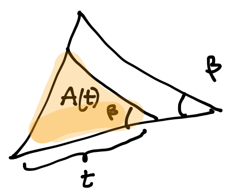
Observe now that



in this situation $\gamma_1 \cap \gamma_2 = \emptyset$ [Gauss-Bonnet again]

in particular γ_2 intersect $[x,y]$

We may now consider the map $t \to A(t)$. we observe (Gauss Bonnet) that this map is increasing from $[0,t_0]$ to $[0,\pi-\alpha-\beta]$. Thus there exist a unique s_0 so that $A(s_0)=\pi-\alpha-\beta-\gamma$.



By Gauss-Bonnet again, the angles will be α,β,γ.

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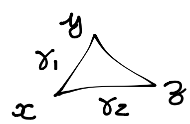
Using isometries we may assume



$(x,y)\in[0,\infty]$; $x=i$ and $\dot\gamma_1(0), \dot\gamma_2(0)$ is oriented then the previous construction shows the uniqueness of x,z,y : we can repeat it ▶