Hyperbolic planes and geodesics — Sheet 1

0.1. **Circle in $\mathbb{CP}^1$.** Show that $\Im[a,b,x,c]=0$ is the equation (in x) of the circle through a,b,c.

0.2. **Géodesics and cross ratios.** give the equation of the unique geodesic through x and y $\partial_\infty \mathbf{H}^2$ using the cross ratio: one could use the fact the the corresponding circle is invraiant under conjugation.

0.3. **The vertical geodesic.** Give the arc length parametrisation of the vertical axis from 0 to ∞.

0.4. **Geodesic and shortest length.**

(1) let $x=(0,u)$ and $y=(0,v)$ two points on the vertical axis. Show that if c is a curve from x to y then
$$\text{length}(c) \geq \left|\log \frac{u}{v}\right|.$$
with equality if and only if c is the gedesic segment from x to y.
(2) Show that the geodesic arc from x to $y$is the shortest length curve from x to y.
(3) Show – using homographies – that this is true for all x and y.
(4) Show that $d(x,y)=d(x,z)+d(z,x)$ if and only if the three points x, y and z are on the same geodesic with z between x and y.
(5) Show that geodesics parametrized by arc length are characterized by the equation $d(\gamma(t),\gamma(s))=|t-s|$ for all t and s.

0.5. **Distances et birapport.** Soit γ be une géodésique ayant comme point à l'infini a et b. Soit x et y deux points de γ, tels que (a,x,y,b) soit orienté. Soit X_1 et X_0, respectivement Y_0 et Y_1, les extrémités des geodesiques orthogonales en γ à x, respectivement y, telles que (a,X_i,Y_i,b) soit orienté.

En considérant tout d'abord le cas où γ est l'axe vertical. Montrez que
$$d(x,y)=\log[a,b,y,x]=\log[a,b,Y_i,X_i],\ [X_0,Y_0,X_1,Y_1]=\tanh(d_{\mathbf{H}^2}(x,y).$$

0.6. **Distances entre deux géodésiques.** Soit $[a,b]$ et $[c,d]$ deux géodésiques, où a, b, c, d appartiennent à $\partial_\infty \mathbf{H}^2$.

(1) on suppose tout d'abord que $[a,b]$ at $[c,d]$ s'intersectent. Montrez que l'angle d'intersection est orthogonal si et seulement si $[a,b,c,d]=-1$.
(2) on suppose à partir de maintenant que $[a,b]$ et $[c,d]$ ne s'intersectent pas. Montrez qu'il existe une unique géodésique orthogonale au deux.
(3) Donnez la longueur ℓ du segment orthogonal entre les deux en fonction de $\log[a,b,c,d]$.
(4) Montrez que pour tout x et y de $[a,b]$ at $[c,d]$ et c, d respectivement: $d(x,y)\leq \ell$ avec égalité si et seulement si la géodésique passant par x et y est orthogonale au deux.

0.7. **Hexagon.** (H) Given any triple of positive numbers A,B,C prove that there exists a unique – up to isometries – hexagon with right angles, with vertices $(a_1,a_2,\ldots,a_6)$ such that $d(a_1,a_2)=A$, $d(a_3,a_4)=B$ and $d(a_5,a_6)=C$. (you should better use the last question...).

0.8. **The boundary at infinity and geodesics.** (E) Let γ and η be two parametrised geodesics.

(1) prove that $\limsup_{t\to+\infty} d(\gamma(t),\eta(t))<\infty$ iff $\gamma(+\infty)=\eta(+\infty)$.
(2) Prove that if $\gamma(+\infty)=\eta(+\infty)$, then there exist a positive constant C, constants K and A, so that for all $t>0$: $d(\gamma(t+K),\eta(t))\leq A.e^{-Ct}$.

0.9. **The Poincaré disk model (with an application to the angles of geodesics).**

(1) Show that there exists a homography in $\mathsf{PSL}(2,\mathbb{C})$, denoted f so that $f(i)=0$, and $f(0)=-1$, $f(\infty)=-1$. Give a formula for f.
(2) Show that $f(\partial_\infty \mathbf{H}^2)=\{z,|z|z=1\}$, and $f(\mathbf{H}^2)=\mathbf{D}=\{z,|z|<1\}$.
(3) Show that if d is the distance on $\mathbf{D}$ given by $d(z,w)-=d_{\mathbf{H}^2}(f^{-1}(z),f^{-1}(w)$, then

$$d(x,0)=\log\left(\frac{1+|x|}{1-|x|}\right).$$

(4) Show that the geodesics through 0 are lines
(5) Prove that for two geodesics passing though 0 the angle θ between the two geodesics γ_1 and γ_2 satisfies

$$\sin\left(\frac{\theta}{2}\right)=\lim_{t\to 0}\frac{1}{2t}d(\gamma_1(t),\gamma_2(t)). \quad (1)$$

hint make the computation whenever the mediatrix between γ_1 and γ_2 is the vertical axis;
(6) for two geodesics passing though i in the half-plane model, show that Equation (1) holds for θ the angle of the geodesics.
(7) Show that the last assertion holds for any point x in $\mathbf{H}^2$

0.10. **Metrics balls.** Show that the center of the Euclidean ball $B(x,r)$ defined by the hyperbolic ball $B_{\mathbf{H}^2}(i,R)$ is on the imaginary axis. Compute x and r in terms of R.

0.11. **Convex sets.** Show that the intersection of two convex sets is convex.

0.12. **The hyperbolic area.**

(1) Let f be an homography. Show that $\det(\mathrm{D}_z f)=\frac{\Im(f(z))^2}{\Im(z)^2}$
(2) Let define the hyperbolic area of a set C in the upper half plane model as $\mathrm{Area}(C)=\int_C \frac{\mathrm{d}x\mathrm{d}y}{y^2}$. Show that $\mathrm{Area}(C)=\mathrm{Area}(f(C))$ for any homography f.

0.13. **Gauss–Bonnet for polygons.** Let P be a polygon. For any vertices x of P let α_x be the exterior angle of the two sides (that is $\pi-\beta_x$ where β_x is the internal angle). Show that, if X is the set of vertices of P,

$$\mathrm{Area}(P)=\sum_{x\in X}\alpha_x-2\pi.$$

Hint: Use the Gauss–Bonnet Formula for triangle and a recursion formula.

0.14. **Angles and triangles.**

(1) Let γ be a geodesic and $x\in\partial_\infty\mathbf{H}^2$. Let $\alpha(t)$ be the angle between γ and the geodesic through x and $\gamma(t)$. Show that α is injective (*Hint*: use Gauss–Bonnet). Show now that given α, β and γ with $\alpha+\beta+\gamma<\pi$, then there exists a unique (up to isometries) triangle with angle α, β and γ.
(2) Prove that given α and γ with $\alpha+\gamma<\pi$, there exists a unique triangle $T(\alpha,\gamma)$ up to isometries, with one vertex at infinity and internal angles α and γ.
(3) Using a similar idea, use that there exists a unique triangle with internal angles α, β and γ with $\alpha+\beta+\gamma<\pi$.