II- Groups of isometries

0.1. **Reflections.** Let γ be a geodesic

(1) For any $x \in \mathbf{H}^2$ not in γ, prove that there exists a unique $z \in \gamma$ such that the angle $\alpha(z)$ of $[x, z]$ with γ is $\pi/2$. *Hint* consider the function $z \to \alpha(z)$ show that it is injective and describe its range.
(2) Given x, let η be the geodesic orthogonal to γ though x so that $\eta(0) \in \gamma$. We define the reflection σ with respect to γ by $\sigma(x) = \eta(-t)$ if $\eta(t) = x$. Show that σ is an isometry. *Hint:* consider first the case when γ is the vertical axis.
(3) Show that σ reverses the orientation.
(4) Let $x \neq y \in \mathsf{H}^2$. Let

$$\gamma(x, y) = \{z \mid d(z, x) = d(z, y)\}$$

Assume first that $x = -\overline{y}$. Using an isometry interchanging x and y (and Euclidean geometry) show that $\gamma(x, y)$ is the vertical axis. Proceed to show that in general, $\gamma(x, y)$ is a geodesic. Then show that

$$H(x, y) = \{z \mid d(z, x) < d(z, y)\},$$

is convex

0.2. **Center of radius.**

(1) Let x and y be distinct points in $\mathbf{H}^2$. Let R be a positive number. Show that there exists $z \in \mathbf{H}^2$ and $S < R$ so that $B(x, R) \cap B(y, R) \subset B(z, S)$: consider first the case when x and y are on the vertical axis.
(2) Let K be a bounded set in $\mathbf{H}^2$, let

$$R = \inf\{S \mid \exists x, K \subset B(x, S)\}\,.$$

Show that there exists a unique $y \in \mathbf{H}^2$ so that $K \subset B(y, R)$.

0.3. **Tilings.** let $P_1, \ldots, P_p$ be polygons in $\mathbf{H}^2$. A *tiling* of a subset E in $\mathbf{H}^2$ – with *fundamental tiles* $P_1, \ldots, P_k$– is a decomposition $E = \cup_{i \in I} Q_j$, so that

- Q_j are closed convex polygons,
- for all i and j, the interior of Q_i and Q_j are disjoint if $i \neq j$.
- for all i there exists $g \in \mathbf{H}^2$ and k so that $Q_j = gP_k$.

An isometry g preserves the tiling if there is exists a bijection h of I so that $g(Q_i) = Q_{h(j)}$

(1) show that the set Γ of isometries preserving the tiling is a subgroup of the isometry group of $\mathbf{H}^2$.
(2) Let $\{\gamma_m\}_{m \in \mathbb{N}}$ be a sequence of elements of Γ converging to the identity. Show then that for all i, there exists n_i, so that for all $p > n_i$, $\gamma_p(Q_i) = Q_i$.
(3) Show that if $E = \mathbf{H}^2$, assuming all the fundamental tiles are bounded, using the "center of radius" exercise, show that for n large enough g_n fixes three points not on a geodesic.
(4) Show that if $E = \mathbf{H}^2$, assuming all the fundamental tiles are bounded, show that Γ is discrete.

A tiling is *periodic* if there is only one fundamental tile P and the group of symmetries Γ is so that $\mathbf{H}^2 = \cup_{\gamma \in \Gamma} \gamma P$ is a tiling.

0.4. **Triangle group.** Let p, q, r be three positive integers such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \leq 1$. Assume here to simplify that p, q, r are pairwise distinct.

(1) prove they there exists two hyperbolic triangles T_1 and whose angles at the vertices are π/q, π/r, π/p.
(2) Let $E = \cup Q_i$ be a connected set tiled with fundamental tiles T_1 and T_2. We say E si adapted if
  (a) $Q_i \cap Q_j$ is empty or an edge or a vertex of Q_i and Q_j
  (b) furthermore whenever Q_i and Q_j meet at a vertex then the angles are the same.
  (a) Prove that an adapted tiling of $\mathbf{H}^2$ is unique up to isometries.
  (b) prove that given E, there is always a bigger set F with an adapted tiling that contains the tiling of E.
  (c) Show that $\mathbf{H}^2$ admits an adapted tiling.
(3) let $T(p, q, r)$ be the group generated by the reflections σ_i along the side of these triangles. Prove that $T(p, q, r)$ is discrete. *Hint* show that $T(p, q, r)$ preserves a tiling.

0.5. **Dirichlet fundamental domain.** Let Γ be a discrete subgroup of $\mathsf{PSL}(2, \mathbb{R})$ acting freely and properly without fixed points on the hyperbolic plane. For any x_0 in the hyperbolic plane. Let

$$\Delta := \{x \mid \forall \gamma \in \Gamma \setminus \{1\}, d(x, x_0) < d(\gamma(x_0), x)\}$$

(1) prove that Δ is convex, open and non empty.
(2) prove that $\forall \gamma \in \Gamma \setminus \{1\}$, $\gamma(\Delta) \cap \Delta = \emptyset$ and $\bigcup_{\gamma \in \Gamma} \gamma(\overline{\Delta}) = \mathsf{H}^2$.
(3) Assume that if $\Gamma \backslash \mathbf{H}^2$ is compact and let R so that $\bigcup_{\gamma \in \Gamma} B(\gamma(x), R))$. Let $S = \{\gamma \in \Gamma \mid d(x, \gamma(x) \leq R\}$ and

$$\Delta^S := \{x \mid \forall \gamma \in S \setminus \{1\}, d(x, x_0) < d(\gamma(x_0), x)\}$$

  (a) Prove that S is finite,
  (b) Prove that $\Delta^S \cap \{y \mid d(x, y) = R\}$ is empty
  (c) Show that $\Delta^S \subset B(x, R)$,
  (d) Show that if $\gamma \notin S$,

$$\Delta^S \subset \{y \mid d(x, y) < d(\gamma(x), y)\}.$$

  Show that then $\Delta^S = \Delta$ and is thus a finite sided bounded polygon.
(4) Prove that $\mathbf{H}^2$ is tiled with Δ as a fundamental tile and Γ as symmetry group.

0.6. **Torsion free subgroup.** A group is torsion free, if there is no non trivial element whose power is trivial.

(1) Show that a torsion free discrete subgroup of $\mathsf{PSL}(2, \mathbb{R})$ acts properly discontinuously on H^2.
(2) Find an explicit torsion free subgroup of finite index of $\mathsf{SL}(2, \mathbb{Z})$ as a kernel of a map onto $\mathsf{SL}(2, \mathbb{Z}/p.\mathbb{Z})$.