Hyperbolic surfaces

0) Characterization of geodesics

<u>Exercise</u>: 1) $d(x,y)+d(y,z)=d(x,z)$

then $y\in[x,z]$ (use broken geodesics)

2) $t\to\gamma(t)$ is a geodesic arc if and only if

$$d(\gamma(t),\gamma(s))=|t-s|$$

1) Model of a metric space

Let (E,d_E) and (F,d_F) be two metric spaces

Let $x\in E$, $y\in F$. A local isometry from (E,x) to (F,y) is a bijective isometry φ: from $B(x,R)\to B(y,R)$; a local isometry $E\to F$

<u>Ex</u> ① $\forall x,y\in H^2$, there exists a local isometry $(H^2,x)\to(H^2,y)$

in other words H^2 is locally homogeneous.

② Let P be a closed half plan in H^2; that is

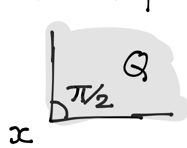
$$P=\{(x,y)\in H^2\ ,\ x\geq 0\}\ ;\ \partial P=\{(0,y)\in H^2\}$$

Show that if $x\in\partial P$, then (P,x) is not isometric to (H^2,x)

◀  indeed ∂P satisfies with its parametrisation

thus $\varphi(\partial P)$ is a geodesic • But $B(y,R)-\varphi(\partial P)$ is non connected ▶

Use a similar idea to prove that



(Q,x) is not isometric to (H,y) or (P,y)

Definitions (i) a hyperbolic surface is a metric space M so that M is modeled on $\mathbb{H}^2$

(ii) a hyperbolic surface with totally geodesic boundary is a metric space modelled on $\mathcal{P}$

(iii) a hyperbolic surface with totally geodesic boundary and right angles is a metric space modelled on $\mathcal{Q}$

<u>ex</u> : hexagon with right angles

<u>rk</u> in that case, interior points x : $(M,x) \sim (\mathbb{H}^2, y)$

boundary pts $(M,x) \sim (\mathcal{P}, y)$ $y \in \partial \mathcal{P}$

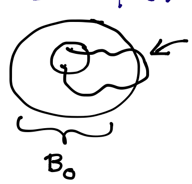
corner pt (of vertices) $(M,x) \sim (\mathcal{Q}, x_0)$

2. Hyperbolic surfaces as metric spaces

① length of a curve

② the Riemanian distance d_R

<u>lemma</u> | (S, d_R) is locally isometric to (S, d)

◀ 

$\ell(c) \geqslant 2R$; thus $d_R(x,y) \leqslant \inf (\ell(c), c \in B_0)$
$= d_H(x,y)$ ▶

We will now always assume S is equipped with its Riemannian distance

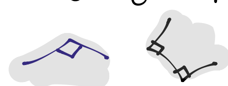
<u>Examples</u> ① $\Gamma \subset PSL(2,\mathbb{R})$ acts properly discontinuously

if $\forall x \in \mathbb{H}^2$; $\exists R$ so that

$$\#\{\gamma \in \Gamma / \; B(x,R) \cap B(\gamma x, R) \neq \phi\} = 1$$

Theorem for such a Γ, H^2/Γ has the structure of a hyperbolic surface.

② gluing & pasting

Theorem  is a hyperbolic surface so that $i : (S_1 \hookrightarrow S_1 \cup S_2$ is a local isometry

Hyperbolic Surfaces as quotients

Our goal is to prove that if Γ is a discrete torsion free subgroup of $PSL(2,\mathbb{R})$ then $\Gamma \backslash H^2$ has the structure of a hyperbolic surface.

Let $x \mapsto [x]$ be the projection from $H^2 \to \Gamma \backslash H^2 := S$

$$d_S([x],[y]) := \inf \left(d(z,w) \mid z \in [x], w \in [y]\right)$$
$$= \inf \left(d(\gamma x, \eta y \mid \gamma, \eta \in \Gamma\right)$$
$$= \inf(d(x, \eta y \mid \eta \in \Gamma)$$

Proposition: $\forall x \quad \exists R$ such that $\pi : B(x,R) \to B([x],R)$ is a bijective isometry.

◀ We know that there exists R_0 so that $\forall \gamma$,

$$B(x,R_0) \cap \gamma B(x,R_0) = \emptyset \text{ if } \gamma \neq Id$$

Let $R = \frac{R_0}{8}$, let us show that $\pi : B(x,R) \to B([x],R)$ is an isometry

Let $z, w \in B(x,R)$. Then

$$d_S([z],[w]) = \inf (d(z,\gamma w) \mid \gamma \in \Gamma)$$

If $\gamma \neq Id$, Then $\gamma(B(x,R)) \cap B(x,R) = \emptyset$. In particular since

$\gamma w \in \gamma B(x,R)$; $\gamma w \notin B(x,R)$ thus :

$$d(x,\gamma w) > R \text{. Then}$$

$$d(z,\gamma w) \geqslant d(\gamma w, x) - d(x,z) \geqslant R - R_0 \geqslant 7R_0 > 2R_0 \geqslant d(z,w)$$

Thus $\quad d_S([z],[w]) = \inf(d(z,\gamma w) \mid \gamma \in \Gamma) = d(z,w)$

It follows that π is an isometry and in particular injective.

It remains to prove π is surjective

Let z, so that $d_S([x],[z]) < R$. Recall that

$d_S([x],[z]) = \inf(d(x,\gamma z) \mid \gamma \in \Gamma)$. Thus there exists some γ

so that $d(x,\gamma z) < R$, thus $\gamma z \in B(x,R)$. Then $[\gamma z] = [z]$ ▶

Corollary (S, d_S) is a hyperbolic surface.

Corollary : Every closed Riemann surface with genus $\geqslant 2$ admits a hyperbolic structure (with the same notion of angles)

◀ $(S,J) = \Gamma \backslash \mathbb{H}^2$ with $\Gamma \subset PSL(2,\mathbb{R})$. The result follows ▶

Corollary : Every closed oriented hyperbolic surface $\neq T2$ is Biholomorphic to : $S = \Gamma \backslash \mathbb{H}^2$

Γ = monodromy group of the structures

◀ Let S be a hypebolic surface. The surface inherits a structure of a manifold: a chart is given by isometry: $B_S(x,R) \to B_{\mathbb{H}^2}(x,R)$

Change of charts are given by orientation preserving isometries: but we show that every orientation preserving isometry is (at least locally) the restriction of an homography. ; In particular it is holomorphic. Thus the system of charts above defines the structure of a 1-dimensional complex manifold. ▶

Geodesics

1. Definitions

Let S be a hypebolic surface. A geodesic is a parametrized curve locally minimizing the length: $\gamma :]a,b[\longrightarrow S$

$$\forall t\ \exists \varepsilon>0,\ s_1,s_2 \in]t-\varepsilon, t+\varepsilon[$$

$$d(\gamma(s_1),\gamma(s_2)) = |s_1 - s_2|$$

As an example let $\pi : \mathbb{H}^2 \to \mathbb{H}^2/_\Gamma = S$. If η is a geodesic in $\mathbb{H}^2$ then $\pi\circ\eta$ is a geodesic in S

Proposition. Let γ_1 and γ_2 two geodesics $]a,b[\to S$ assume $\exists\, c,d \in]a,b[$ with $c<d$ such that $\gamma_1|_{]c,d[} = \gamma_2|_{]c,d[}$ then $\gamma_1|_{]a,b[} = \gamma_2|_{]a,b[}$

Proposition: Let γ be a geodesic in $S = \mathbb{H}^2/_\Gamma$. Then there exists a geodesic η in $\mathbb{H}^2$, such that $\pi\circ\eta = \gamma$

◀ Let $\gamma(t_0) \in S$, let $x \in \mathbb{H}^2$ so that $\pi(x) = \gamma(t)$; let $R>0$ so that $\pi : B(x,R) \to B(\gamma(t),R)$ is an isometry.

let $]c,d[$, $c<d$ so that $\gamma]c,d[\subset B(\gamma(t),R)$

$\pi^{-1}\circ\gamma$ is a geodesic arc and thus $\exists$ a geodesic η

such that $\eta|_{[c,d]} = \pi^{-1}\circ\gamma$.

then $\pi\circ\eta = \gamma$ [using the previous proposition]

2. Geodesically complete

A hyperbolic surface is geodesically complete if $\forall$ geodesic arc η :

$[0,b]\to S$, there exists $\eta_0 :]-\infty,+\infty[\to S$ such that $\eta_0|_{]0,b[} = \eta$

<u>prop</u>: If (S,d) is complete as a metric space. Then S is geodesically complete

<u>Ex</u>. If (S) is compact then it is geodesically complete.

Proposition [Existence of distance minimizing geodesic]

let S be a geodesically complete hyperbolic surface. let $x, y \in S$.

then there exists a geodesic arc : $\gamma : [0,L] \to S$ so that $\gamma(0)=x$, $\gamma(L)=y$

$d(x,y) = L$.

◀ 

let $\gamma : [0,+\infty] \to S$ so that

$\gamma(0)=x$; $\gamma(R)=z$ & $\forall u \in S_x(R)$; $d(u,y) \geq d(z,y)$

let $L = d(x,y)$. We prove that $\forall t \in [R,L]$;

$L = d(\gamma(t),y) + t$ (for $t=L$;

let $W = \{t \in [R,L] \mid d(\gamma(t),y) + t = L\}$

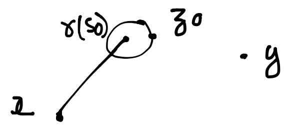
(a) $R \in W$: $L \leqslant d(z,y) + R$

If c is a curve from x to y intersecting $S_x(R)$ at u

then $\ell(c) \geqslant R + d(y,u) \geqslant R + d(y,z) \geqslant d(x,y)$. thus

$$d(x,y) \geqslant R + d(y,z) \geqslant d(x,y) \text{ and thus we have}$$
$$R + d(y, \eta(R)) = d(x,y)$$

(b) let $s_0 = \sup(t \mid t > R \text{ et } \forall s \leqslant t \,;\, s \in W\}$



let $\varepsilon \leqslant d(\gamma(s_0), y)$ so that $B(\gamma(s_0), \varepsilon)$ is isometric to a hyperbolic ball.

z_0 tq $\forall u \in S_{\gamma(s_0)}(\varepsilon)$ $d(u,y) \geqslant d(z_0, u)$

then $d(y, \gamma(s_0)) = \varepsilon + d(z_0, y)$ $\qquad L - s_0 - \varepsilon = d(z_0, y)$.

Thus $d(x, z_0) \geqslant d(x,y) - d(y, z_0) = L - (L - s_0 - \varepsilon) = s_0 + \varepsilon$

since $d(x, z_0) \leqslant s_0 + \varepsilon$ we have $d(x, z_0) = s_0 + \varepsilon$

In particular if c is the broken geodesic $\gamma[0, s_0] \cup [\gamma(s_0), z_0]$, c is actually a geodesic. Thus $s_0 + \varepsilon \in W$. It follows that $s_0 = L$ ▶

Corollary $\forall x \in \mathbb{H}^2$, $\bigcup_{\gamma \in \Gamma} B(x, \operatorname{diam}(S)) = \mathbb{H}^2$

$\cdot \sim \cdot$

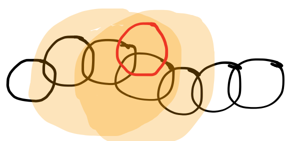
2- Convex hyperbolic surfaces

Theorem. let S be a simply connected hyperbolic surface s.t every two points can be joined by a geodesic, then S is isometric to a convex open set of $\mathbb{H}^2$.

Let S be a hyperbolic surface. A path of balls is a finite sequence of balls B_i in S such that (i) B_i isometric to a hyperbolic ball.

(iii) $\exists B_i'$ isometric to a hyperbolic ball so that $B_i' \supset B_{i-1} \cup B_i \cup B_j$

(iv) $B_i \cap B_{i+1} \neq \emptyset$



a path of balls goes from x to y if $B_1 \ni x$, $y \in B_n$

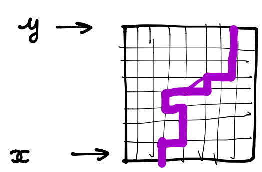
Two path of balls are homotopic, if we can pass from one to the the other by a family of path of balls B_i where B_i is obtained from B_{i+1} by either adding a ball or removing a ball.

Path of balls and curves. A curve is covered by a path of balls if we have a subdivision so that $c(t_i, t_{i+1}) \subset B_i$. Every curve is covered by a path of balls,

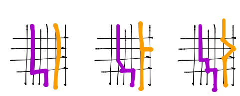
every path of balls covers a curve.

If c_0 and c_1 are homotopic then there exists homotopic paths of balls connecting them

◀ Let $H: [0,1]\times[0,1] \longrightarrow S$ be a homotopy we can find a subdivision of $[0,1]\times[0,1]$ by smaller squares, such that $H(\square) \subset B_{ij} \approx$ a ball in $\mathbb{H}^2$ and $H(\boxplus) \subset \tilde{B}_{ij} \approx$ a ball in $\mathbb{H}^2$. , all top and bottom horizontal rows are identi



then every path on the grid give rise to a path of curves and the homotopy is given by the elementary moves. ▶



Proposition : If $\mathcal{B}$ and $\mathcal{B}'$ covers the same path they are homotopic

◀ We can find a « finer » path of balls $\hat{\mathcal{B}} = B_1 \cdots\cdots B_n$

with a subdivision $1 = i_1 < i_2 < \cdots < i_k$

so that $\hat{B}_{i_1} \cdots\cdots \hat{B}_{i_2 - 1} \subset B_{i_1}$

then we may show that $\hat{\mathcal{B}}$ homotopic to $\mathcal{B}$

we may choose $\hat{\mathcal{B}}$ so that $\hat{\mathcal{B}}$ is also finer than $\mathcal{B}'$: the result follows. ▶

A sequence of isometries $f_i : B_i \hookrightarrow \mathbb{H}^2$ are compatible if

$$f_i |_{B_i \cap B_{i+1}} = f_{i+1} |_{B_i \cap B_{i+1}} .$$

Proposition : given $f_1 : B_1 \hookrightarrow \mathbb{H}^2$ there exists a unique sequence of compatible isometries : f_n is the final isometry

Proposition if $\mathcal{B}$ and $\mathcal{B}'$ are homotopic, if $f_1|_{B_1}$ coincide with $g_1|_{B_1'}$ in the neighbourhood of x, then f_n and g_n coincide in the neighbourhood of y

◀ Proof of the theorem : we can then find a map $\phi : S \to \mathbb{H}^2$ defined as follows. Let $x \in S$, f defined $B(x, R) \to \mathbb{H}^2$

then the corresponding final isometry f_+^y on a neighbourhood of y.

Coincide on small balls with f_+^z if z is on some ball at y isometric to a hyperbolic. Then we set $\phi(y) = f_+^y(y)$. It coincides with f_+^y on some small set and is thus a local isometry.

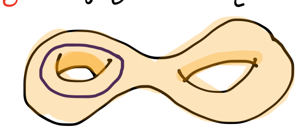
Finally if $x \neq y$, let $\gamma : x \rightsquigarrow y$ geodesic; $\phi(\gamma)$ is a geodesic thus $\phi(x) \neq \phi(y)$. Thus ϕ is injective ▶

3. Closed geodesics

Let $\pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma = S$. Assume that S is compact

Proposition Let γ be an element of Γ. Then there exists a unique geodesic $\eta :]-\infty, +\infty[\to \Gamma$ so that $\gamma \cdot \eta(t) = \eta(t+L)$

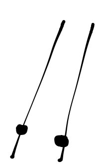
L is the length of γ, $\pi \circ \eta$ is the closed geodesic associated to γ



◀ Uniqueness Let η_1, η_2 be two geodesics such that

$$\gamma(\eta_1(t)) = \eta_1(t+L_1)\ ; \quad \gamma(\eta_2(t)) = \eta_2(t+L_2)$$

- Let us first prove that $L_1 = L_2$,



Observe that $\eta_i(nL_i) = \gamma^n \eta(0)$

thus $d(\eta_1(nL_1), \eta_2(nL_2)) = k = d(\eta_1(0), \eta_2(0))$

It follows that

$$nL_2 = d(\eta_2(0), \eta_2(nL_2)) \leqslant 2k + d(\eta_1(0), \eta_1(nL_1)) = 2k + nL_1$$

thus $L_2 \leqslant L_1$. Symmetrically we prove $L_1 \leqslant L_2$. Thus $L_1 = L_2 =: L$

Let $K = \sup(d(\eta_1(s), \eta_2(s)) \mid s \in [0, L])$

Then $\forall t$; $d(\eta_1(t), \eta_2(t)) \leqslant K$

thus $\eta_1(+\infty) = \eta_2(+\infty)$; $\eta_1(-\infty) = \eta_2(-\infty)$. It follows $\eta_1(s) = \eta_2(s+M)$ for some cst M ●

<u>Existence</u> : let $\overline{B}$ be a closed ball so that

$$\bigcup_{\eta \in \Gamma} \eta \overline{B} = \mathbb{H}^2 \quad (\text{use } R = \text{diam } S)$$

let $L = \text{Inf}\,(d(z, \gamma z) \mid z \in \mathbb{H}^2)$

① $\exists z_0$ such that $d(z_0, \gamma z_0) = L$

let z_i so that $d(z_i, \gamma z_i) \to L$

then $z_i \in \eta_i B$; and it follows that $z_i = \eta_i W$, $W \in B$

and $d(w_i, \eta_i^{-1} \gamma \eta_i (w_i)) \to L$ thus $\eta_i^{-1} \gamma \eta_i\, B(R+L) \cap B(R+L) \neq \phi$

It follows that $\eta_i^{-1} \gamma \eta_i = \eta_0^{-1} \gamma \eta_0$. We may extract a subsequence for $w_i \to w_0$. And we have $d(\eta_0 w_0, \gamma(\eta_0 w_0)) = L$.

② let η be the geodesic such that $\eta(0) = z_0$, $\eta(L) = \gamma z_0$

It follows that $\gamma(\eta(\varepsilon)) = \eta(L+\varepsilon)$

Indeed : $d(\eta(L-\varepsilon), \gamma(\eta(\varepsilon))) \geqslant L - d(\eta(\varepsilon), \eta(L-\varepsilon)) = L - (L - 2\varepsilon) = 2\varepsilon$

thus $\gamma(\eta(\varepsilon)) = \eta(L+\varepsilon)$ for all ε small enough

thus $\forall s$, $\gamma(\eta(s)) = \eta(L+s)$ ▶

If Γ is so that $\mathbb{H}^2/\Gamma$ is compact. the collection

$$\{L(\gamma)\} = \{\text{periods of } \gamma\}$$

is called the length spectrum

4. embedded closed geodesics

<u>Thm</u> let S be a hyperbolic surface