The Laplacian on hyperbolic surfaces

1. The Laplacian on Riemannian surfaces

Let S be a surface equipped with a Riemannian metric $\langle | \rangle$

Assume S is oriented, let J be the corresponding almost complex structure, characterised by $\langle u|v\rangle = \omega(u, Jv)$, where ω is the area form associated to g which is characterised by $\omega(e_1, e_2) = 1$ if (e_1, e_2) is an oriented orthonormal basis. Let $*$ be the Hodge operator : $\Omega^2(S) \to C^\infty(S)$ defined by $* f\omega = f$

then the laplacian on S is given by $\Delta f = * \, d(df \circ J)$

proposition let (x,y) be isothermal coordinates defined on U.

· let $\Omega \in C^\infty(U)$ so that $\omega = \Omega \, dx \cdot dy$, then

$$\Delta f = -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\left(\frac{1}{\Omega}\right)$$

◀ We have $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. Since $J\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}$; we have $dy \circ J = dx$, $dx \circ J = -dy$. Thus $df \circ J = -\frac{\partial f}{\partial x}dy + \frac{\partial f}{\partial y}dx$, and $d(df \circ J) = \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)dx \wedge dy$. Finally $*(dx \wedge dy) = \frac{1}{\Omega}$, the result follows

Proposition . If S is a closed surface

$$\int_S f \Delta g \, \omega = -\int df \wedge dg \circ J = \int \langle \nabla f | \nabla g \rangle \, \omega$$

Recall here that $\langle \nabla f | u \rangle = df(u)$

◀ by definiton $\int_S f \Delta g \, \omega = \int_S f \; d(dg \circ J) = -\int_S df \wedge dg \circ J$ (by Stokes)

Assume that e_1, e_2 are orthonormal, then

$$df \wedge dg \circ J\,(e_1,e_2) = df(e_1)\cdot dg(Je_2) - df(e_2)\,dg(Je_1)$$
$$= -df(e_1)dg(e_1) - df(e_2)dg(e_2) = -\langle \nabla f | \nabla g\rangle = -\langle \nabla f | \nabla g\rangle\, \omega(e_1,e_2).$$

Thus the result follows ▶

Corollary. Assume S is closed then $\int f \Delta f \geqslant 0$ with equality only if f is locally constant.

2. the heat kernel.

On a Riemannian surface S, a fondamental solution of the heat equation, or a heat kernel is a function

$$[0,\infty[\times S\times S \to \mathbb{R},\ (t,u,w) \mapsto p_t(u,w)$$

so that (i) $\Delta_z\, p_t(z,w) + \frac{\partial}{\partial t} p_t(z,w) = 0$

(ii) $p_t(z,w) = p_t(w,z)$

(iii) $\forall f \in C_c^\infty(S)$ [smooth with compact support] then

$$(p_t * f)(w) := \int_S p_t(z,w)\, f(w)\, d\omega(w) \longrightarrow f \text{ when } t \to 0$$

proposition: If $f \in C_c^\infty(S)$ then $f_t := p_t * f$ satisfies

(i) $f_0 = f$; $\frac{\partial f_t}{\partial t} + \Delta f_t = 0$

Example: we have an explicit heat kernel on H^2, which is given by

$$p_t^0(z,w) = \frac{\sqrt{2}}{(4\pi t)^{3/2}}\, e^{-\frac{t}{4}} \int_{d(z,w)}^{+\infty} \frac{r\, e^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(d(z,w))}}\, dr$$

3 a parte: Integration over fundamental domains.

let $S = \mathbb{H}^2/\Gamma$ and $F \subset \mathbb{H}^2$ a fundamental domain for the action of Γ:

$\gamma \mathring{F} \cap \mathring{F} \neq \emptyset \Rightarrow \gamma = id$; $\mu(F \setminus \mathring{F}) = 0$, $\bigcup_{\gamma\in\Gamma} \gamma F = \mathbb{H}^2$. let $\pi : \mathbb{H}^2 \to S$

then

$$(i) \int_S f(x)\,d\omega(x) = \int_F (f\circ\pi)\,d\omega(x)$$

Assume now that $f \in C(H^2)$ so that $\sum_{\gamma\in\Gamma}' f\circ\gamma = g$ is well defined (for instance $f \in C_c(\mathbb{H}^2)$). Observe now that $g(\gamma y) = g(y)$, thus $\exists\, h \in C(S)$ so that $g = h\pi$. then

$$(ii) \int_S h(x)\,d\omega(x) = \int_{\mathbb{H}^2} f(x)\,d\omega(x)$$

Indeed $\int_{\mathbb{H}^2} f(x)\,d\omega(x) = \sum_{\gamma\in\Gamma}' \int_{\gamma F} f(x)\,d\omega(x) = \sum_{\gamma\in\Gamma}' \int_F f(\gamma^{-1}(x))\,d\omega(x) = \int_F g(x)\,dx = \int_S h(x)\,dx$

Finally let $\Gamma_0 < \Gamma$; let F be a fundamental domain for Γ, then let us pick a set $A \subset \Gamma$, which contains exactly one representative of each class in Γ/Γ_0, then

$\bigcup_{\alpha\in A} \alpha F$ is a fundamental domain for Γ_0

4. Spectral theorem & closed surfaces.

Spectral thm for the Laplacian.

Assume S is closed. Then there exists a non decreasing sequence $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and functions $\varphi_i \in C^\infty(S)$ so that.

(i) $\Delta\varphi_i = \lambda_i \varphi_i$

(ii) $\{\varphi_i\}_{i\in\mathbb{N}}$ is an Hilbert basis of $L^2(S)$

(iii) $p_t(z,w) = \sum_{p=0}^{\infty}{}' e^{-\lambda_p t}\varphi_p(z)\varphi_p(w)$ is a heat kernel

(iv) the heat kernel is unique

ⓐ Recall that $L^2(S)$ is the Hilbert completion of $C_c^\infty(S)$ with the norm

$\|f\| = \int_S |f|^2 \cdot \omega$.

ⓑ we have $e^{-\Delta t} f = p_t * f$, and $e^{-\Delta t}$ is bounded for all t.

proposition $\quad p_t(z,w) = \sum'_{\gamma\in\Gamma} p^0_t(z,\gamma w)$

$\triangleleft$ <u>Idea</u> of the proof

a) $\forall g\in SL(2,\mathbb{R})$; $p_t(gz,gw) = p_t(z,w)$

b) thus let $q_t(z,w) := \sum'_{\gamma\in\Gamma} p^0_t(z,\gamma w)$

then $q_t(\zeta z, \eta w) = \sum'_{\gamma\in\Gamma} p^0_t(\zeta z, \gamma\eta w) = \sum'_{\gamma\in\Gamma} p^0_t(z, \zeta^{-1}\gamma\eta(w)) = \sum'_{\gamma\in\Gamma} p^0_t(z,\gamma w)$.

$= q_t(z,w)$

thus $q_t(z,w)$ is defined on $S\times S$. Finally it satisfies

$$\Delta_z q_t(z,w) = \sum'_{\gamma\in\Gamma} \Delta_z p^0_t(z,\gamma w) = -\sum'_{\gamma\in\Gamma} \frac{\partial}{\partial t} p^0_t(z,\gamma w)$$
$$= -\frac{\partial}{\partial t} q_t(z,w)$$

Provided that 1) q_t make sense, 2) we can interchange $\sum'$ and Δ_z, $\sum'$ and $\frac{\partial}{\partial t}$.

which we admit. Finally

$$q_t * f = \sum'_{\gamma\in\Gamma} \int_F p^0_t(z,\gamma w) f(w)\, dw = \int_F p^0_t(z,\gamma w) f(\gamma w)\, dw$$

where F is a fundamental domain of the action of Γ

observe that $\mu(\bar F\setminus F) = 0$ and $\bigcup_{\gamma\in\Gamma} \gamma F = \mathbb{H}^2$, $\gamma\mathring{F}\cap\mathring{F}$ is empty unless $\gamma = id$.

thus $q_t * f = \int_{\mathbb{H}^2} p^0_t(z,w)\cdot f = p^0_t * f \longrightarrow f$ when $t\to 0$.

By uniqueness $q_t = p_t$ $\quad\triangleright$

5. Primitive elements in the fundamental group

let us say that $\gamma\in\Gamma$ is primitive if $\nexists p>1$ so that $\zeta^p = \gamma$ with $\zeta\in\Gamma$

proposition. Moreover if ζ and ξ are primitive, if $\exists p,q>0$ so that $\zeta^p = \xi^q$ then $p=q$

If ζ is primitive then $Z(\zeta) = Z(\zeta^n) = \langle\zeta\rangle$

Corollary. If $\gamma \in \Gamma$, there exists a unique primitive η so that $\gamma = \eta^q$ with $q > 0$

◀ Sketch: let $\gamma \in \Gamma \setminus \{Id\}$, we may choose a model so that $\gamma z = \alpha z$
then $Z_{PSL(2,\mathbb{R})}(\gamma) = \{ z \mapsto \alpha z \mid \alpha \in \mathbb{R} \} \approx \mathbb{R}$. Then $Z_\Gamma(\gamma)$ is a discrete subgroup of $\mathbb{R} \approx Z_{PSL(2,\mathbb{R})} \Gamma$; thus $Z_\Gamma(\gamma) = \langle \zeta \rangle$, we choose ζ uniquely

so that $\gamma = \zeta^p$ with $p > 0$. Finally if $\gamma = \xi^q$ then $\xi \in Z_\Gamma(\gamma) = \langle \zeta \rangle$.
the result follows from these remarks ▶

6. A baby trace formula: McKean formula

Let $p_t : L^2(S) \to L^2(S)$ defined by $p_t \cdot f = \int p_t(x,y) \cdot f(y)\, d\omega(y)$

One can prove

Proposition (i) p_t is a bounded operator

(ii) $tr(p_t) := \sum_1 e^{-\lambda_i t}$ est bien defini

Observe now that $\quad tr(p_t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} = \int_S p_t(x,x)\, d\omega(x)$

thus we obtain the pretrace formula

$$tr(p_t) = \int_S \sum_1 p_t^0(x,\gamma x)\, dx$$
$$= \sum_{\gamma \in \Gamma} \int_F p_t^0(x,\gamma x)\, dx$$

we now want to regroup this terms in conjugacy classes denoting $[\gamma]$ the conjugacy class and $[\Gamma]$ the set of conjugacy classes.

$$= \sum_{\eta \in [\Gamma]} \Big(\sum_{\gamma \in \eta} p_t^0(x,\gamma x) \Big)$$

Then we can write

$$\sum_{\gamma \in [\Gamma]} \Big(\sum_{\eta \in \gamma} p_t^0(x,\eta x) \Big) = \sum_{\gamma \text{ primitive conjugacy classes}} \overbrace{\Big(\sum_{n \geq 0} \sum_{\eta \in \gamma^n} \underbrace{p_t^0(x,\eta x)}_{B\gamma} \Big)}^{A^\gamma} + \overbrace{p_t^0(x,x)}^{C}$$

By the class formula $[\gamma^n] \simeq \Gamma/Z(\gamma^n) = \Gamma/\langle\gamma\rangle$ (if γ is primitive)
thus

$$B_n^\gamma = \sum_{\alpha\in\Gamma/Z_\gamma}' p_t^0(x, \alpha^{-1}\gamma^n\alpha\cdot x)$$

taking integrand we get

$$\int_F B_n^\gamma(x)\, d\omega(x) = \sum_{\alpha\in\Gamma/\langle\gamma\rangle}' \int p_t^0(x,\alpha^{-1}\gamma^n x)\, d\omega x$$

$$= \sum_{\alpha\in\Gamma/\langle\gamma\rangle}' \int_F p_t^0(\alpha x, \gamma^n \alpha x)\, d\omega(x)$$

$$= \sum_\alpha' \int_{\alpha^{-1}F} p_t^0(x,\gamma^n x)\, d\omega(x)$$

$$= \int_{\bigcup_\alpha \alpha^{-1}F} p_t^0(x,\gamma^n x)\, d\omega(x)$$

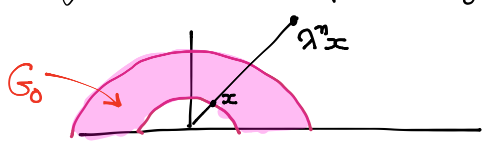
but now $G := \bigcup_{\alpha\in Z/Z_\gamma}\alpha^{-1}F$ is a fundamental domain for the action of $\langle\gamma\rangle$:

[Class formula] It follows that

$$\int_S \sum_{n\geq 0}'\left(\sum_{\eta\in\gamma^n}' p_t^0(x,\eta x)\, d\omega(x)\right) = \int_G \sum_{n\in\mathbb{N}^*}' p_t^0(x,\gamma^n x)\, dx$$

But $\Psi(x) = \sum_{n\in\mathbb{N}}' p_t^0(x,\gamma^n x)\, dx$ is invariant under γ thus $= \int_{H^2/\Gamma}\Psi\, dx$

It follows that we can replace G by a better looking fundamental domain namely



Recall that $p_t(x,\gamma^n(x)) = f_t(d(x,\gamma^n x))$

then $d(x,\gamma^n x) =$ is given by some explicit formula and a computation lead to

$$\int_{G_0} P_t(x,\lambda^n x)\, d\omega(x) = \left(\frac{2\lambda}{\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}}} \cdot e^{-\log(\lambda)^2 \frac{n^2}{4t}} \right) \frac{1}{2}(4\pi t)^{-\frac{1}{2}} e^{-\frac{t}{4}}$$

thus rewriting using $\ell(\gamma)$, we have

$$\int_{G_0} P_t(x,\gamma^n x) d\omega(x) = \frac{e^{\ell(\gamma)}}{sh(n\ell(\gamma))} \cdot e^{-\ell(\gamma)^2 \frac{n^2}{4t}} \left(\frac{1}{2}(4\pi)^{-\frac{1}{2}} e^{-\frac{t}{4}} \right)$$

We still have the term in $(*)$ coming from integrating of C, since C is constant

$$\int_F P_0(x,x) d\omega(x) = \mathrm{Aire}(F)\, C$$

We finally obtain after explicitly writing C

McKean Formula

$$\sum_i e^{-\lambda_i t} = \mathrm{Aire}(F)\,(4\pi)^{-\frac{3}{2}} e^{-t/4} \int_0^\infty \frac{r e^{-r^2/4t}}{\sinh(r/2)} dr$$

$$+ \sum_{\lambda \in \mathcal{G}} \sum_{n>0} \frac{e^{n\lambda}}{sh(n\lambda)} e^{-\frac{\lambda^2 n^2}{4t}} \left(\frac{1}{2} 4\pi\right)^{-\frac{1}{2}} e^{-\frac{t}{4}}$$

Where $\mathcal{G}$ is the primitive length spectrum

This is a special case of the Selberg trace formula.