ENTROPY AND AFFINE ACTIONS FOR SURFACE GROUPS

FRANÇOIS LABOURIE

ABSTRACT. We give a short and independent proof of a theorem of Danciger and Zhang: surface groups with Hitchin linear part cannot act properly on the affine space. The proof is fundamentally different and relies on ergodic methods.

1. **Introduction.** Our goal is to give an independent proof, based on thermodynamical ideas, of a recent theorem by Danciger and Zhang [7].

Theorem 1.1. *Assume that a surface group acts on the affine space so that its linear part is a Hitchin representation. Then its action on the affine space is not proper.*

The proof given in this article, which also gives results of independent interest (Theorems 4.1 and 5.2) , uses ergodic theory and hyperbolic dynamics: entropy, Sinai–Ruelle–Bowen measures. We hope that the use of this type of methods will open a novel approach on the study of proper affine actions, expanding previous work of Goldman, Margulis and the author.

Being very optimistic, as an approach to the Auslander conjecture, one could hope that, in the spirit of Kahn–Markovic [16] and Kahn–Labourie–Mozes [15], the presence of free groups could help in building surfaces groups close to being Fuchsian inside groups acting cocompactly on the affine space.

A *surface group* is the fundamental group of a closed connected oriented surface of genus at least 2. A *Hitchin representation* [14] is a representation that can be deformed into a *Fuchsian representation*, that is a discrete representation with values in an irreducible $\mathsf{SL}(2, \mathbb{R})$.

A conjecture, attributed to Auslander [2], states that if a group Γ acts properly and cocompactly on the affine space then it does not contain a non abelian free group. This conjecture has been proved up to dimension 6 by Abels, Margulis and Soifer in [1]. On the other hand, Margulis work in [21] has exhibited free groups acting properly on the affine space. Work of Goldman, Margulis and the author [11], further extended by Ghosh and Treib [10], has shown how to characterise proper actions of a hyperbolic group using the *Labourie–Margulis diffusion*, which is an extension to measures – introduced in [17] – of the *Margulis invariant* introduced by Margulis in [22]. As for surface groups, they were shown by Mess [23] to admit no proper

affine actions on the affine 3-space. An alternate proof was given by Goldman and Margulis [12] and then by the author [17] with the extension to groups whose linear part is Fuchsian. On the other hand, Danciger, Guéritaud and Kassel [6] exhibited examples of proper affine actions of surface groups, or more generally all Coxeter groups, in higher dimensions. For a survey and similar considerations see [28] and [5]

I thank Sourav Ghosh, Fanny Kassel, Andrés Sambarino, Ilia Smilga and Tengren Zhang for valuable discussions and input. An anonymous referee also helped improve a lot the final version of this article.

1.1. *A sketch of the proof.* As an initial observation, we observe that the problem reduces to the case of representations whose linear part is in $\mathsf{SO}(p, p-1)$. Indeed, according to Andrés Sambarino [?][Corollary 1.5] preceded by oral communications by Guichard, the Zariski closure G of a Hitchin representation in $\mathsf{SL}(m, \mathbb{R})$ always contains the irreducible $\mathsf{SL}_2(\mathbb{R})$, and, if non Zariski dense, is contained in either $\mathsf{Sp}(2p)$ or $\mathsf{SO}(p, p - 1)$, depending if $m = 2p$ or $m = 2p - 1$. Recall finally that if an element of the affine group acts properly on the affine space, then 1 is an eigenvalue of its linear part. Thus 1 is in the spectrum of any element in the Zariski closure of its linear part. It follows that the representation is in odd dimension and non Zariski dense in $\mathsf{SL}(2p - 1)$, thus contained in $\mathsf{SO}(p, p - 1)$.

After this initial observation, the proof follows the thermodynamic theme introduced in [17]. A sketch is as follows

From now on, let Γ be a surface group whose linear part is a Hitchin representation in $\mathsf{SO}(p, p - 1)$. The *Labourie–Margulis diffusion M* is a continuous function on the space of finite measures invariant by the geodesic flow of the surface, associated to the representation on the affine space [17]. According to a generalization of [17, 11] due to Ghosh and Treib [10, Theorem 7.1 and Definition 4.4], if there exists a measure μ so that $M(\mu) = 0$, then the action on the affine space is not proper.

As a first step in the proof, we embed the Lie algebra of $\mathbb{R}^{p, p-1} \rtimes \mathsf{SO}(p, p - 1)$ as a subalgebra of the Lie algebra of $\mathsf{SO}(p, p)$. Thus an affine representation is seen as a (special) infinitesimal deformation of the linear part of the representation of $\mathfrak{so}(p, p - 1)$ in $\mathsf{SO}(p, p)$. As in [12, 9], we now interpret in lemma 6.2 the Margulis invariant as a variation of the pth eigenvalue (or the $(p + 1)^{st}$), while the other eigenvalues remain constant.

As a consequence of the Abramov lemma and the definition of equilibrium states as done in [26], we can now interpret, in lemma 6.4, the Margulis invariant as the variation of the topological entropy of the *last root flow*, a flow for which the length of the closed orbit

associated to γ is the logarithm of the product of the $(p-1)^{st}$ and p^{th} eigenvalues of γ.

A recent series of results by Pozzetti, Sambarino and Wienhard [25] implies among other things that this entropy is constantly equal to 1. We prove this result independently in Theorem 5.2 by proving that the *isotropic limit curve* is smooth and use an idea due to Potrie–Sambarino [24] to obtain the same result. This is a parallel to [25, Theorem 9.9].

This smoothness, obtained in Theorem 4.1 now follows from a general lemma about proximal bundles – lemma 4.3 – and a transversality property – proposition 3.6 – that we prove for Fuchsian representations in $\mathsf{SO}(p,p)$. This transversality property is a consequence of Lusztig positivity [20] as used in [8] and we wonder whether this property could characterise Hitchin representations in $\mathsf{SO}(p,p)$ within Anosov representations.

Combining these simple ideas, on obtains that the Labourie–Margulis diffusion for the Bowen–Margulis measure of the last root flow is zero and thus concludes the proof of the Theorem by Danciger and Zhang.

2. **Isotropic flags and the geometry of** $\mathsf{SO}(p,p)$**.** Let F be a vector space equipped with a metric Q of signature (p,p), let $\mathsf{SO}(p,p)$ be its isometry group. For every vector space V in F we denote by V° its orthogonal with respect to the quadratic form. An *isotropic space* is a vector space on which the restriction of Q vanishes, a *maximal isotropic plane* is an isotropic plane of dimension p. We denote by $\mathbf{L}$ the space of maximal isotropic planes.

Recall that the action of $\mathsf{SO}(p,p)$ on $\mathbf{L}$ has two orbits, which are both connected components of $\mathbf{L}$. To distinguish them, let us a p-plane M, on which the restriction of Q is positive definite, as well as orientations on M and M°. Any p-isotropic plane P is then the graph of a linear anti-isometry A from M to M°. We say P is *positive* when A preserves the orientation and *negative* otherwise. We denote by $\mathbf{L}^+$ the space of positive p-isotropic planes and $\mathbf{L}^-$ the space of negative p-isotropic planes. Any $(p-1)$-isotropic plane is contained in exactly one positive isotropic p-plane and one negative isotropic p-plane.

An *isotropic flag* is a collection of isotropic planes $L = (L_i)_{1 < i \leqslant p}$ so that $L_i \subset L_{i+1}$, $\dim(L_i) = i$. An isotropic flag L can be positive or negative depending on L_p. We denote by $\mathbf{Fl}$ the space of positive isotropic flags. The group $\mathsf{SO}(p,p)$ acts transitively on $\mathbf{Fl}$ and the stabiliser of a point is the minimal parabolic subgroup of $\mathsf{SO}(p,p)$. Observe also that L_p is determined by L_{p-1}. Two isotropic flags L and M are *transverse* if for all i, we have $M_i \oplus L_i^\circ = L_i \oplus M_i^\circ = F$.

A p-tuple of lines $E = (E_i)_{i=1,\ldots,p}$ is *isotropic* if $E_1 + \ldots + E_p$ is maximal isotropic. The isotropic flag Flag(E) *associated to* E is

$$\mathrm{Flag}(E) := (L_1, \ldots, L_p) \, , \tag{1}$$

where $L_i = E_1 + \ldots + E_i$. Two p-tuples of lines $E = (E_i)_{i=1,\ldots,p}$ and $\overline{E} = (\overline{E}_i)_{i=1,\ldots,p}$ are *Q-paired* if they are both isotropic and Q restricted to $E_i \oplus \overline{E}_j$ is zero for $i \neq j$ and non degenerate otherwise. We then have

Proposition 2.1. [TRANSVERSE FLAGS] *The map that sends $(E, \overline{E})$ to $(\mathrm{Flag}(E), \mathrm{Flag}(\overline{E}))$ is an $\mathsf{SO}(p,p)$-equivariant bijection from the space of Q-paired p-tuples of lines to the set of transverse flags.*

Let us conclude with a description of the tangent space to **L**: Let M_0 and M_1 be two transverse isotropic planes. Let M be an isotropic p-plane transverse to to E_1, so that M is the graph of $f \in \mathrm{Hom}(M_0, M_1)$. Let ω_M be the 2-form on M_0 given by $\omega_M(u,v) = Q(u, f(v))$. We observe that $Q(u, f(v))$ is antisymmetric if and only if the graph of f is isotropic. This leads to:

Proposition 2.2. [IDENTIFICATION] *Let M_0 and M_1 be two transverse isotropic p-planes. The map $M \to \omega_M$ is a diffeomorphism between the space of isotropic planes transverse to M_1 and $\Lambda^2(M_0^*)$. In particular, $\mathsf{T}_{M_0}\mathbf{L} = \mathrm{Hom}(M_0, M_1)$ identifies with $\Lambda^2(M_0^*)$.*

3. **Anosov representations for $\mathsf{SO}(p,p)$ and $\mathsf{SO}(p,p-1)$.** Let Σ be a closed hyperbolic surface, X its unit tangent bundle and $(\varphi_t)_{t\in\mathbb{R}}$ its geodesic flow. We also write $\Gamma := \pi_1(\Sigma)$.

Let ρ be a representation of Γ in $\mathsf{SO}(p,p)$ that we see acting on a vector space F equipped with a quadratic form $\langle | \rangle$ of signature (p,p). Since $\pi_1(X)$ is an extension of $\pi_1(\Sigma)$, we will consider ρ also as a representation of $\pi_1(X)$ and denote $\mathcal{F}_\rho$ the associated flat bundle on X and F_x, the fibre of $\mathcal{F}_\rho$ at a point x in X.

Observe that $(\varphi_t)_{t\in\mathbb{R}}$ lifts to a flow $(\Phi_t)_{t\in\mathbb{R}}$ acting on $\mathcal{F}_\rho$ by vector bundle automorphisms which are parallel along the geodesic flow

Definition 3.1. [ANOSOV REPRESENTATIONS FOR $\mathsf{SO}(p,p)$] *We say ρ is Borel Anosov for $\mathsf{SO}(p,p)$, if the bundle $\mathcal{F}_\rho$ splits into $2p$ continuous line bundles $\mathcal{E}_i, \overline{\mathcal{E}}_i$ with $1 \leqslant i \leqslant p$, with the following properties:*

(1) The line bundles $\mathcal{E}_i$ and $\overline{\mathcal{E}}_i$ are invariant under $(\Phi_t)_{t\in\mathbb{R}}$ and Q-paired,
(2) The flow $(\Phi_t)_{t\in\mathbb{R}}$ contracts the bundles

$$\mathcal{E}_i^* \otimes \overline{\mathcal{E}_p} \qquad \text{when } i < p, \tag{2}$$

$$\mathcal{E}_i^* \otimes \mathcal{E}_j \qquad \text{when } j < i, . \tag{3}$$

We recall that a flow $(\Phi_t)_{t\in\mathbb{R}}$ *contracts a bundle* $\mathcal{E}$ over a compact manifold if there exist a continuous metric and positive constants a and b, so that for all positive t, for all u in $\mathcal{E}$, we have $\|\Phi_t u\| \leqslant a e^{-bt} \|u\|$. To be contracting on a compact manifold is independent of the parametrisation of the flow or the choice of the metric.

Let C_γ be a closed orbit of the flow on X of length ℓ_γ associated to an element γ in Γ. Then $\rho(\gamma)$ is conjugated to the endomorphism

Φ_{ℓ_γ} of F_x, and in particular $(\mathcal{E}_i)_x$ and $(\overline{\mathcal{E}}_i)_x$ are eigenlines of Φ_{ℓ_γ}. We denote by $\lambda_i(\rho(\gamma))$ and $\overline{\lambda}_i(\rho(\gamma))$ the corresponding eigenvalues, which are also eigenvalues of $\rho(\gamma)$. The Anosov condition gives $\lambda_i \cdot \overline{\lambda}_i = 1$ and the following ordering of eigenvalues

$$\lambda_1 > \lambda_2 > \ldots > \sup\{\lambda_p, \lambda_p^{-1}\} \geqslant \inf\{\lambda_p, \lambda_p^{-1}\} > \ldots > \overline{\lambda}_1 .$$

3.1. *Limit curves.* Let ρ be an Anosov representation for $\mathsf{SO}(p, p)$. We may lift the bundle E_ρ to a trivial bundle over the unit tangent bundle Y of the hyperbolic plane. The line bundles $\mathcal{E}_i$ and $\overline{\mathcal{E}}_i$ also lift and since they are invariant under $(\Phi_t)_{t \in \mathbb{R}}$, they are constant along each orbit of $(\phi_t)_{t \in \mathbb{R}}$. Let us then consider the maps

$$E_i : (x, y) \mapsto E_i(x, y) := (\mathcal{E}_i)_z , \quad \overline{E}_i : (x, y) \mapsto \overline{E}_i(x, y) := (\overline{\mathcal{E}}_i)_z ,$$

where (x, y) in a pair of distinct points in the boundary at infinity $\partial_\infty \mathbf{H}^2$ of the hyperbolic plane $\mathbf{H}^2$, and z is a point in a the geodesic defined by (x, y). z in a point in the geodesic defined by the pair of distinct points

Proposition 3.2. [LIMIT CURVE]

We have $E_i(x, y) = \overline{E}_i(y, x)$. Moreover the isotropic flag $\xi(x, y)$ given by $(E_1(x, y), E_2(x, y), \ldots E_p(x, y))$ only depends on x.

The map $\xi : x \mapsto \xi(x) := \xi(x, y)$ from $\partial_\infty \pi_1(\Sigma)$ to $\mathbf{Fl}$ xis *the limit curve* of the Anosov representation.

Proof. The continuity of the maps E_i follows from the continuity of the bundles $\mathcal{E}_i$. By density, it is enough to check the first identity for $(x, y) = (\gamma^+, \gamma^-)$ where γ^+ and γ^- are respectively the attracting and repelling points of an element γ of Γ. The result follows by the identification of E_i with eigenlines of $\rho(\gamma)$. Similarly, for the second identity we know that $\xi(\gamma^-, \gamma^+)$ is an attracting point of $\rho(\gamma)$. It follows that $\xi(\gamma^-, y) = \rho(\gamma)^n \xi(\gamma^-, \gamma^n y)$. Since

$$\lim_{n \to \infty} \gamma^n(y) = \gamma^+ ,$$

it follows that if $y \neq \gamma^-$

$$\xi(\gamma^-, y) = \lim_{n \to \infty} \rho(\gamma)^n \xi(\gamma^-, \gamma^n y) = \xi(\gamma^-, \gamma^+) .$$

This concludes the proof $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using proposition 2.1, we can recover the maps E_i using the limit curve ξ. Let us finally define the *isotropic limit curves* Θ and $\overline{\Theta}$ from $\partial_\infty \pi_1(\Sigma)$ to $\mathbf{L}$ as

$$\Theta := \bigoplus_{i=1}^{p} E_i \quad , \quad \overline{\Theta} := \bigoplus_{i=1}^{p} \overline{E}_i . \qquad\qquad (4)$$

3.2. *Hitchin representations in* $\mathsf{SO}(p, p-1)$. By [18], if ρ is a Hitchin representation in $\mathsf{SL}_{2p-1}(\mathbb{R})$, we have a decomposition of the associated bundle

$$\mathcal{V}_\rho = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_{2p-1} \, ,$$

such that the line bundles $\mathcal{V}_i$ are invariant by the flow and the flow contracts $\mathrm{Hom}(\mathcal{V}_i, \mathcal{V}_j)$ for $i > j$. If furthermore the representation is with values in $\mathsf{SO}(p, p-1)$, the flow preserves a quadratic form of signature $(p, p-1)$ and the restriction of this quadratic form to the trivial bundle $\mathcal{V}_p$ (equipped with a trivial action of the flow) is negative definite, while the restriction to the other line bundles $\mathcal{V}_i$ is zero.

Proposition 3.3. *Any Hitchin representation ρ with values in $\mathsf{SO}(p, p-1)$ is Borel Anosov for $\mathsf{SO}(p, p)$. Thus any small deformation of ρ is also Borel Anosov for $\mathsf{SO}(p, p)$.*

Proof. Taking $\mathcal{F}_\rho = \mathcal{V}_\rho \oplus \mathbf{R}$ – where $\mathbf{R}$ is the trivial line bundle with the opposite of the euclidean metric – equipped with the product metric, we obtain the decomposition as wished by taking for $i < p$, $\mathcal{E}_i = \mathcal{V}_i$ and $\overline{\mathcal{E}}_i = \mathcal{V}_{2p-i}$ and finally $\mathcal{E}_p$ and $\overline{\mathcal{E}}_p$ to be the two lightlike lines in $\mathcal{V}_p \oplus \mathbf{R}$.

The last statement follows from the stability of Anosov representations [18, 13]. $\qquad\square$

3.3. *The principal* $\mathsf{SL}_2(\mathbb{R})$-*representations.* In this section, we will give an explicit description of the map E_i in the case of Fuchsian representations. Let

$$A = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \tag{5}$$

Recall that the $(2p-1)$-*dimensional irreducible representation* of $\mathsf{SL}_2(\mathbb{R})$ preserves a quadratic form $\langle | \rangle$ of signature $(p, p-1)$. Moreover there exists a basis $\varepsilon_1, \ldots, \varepsilon_{2p-1}$ so that, writing $\overline{\varepsilon}_i := \varepsilon_{2p-i}$ and $\alpha_{k,m} := \langle A(\varepsilon_k) \mid \overline{\varepsilon}_m \rangle$, for all $z \neq 0$ and k, m in $\{1, \ldots, 2p-1\}$, we have

$$\langle \varepsilon_k \mid \overline{\varepsilon}_m \rangle = \delta_{k,m} \quad , \quad \Lambda(\varepsilon_m) = \lambda^{2p-2m} \varepsilon_m$$
$$\alpha_{k,m} \neq 0 \ \text{ if } m \geqslant k \quad , \quad \alpha_{k,m} = 0 \ \text{ if } m < k \, .$$

The *principal representation* J of $\mathsf{SL}_2(\mathbb{R})$ in $\mathsf{SO}(p, p)$ is described as follows: let V be a vector space on which $\mathsf{SL}_2(\mathbb{R})$ acts irreducibly preserving a quadratic form of signature $(p, p-1)$; let $(\varepsilon_1, \ldots \varepsilon_{2p-1})$ be the basis of V as above; let L be a line generated by a vector f. We take the trivial representation of $\mathsf{SL}_2(\mathbb{R})$ on L. We introduce now the base $(e_1, \ldots, e_p, \overline{e}_1, \ldots, \overline{e}_p)$ of $E := V \oplus L$ where

$$\forall i < p, \ e_i := \varepsilon_i \, , \ \overline{e}_i := \overline{\varepsilon}_i = \varepsilon_{2p-i} \qquad e_p := \varepsilon_p - f \, , \ \overline{e}_p = \varepsilon_p + f \, .$$

Then the group $\mathsf{SL}_2(\mathbb{R})$ preserves the quadratic form given in these coordinates by

$$\langle e_i \mid e_j \rangle = \langle \bar{e}_i \mid \bar{e}_j \rangle = 0 \, , \quad \langle e_i \mid \bar{e}_j \rangle = \delta_{i,j}$$

By convention, $(e_1, \ldots, e_p)$ generates a positive isotropic space.

3.4. *The Fuchsian representations in* $\mathsf{SO}(p, p-1)$ *and* $\mathsf{SO}(p,p)$. Let Σ be equipped with a hyperbolic structure and $\partial_\infty \pi_1(\Sigma)$ is identified with $\mathbf{Fl}P^1(\mathbb{R})$. A *Fuchsian representation* of Γ in $\mathsf{SO}(p,p)$ is a representation ρ of the form $J \circ \nu$ where ν is a discrete representation of Γ in $\mathsf{SL}_2(\mathbb{R})$. Let for $i \leqslant p$, the lines $E_i(x_0, y_0)$, respectively $\overline{E}_i(x_0, y_0)$ be generated by e_i, respectively $\bar{e}_i$, where $(x_0, y_0) = ([1:0], [0:1])$ are elements of $\partial_\infty \pi_1(\Sigma)$. Then, since the stabiliser of (x_0, y_0) is the group generated by Λ, and Λ preserves both $E_i(x_0, y_0)$ and $\overline{E}_i(x_0, y_0)$, we define coherently

$$E_i(Ax_0, Ay_0) := A(E_i(x_0, y_0)) \, , \quad \overline{E}_i(Ax_0, Ay_0) := A(\overline{E}_i(x_0, y_0)) \, .$$

Then we have that for all x and y,

$$E_i(Ax, Ay) = A(E_i(x, y)) \, , \quad \overline{E}_i(Ax, Ay) = A(\overline{E}_i(x, y)) \, .$$

One now immediately checks the proposition

Proposition 3.4. *If* Γ *is a Fuchsian group in* $\mathsf{PSL}(2, R)$, $J(\Gamma)$ *is an Anosov representation for* $\mathsf{SO}(p,p)$, *whose limit curve is* $\xi(x) = \mathrm{Flag}(E(x, y))$, *where* $E(x, y) := (E_i(x, y))_{i \in \{1 \ldots 2p-1\}}$ *– see notation* (1).

The following transversality property will play a crucial role in the sequel.

Proposition 3.5. [TRANSVERSALITY] *For all triple of pairwise distinct points* (x, y, z) *in* $\partial_\infty \pi_1(\Sigma)$

$$\Theta(z) \quad \pitchfork \quad \left(E_p(x, y) \oplus (E^\circ_{p-1}(x, y) \cap \overline{\Theta}(y)) \right) \, . \tag{6}$$

Proof. It is enough to consider the case $x := [0:1]$, $y := [1:0]$ and $z := [z:1] = A([0:1])$ where A is as in Equation (5). Let now

$$u = -b_p f + \sum_{m=1}^{p} b_m \varepsilon_m \in \Theta(x) \quad , \quad \text{so that} \quad A(u) \in \Theta(z) \, .$$

Assume furthermore that

$$A(u) \in F(x, y) \, , \quad \text{where} \quad F(x, y) := E_p(x, y) \oplus (E^\circ_{p-1}(x, y) \cap \overline{\Theta}(y)) \, .$$

By construction, $F^\circ(x, y)$ is generated by $\left\{ \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_{p-1}, \bar{\varepsilon}_{p+1}, \right\}$. Thus for $k \leqslant p+1$ and $k \neq p$, $\langle A(u) \mid \bar{\varepsilon}_k \rangle = 0$, in other words

$$0 = \sum_{m=1}^{k} \alpha_{m,k} b_m \, .$$

The matrix corresponding to this system is upper triangular with non zero coefficients, it follows that for all $1 \leqslant m \leqslant p$, we have $b_m = 0$. Thus $\Theta(z) \cap F(x, y) = \{0\}$. $\qquad\square$

Corollary 3.6. *Let ρ be a representation close to a Fuchsian representation. Then the transversality property* (6) *holds*

Proof. This follows from the continuity of limit curves as a dependence of the representation [13, 3] and the fact that Γ acts cocompactly on the space of triples of pairwise distinct points in $\partial_\infty \pi_1(\Sigma)$. $\qquad\square$

4. **The isotropic limit curves and the Smoothness Theorem.**

Theorem 4.1. [SMOOTHNESS THEOREM] *Let* $\mathsf{SO}(p,p)$ *be Anosov representation satisfying the Transversality Property* (6)*. Let Θ be the isotropic limit curve, defined in equation* (4)*. Then the image of Θ is a smooth curve M.*

Moreover, if (x, y) is a pair of distinct points in $\partial_\infty \pi_1(\Sigma)$, then, using the identification of proposition 2.2*, we have*

$$\mathsf{T}_{\Theta(x)}M = \Lambda^2(E^*_{p-1}(x, y) \oplus E^*_p(x)) \, .$$

4.1. *Proof of the Smoothness Theorem* 4.1. We will denote in general by V_x the fibre at $x \in X$ of a vector bundle $\mathcal{V}$ over a compact base X. Let $(\varphi_t)_{t \in \mathbb{R}}$ be a flow on X which lifts to a flow $(\Phi_t)_{t \in \mathbb{R}}$ of bundle automorphisms on V.

Definition 4.2. [PROXIMAL BUNDLE] *We say the lift $(\Phi_t)_{t \in \mathbb{R}}$ is* proximal *if there exists a continuous $(\Phi_t)_{t \in \mathbb{R}}$-invariant proximal decomposition $\mathcal{V} = \mathcal{Z} \oplus \mathcal{W}$ so that*

   (1) the subbundle $\mathcal{Z}$ has rank one,
   (2) the flow contracts the subbundle $\mathcal{Z}$ and the bundle $\mathrm{Hom}(\mathcal{Z}, \mathcal{W}) = \mathcal{Z}^* \otimes \mathcal{W}$.

The following lemma is crucial in the smoothness part of the result.

Lemma 4.3. [PROXIMALITY AND SMOOTHNESS] *Let Z_σ a never vanishing continuous section of $\mathcal{Z}$. Let W_σ be a continuous section of $\mathcal{W}$.*

Then there exist positive constants A and λ so that for all positive t

$$\|\Phi_t(W_\sigma)\| \leqslant A\|\Phi_t(Z_\sigma)\|^{\lambda+1} \, .$$

Proof. Let us choose an auxiliary metric on $\mathcal{Z}$ and $\mathcal{W}$, since the flow is contracting on $\mathcal{Z}$, and $\mathcal{Z}$ is a line bundle, we may reparametrise the flow so that for every u in $\mathcal{Z}$,

$$\|\Phi_t(u)\| = e^{-t}\|v\| \, . \tag{7}$$

Then the contraction property on $\mathcal{Z}^* \otimes \mathcal{W}$ tells that there is a positive constants B and λ so that for all w in $\mathrm{Hom}(\mathcal{Z}, \mathcal{W}) = \mathcal{Z}^* \otimes \mathcal{W}$

$$\|\Phi_t(w)\| \leqslant Be^{-\lambda t}\|w\| \, .$$

Thus for any (u, v) in $\mathcal{Z} \times \mathcal{W}$, where u is non zero, and all positive t,

$$\|\Phi_t(v)\| \leqslant Be^{-\lambda t} \cdot \frac{\|\Phi_t(u)\|}{\|u\|} \cdot \|v\| = B \frac{\|\Phi_t(u)\|^{1+\lambda}}{\|u\|^{1+\lambda}} \cdot \|v\| \,.$$

Let now

$$\begin{aligned} C &:= \inf\{\|Z_\sigma(x)\| \mid x \in X\} \,, \\ D &:= \sup\{\|W_\sigma(x)\| \mid x \in X\} \,, \end{aligned}$$

and recall that by hypothesis C is positive. Let now x be a point in X, then for any positive t,

$$\begin{aligned} \|\Phi_t(W_\sigma(x))\| &\leqslant B \left(\frac{\|\Phi_t(Z_\sigma(x))\|}{\|Z_\sigma(x)\|} \right)^{\lambda+1} \|W_\sigma(x)\| \\ &\leqslant BDC^{-(\lambda+1)} \|\Phi_t(Z_\sigma(x))\|^{\lambda+1} \,. \end{aligned}$$

Thus we obtain the lemma with $A = BDC^{-(\lambda+1)}$. $\qquad\square$

4.2. *Curves in bundles.* The isotropic limit curves Θ and $\overline{\Theta}$ give rise to two continuous, flow invariant maximal isotropic and transverse subbundles (also denoted Θ and $\overline{\Theta}$) of E_ρ. We see these subbundles as sections (also denoted Θ and $\overline{\Theta}$) of $\mathcal{L}_\rho$ the associated bundle over X to the Grassmannian of totally isotropic planes $\mathbf{L}$ in E

Let us choose an orientation on Σ and thus a complex structure associated to the hyperbolic structure, as well as an orientation on $\partial_\infty \pi_1(\Sigma)$.

From hyperbolic geometry, we have a Γ-equivariant map h from the unit bundle of $\mathbf{H}^2$ to $\partial_\infty \pi_1(\Sigma)$, which associates to a unit vector u, the end point of the geodesic given by Ju, where J is the complex structure on Σ associated to the hyperbolic metric.

Thus we obtain a section σ of $\mathcal{L}_\rho$ given by

$$\sigma(u) := \Theta(h(u)) \,.$$

For a representation close to being Fuchsian, since $\Theta(z)$ is transverse to $\overline{\Theta}(w)$ for $z \neq w$ by the Anosov property (*cf* definition 3.1 (1)), we will consider σ as a section of the vector bundle

$$\mathcal{T} := \mathsf{T}_\Theta \mathbf{L} \subset \mathrm{Hom}(\Theta, \overline{\Theta}) \,.$$

that we freely identify with $\Lambda^2(\Theta^*)$, using proposition 2.2 by an identification that respects the lifts of the flow. Let us prove:

Proposition 4.4. *The decomposition $\mathcal{T} = \mathcal{Z} \oplus \mathcal{W}$ is a proximal vector bundle decomposition where*

$$\mathcal{Z} := \Lambda^2\left(E_{p-1}^* \oplus E_p^*\right) \,, \qquad \mathcal{W} := \{\omega \in \Lambda^2(\Theta^*) \mid \omega|_{E_p \oplus E_{p-1}} = 0\} \,. \tag{8}$$

Proof. By the Anosov property, the flow contracts

$$\Lambda^2(\Theta^*) = \bigoplus_{p \geqslant i > j} \Lambda^2(E_i^* \oplus E_j^*) \,,$$

and furthermore contracts less on $\mathcal{Z} := \Lambda^2(E_p^* \oplus E_{p-1}^*)$ than on $\mathcal{U}_1 := \Lambda^2(E_i^* \oplus E_j^*)$ when $i < j$ and $j > p$: more precisely there are positive constants λ and K so that for all positive t and all (u, v) in $\mathcal{Z} \oplus \mathcal{U}_1$, then

$$\frac{\|\Phi_t(v)\|}{\|v\|} \leqslant \frac{\|\Phi_t(u)\|}{\|u\|} K e^{-\lambda t} \,.$$

The result follows.   □

Lemma 4.5. *For all u in X, $\sigma(u)$ does no belong to $\mathcal{W}_u$.*

Proof. In the identification $\Lambda^2(\Theta) = \mathsf{TL} \subset \mathrm{Hom}(\Theta, \overline{\Theta})$, $\mathcal{W}$ is a subset of

$$\mathcal{W}_0 := \{ f \mid \forall (w, v) \in E_{p-1} \times E_p \,.\ q(w, f(v)) = q(v, f(w)) = 0 \}$$

But if f belongs to $\mathcal{W}_0$, then $f(E_p)$ is included in $E_{p-1}^\circ \cap \overline{\Theta}$. Thus the graph of f has an intersection of positive dimension with $E_p \oplus (E_{p-1}^\circ \cap \overline{\Theta})$. It follows from the third statement of proposition 3.6 that $\sigma(u)$ does not belong $\mathcal{W}$ for all u in X.   □

Let then Z_σ in $\mathcal{Z}$, and W_σ in $\mathcal{W}$, so that

$$\sigma = W_\sigma + Z_\sigma \,.$$

Then, by the previous lemma, Z_σ is a never vanishing section of $\mathcal{Z}$.

Let us choose some auxiliary Riemannian metric on E and denote η the section of $\mathcal{Z}$ of norm 1 which is so that for all t,

$$\eta = \frac{\Phi_t(Z_\sigma(\varphi_{-t} u))}{\|\Phi_t(Z_\sigma(\varphi_{-t} u))\|} \,.$$

Observe that η gives the orientation of the subbundle $\mathcal{Z}$.

As a corollary of our proximal and smoothness lemma 4.3 we now get,

Corollary 4.6. *There exist positive constants A and λ, such that for all positive t,*

$$\|\Phi_t(W_\sigma)\| \leqslant A \|\Phi_t(Z_\sigma)\|^{\lambda+1} \,.$$

And in particular

$$\lim_{t \to \infty} \left(\frac{\Phi_t(\sigma(\varphi_{-t} u))}{\|\Phi_t(\sigma(\varphi_{-t} u))\|} \right) = \eta \,.$$

We now explain how this corollary implies the Smoothness Theorem 4.1.

Proof of Theorem 4.1. Let us fix two points x and y in $\partial_\infty \Sigma$ and let γ the geodesic joining x to y in the universal cover of Σ, lifting a geodesic γ_0 in Σ.

Then the lift of the decomposition $\mathcal{T} = \mathcal{Z} \oplus \mathcal{W}$ along γ_0, gives a trivial bundle decomposition (for the induced connection) also denoted $\mathcal{T} = \mathcal{Z} \oplus \mathcal{W}$ along γ, where

$$\mathcal{Z} = Z \times \gamma \,, \quad \mathcal{W} = W \times \gamma \,, \quad \mathcal{T} = T \times \gamma \,.$$

Observe that Z, T and W only depends on x and y and we identify T as $\mathsf{T}_{\Theta(x)}\mathbf{L}$ seen as a subspace of $\mathrm{Hom}(\Theta(x), \overline{\Theta}(y))$. In that identification

$$Z = Z(x, y) = \Lambda^2\left(E_{p-1}(x, y)^* \oplus E_p^*(x, y)\right) \,.$$

Let us now fix a point u in γ and a Riemannian metric on the vector space $\mathrm{Hom}(\Theta(x), \overline{\Theta}(y))$, which induces Riemannian metrics on T, Z and W. Observe now that σ lifts as map σ_0 from γ to T, and that the lift of the flow $(\Phi_t)_{t \in \mathbb{R}}$ on the bundle $\mathcal{T}$ acts trivially on the first factor. Thus, corollary 4.6 translates as

$$\lim_{t \to \infty} \left(\frac{\sigma_0(\varphi_{-t}(u))}{\|\sigma_0(\varphi_{-t}(u))\|}\right) = \eta_0 \,,$$

where η_0 is the vector of norm 1 of Z giving the orientation. Interpreting $\mathrm{Hom}(\Theta(x), \overline{\Theta}(y))$ as a subset of $\mathbf{L}$, we then get by the definition of σ_0

$$\lim_{t \to \infty} \frac{\Theta(h(\varphi_{-t}(u))) - \Theta(x)}{\|\Theta(h(\varphi_{-t}(u)) - \Theta(x)\|} = \eta_0 \,.$$

Finally, since the map which associates $h(\varphi_t(u))$ to t is a homeomorphism from $]-\infty, +\infty[$ to the oriented arc from x to y in $\partial_\infty \Sigma$, the previous assertion is equivalent to

$$\lim_{z \to x^+} \frac{\Theta(x) - \Theta(z)}{\|\Theta(x) - \Theta(z)\|} = \eta_0 \,.$$

Taking the opposite orientation on Σ we obtain symmetrically

$$\lim_{z \to x^-} \frac{\Theta(x) - \Theta(z)}{\|\Theta(x) - \Theta(z)\|} = -\eta_0 \,.$$

Hence $\Theta(\partial_\infty \pi_1(\Sigma))$ is a C^1 curve whose tangent space at x is Z. $\qquad \square$

5. **The last root flow and the Entropy Theorem.** The next proposition shows that the Margulis invariants are the variation of the length of the closed orbit of some flow.

Proposition 5.1. *For ρ with values in $\mathsf{SO}(p, p)$ close to a Hitchin representation in $\mathsf{SO}(p, p-1)$, there exists a reparametrisation $(\psi_t)_{t \in \mathbb{R}}$ of $(\varphi_t)_{t \in \mathbb{R}}$, called the* last root flow, *so that the length of the closed orbit of ψ associated to γ is $\log \lambda_p(\rho(\gamma)) + \log \lambda_{p-1}(\rho(\gamma))$.*

Proof. Any such ρ is Borel Anosov by proposition 3.3. Thus, the real line bundle $\mathcal{Z} = \Lambda^2(E_{p-1}^* \oplus E_{p-1}^*)$ is contracted by the flow and its *contraction spectrum* associates to the closed orbit γ the number $\log \lambda_p(\rho(\gamma)) + \log \lambda_{p-1}(\rho(\gamma))$. Then the result follows by [4, proposition 2.4]. □

The entropy theorem is now properly stated as

Theorem 5.2. [ENTROPY THEOREM] *For ρ close enough to a Hitchin representation in* $\mathsf{SO}(p, p-1)$, *the entropy of the last root flow is equal to 1.*

This theorem is also a consequence to [25], which also uses a fundamental idea due to Potrie and Sambarino [24].

Proof. We follow closely Potrie and Sambarino [24], to obtain a proof of the Entropy Theorem 5.2. We observe that the if γ_+ and γ_- are respectively the attracting and repelling fixed points of γ on $\partial_\infty \pi_1(\Sigma)$, then $\Theta(\gamma_+)$ is a fixed point of $\rho(\gamma)$ in the smooth curve $M = \Theta(\partial_\infty \pi_1(\Sigma))$ in **L**. Moreover, since the tangent space to M at $\Theta(\gamma_+)$ is identified with $\Lambda^2(E_p^*(\gamma^+, \gamma^-) \oplus E_{p-1}^*(\gamma^+))$, the derivative of $\rho(\gamma)$ at $\Theta(\gamma_+)$ on M is $\lambda_p(\gamma) \cdot \lambda_{p-1}(\gamma)$. The same discussion as in Potrie–Sambarino using Sinai–Ruelle–Bowen measures gives us the result in the neighbourhood of the Fuchsian representation by Corollary 3.6 and Theorem 4.1 since the isotropic limit curve is C^1. Finally, as in [24], the analyticity of the entropy obtained in [3] implies that the entropy is constant and equal to 1 on the neighbo rhood of the set of Hitchin representations in $\mathsf{SO}(p, p-1)$. □

It seems likely that all representations in the Hitchin component for $\mathsf{SO}(p, p)$ are Anosov, in which case the previous theorem applies to the whole Hitchin component.

6. **Entropy and the Affine Action Theorem.**

6.1. *Affine group and quadratic forms.* Let us consider a representation ρ of a surface group in the affine group of the vector space F whose linear part ρ^0 is a Hitchin representation taking values $\mathsf{SO}(p, p-1)$. We describe the translation part as an element ω in $H^1_{\rho^0}(F)$, defined by the cocycle

$$\gamma \mapsto \omega_\gamma := \rho(\gamma)(0) .$$

Let L be a one-dimensional vector space generated by a vector f. Let us consider the quadratic form on $E \oplus L$, given by $\langle u + | u + xf \rangle = Q(u, u) - x^2$ of signature (p, p).

The corresponding embedding of $\mathsf{SO}(p, p-1)$ into $\mathsf{SO}(p, p)$ is so that we have the $\mathsf{SO}(p, p-1)$-invariant decomposition of the Lie algebra

$$\mathfrak{so}(p, p) = \mathfrak{so}(p, p-1) \oplus F . \tag{9}$$

Accordingly, let us consider $\text{Rep}(\pi_1(\Sigma), \mathsf{SO}(p, p-1))$ as a subset of $\text{Rep}(\pi_1(\Sigma), \mathsf{SO}(p, p))$. Recall that by Hitchin [14], the connected component of $\text{Rep}(\pi_1(\Sigma), \mathsf{SO}(p, p))$ containing the fuchsian representations in $\mathsf{SO}(p, p-1)$ is smooth and its tangent space identifies with $H^1_{\text{ad}(\rho)}(\mathfrak{so}(p, p))$. Thus, the decomposition (9) gives a linear map from $H^1_\rho(F)$ to $\mathsf{T}_{\rho^0} \text{Rep}(\pi_1(\Sigma), \mathsf{SO}(p, p))$.

We will describe this identification more explicitly in the next proposition. Let us first represent elements $\dot\rho$ in $\mathsf{T}_{\rho^0} \text{Rep}(\pi_1(\Sigma), \mathsf{SO}(p, p))$ by cohomology classes of cocycles $\dot\rho : \gamma \mapsto \dot\rho_\gamma$. Let finally $\mathcal{H}$ be the subset of $\mathsf{T}_{\rho^0} \text{Rep}(\pi_1(\Sigma), \mathsf{SO}(p, p))$ defined by

$$\mathcal{H} \ := \ \{\dot\rho \mid \forall u, v \in E, \ \ \forall \gamma \in \Gamma \ \ \langle \dot\rho_\gamma(u) \mid v \rangle = 0\} \ .$$

Then we have

Proposition 6.1. [INTERPRETATION] *The map* $\dot\rho \mapsto \omega_\gamma$, *where* ω_γ *is defined by*

$$\forall v \in E, \ \ Q(\omega_\gamma, v) = \langle \dot\rho_\gamma(f) \mid \rho(\gamma)(v) \rangle$$

is an isomorphism between $\mathcal{H}$ *and* $H^1_{\rho^0}(F)$.

6.2. *Labourie–Margulis diffusion.* Let X be the unit tangent bundle of a hyperbolic surface Σ. Let ρ be a representation of $\pi_1(\Sigma)$ in the affine group. Let ρ^0 be the linear part of ρ. We assume that ρ^0 is a Hitchin representation in $\mathsf{SO}(p, p-1)$. Let ω be the affine part of ρ, that we see as a (closed) form in $\Omega^1(X, \mathcal{V}_0)$, where $\mathcal{V}_0$ is the flat bundle on X associated to ρ^0. By section 3.2, we have the flow invariant decomposition:

$$\mathcal{V}_0 = \sum_{i=1}^{2p-1} \mathcal{V}_i \ .$$

Let ε_p be a section of norm 1 of the spacelike line bundle E_p. Let us choose a parametrisation of the geodesic flow, with generator ζ. Let μ be a measure invariant by the geodesic flow. We define as in [17, 11] the *Labourie–Margulis diffusion*

$$M(\mu) := \int_{\mathsf{U}S} Q(\varepsilon_p, \omega(\zeta)) \, \mathrm{d}\mu \ . \tag{10}$$

Let $(\rho_t)_{t \in \mathbb{R}}$ be a family of representations of Γ in $\mathsf{SO}(p, p)$ associated to ρ, according to our Interpretation Proposition 6.1, so that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho_t = \omega \ , \ \ \rho^0 = \rho_0 \ .$$

For t close to zero, ρ_t is a Borel Anosov representation in $\mathsf{SO}(p, p)$ by proposition 3.3. We can decompose the associated bundle as in

Definition 3.1 into

$$\mathcal{E}_{\rho_t} = \bigoplus_{i=1}^{p} \mathcal{E}_i^t \oplus \bigoplus_{i=1}^{p} \overline{\mathcal{E}}_i^t .$$

This decomposition is given by the limit curves. Since they depend analytically on the representation [3, Theorem 6.1], we may choose an identification of $\mathcal{E}_{\rho_t}$ with $\mathcal{V}_0 \oplus L$, where L is the trivial bundle such that furthermore

(1) the quadratic form is constant,
(2) the bundles $\mathcal{E}_i^t$ and $\overline{\mathcal{E}}_i^t$ are constant and thus denoted $\mathcal{E}_i$ and $\overline{\mathcal{E}}_i$
(3) Finally $\mathcal{E}_i = \mathcal{V}_i$, $\overline{\mathcal{E}}_i = \mathcal{V}_{2p-i+1}$, for $i < p$, $\mathcal{E}_p$ and $\overline{\mathcal{E}}_p$ are the lightlike lines in $\mathcal{V}_p \oplus L$.

Let δ_γ be the geodesic current supported on a closed orbit given by a non trivial element γ of $\pi_1(\Sigma)$. The next lemma is a generalisation of [12, Lemma 2].

Lemma 6.2. *The variations of the eigenvalues are given as follows:*

$$\left.\frac{d}{dt}\right|_{t=0} \lambda_p(\rho_t(\gamma)) = \frac{1}{2}M(\delta_\gamma) \quad, \quad \text{and for all } i < p, \quad \left.\frac{d}{dt}\right|_{t=0} \lambda_i(\rho_t(\gamma)) = 0 .$$

Proof. We can obtain this lemma as a direct application of [19, Lemma 4.1.1], we however reproduce the easy proof in this context. We choose bases e_i and $\bar{e}_i$ of $\mathcal{E}_i$ and $\overline{\mathcal{E}}_i$ respectively, so that

$$e_p = \frac{1}{2}(\varepsilon_p + f) , \ \bar{e}_p = \frac{1}{2}(\varepsilon_p - f) ,$$

where ε_p is a basis of E_p of norm 1 and f a basis of the trivial bundle L of norm -1. Writing $\dot{a} = \left.\frac{d}{dt}\right|_{t=0} a(t)$, then is equal to

$$
\begin{aligned}
M(\delta_\gamma) &= Q(\omega_\gamma, \varepsilon_p) = Q(\dot{\rho}_\gamma(f) \, \varepsilon_p) \\
&= \langle e_p + \bar{e}_p \mid \dot{\rho}_\gamma(e_p - \bar{e}_p) \rangle = 2 \langle \dot{\rho}_\gamma(e_p) \mid \bar{e}_p \rangle \\
&= 2\dot{\lambda}_p .
\end{aligned}
$$

Similarly, by proposition 3.6, $0 = \langle \dot{\rho}_\gamma(\varepsilon_k) \mid \varepsilon_{2p-k} \rangle$ for $k < p$, thus $\dot{\lambda}_k = 0$. $\qquad\square$

Recall that all representations ρ_s are associated to a last root flow $(\psi_t^s)_{t \in \mathbb{R}}$ by proposition 5.1. Such a last root flow is a repamaretrisation of the geodesic flow $(\phi_t)_{t \in \mathbb{R}}$, hence we can write if ζ^s is the generator of ψ^s, $\zeta^s = g^s \zeta^0$, where g^s is a function on X; that is called the *reparametrisation of* $(\psi_t^0)_{t \in \mathbb{R}}$ *by* $(\psi_t^0)_{t \in \mathbb{R}}$. Let us define the function $\dot{g}$ on X

$$\dot{g} := \left.\frac{d}{ds}\right|_{s=0} g^s .$$

Corollary 6.3. *For any measure μ, if g is the variation of the reparametrisation of the last root flow*

$$2 \int_{UX} \dot{g} \, d\mu \;\; = \;\; M(\mu) \, . \tag{11}$$

Proof. Let δ_γ be the geodesic current supported on a closed orbit. Then using the definition for the first equality, the fact that $\lambda_p = 1$ for an $\mathsf{SO}(p, p-1)$ representation and lemma 6.2, we have

$$\int_{UX} \dot{g} \, d\delta_\gamma = \frac{\dot{\lambda}_{p-1}}{\lambda_{p-1}} + \frac{\dot{\lambda}_p}{\lambda_p} = \dot{\lambda}_p = \frac{1}{2} M(\delta_\gamma) \, .$$

Thus the equation (11) holds for all currents supported on closed orbits, hence for all linear combinations of such by linearity, hence for all measures by density of the currents supported on closed geodesics and continuity of the diffusion. $\qquad\square$

6.3. *Abramov lemma.* We will use the thermodynamic formalism and refer to [3] for a general discussion and references. We first recall that given a metric Anosov flow $(\psi_t)_{t \in \mathbb{R}}$ on X, and a flow invariant probability measure m on X we can define its *entropy $h(m)$*. Moreover there is a unique measure μ_{eq} called the *Bowen–Margulis measure* so that

$$h(\mu_{eq}) = \sup \left\{ h(m) \mid m \text{ invariant by } (\psi_t)_{t \in \mathbb{R}} \right\} .$$

The number $h(\mu_{eq})$ is called the *topological entropy*.

Let now $\left(\psi_t^s\right)_{t \in \mathbb{R}}$ be a family of metric Anosov flows on a space X depending on some parameter s in $]-1, 1[$. Let ℓ_γ^s be the length of every closed orbit γ for ψ^s. Let f_s be a family of continuous functions on X so that

$$\ell_\gamma^s = \int_0^{\ell_\gamma^0} f_s \circ \psi_u^0(x) \, du \, ,$$

where x is a point in γ — see [3, Paragraph 3.1] for details. If m is an invariant measure for $\left(\psi_t^0\right)_{t \in \mathbb{R}}$, then

$$m^s := \frac{1}{\int_X f_s m} f_s m \, , \tag{12}$$

is an invariant measure for $\left(\psi_t^s\right)_{t \in \mathbb{R}}$. The Abramov formula [26, Lemma 2.4] is

$$h(m^s) = \frac{1}{\int_X f_s m} h(m) \, . \tag{13}$$

Then as a consequence.

Lemma 6.4. *Assume that $f^s(m)$ is C^1 as a function in s and its derivative is bounded as a function of m. Let h_s be the topological entropy of ψ^s. Let μ_{eq} the Bowen–Margulis measure of $\left(\psi_t^0\right)_{t \in \mathbb{R}}$. Assume that h_s is constant and non zero then*

$$0 = \int_X \left.\frac{\mathrm{d} f_s}{\mathrm{d} s}\right|_{s=0} \mathrm{d}\mu_{eq} . \tag{14}$$

Proof. Let us consider μ_{eq}^s as in formula (12). By Abramov formula for the first equality and the definition of topological entropy for the second

$$\frac{h_0}{\int_X f_s \, \mathrm{d}\mu_{eq}} = h(\mu_{eq}^s) \leqslant h_s = h_0 .$$

Thus for all s,

$$\int_X f_s \mathrm{d}\mu_{eq} \geqslant 1 = \int_X \mathrm{d}\mu_{eq} = \int_X f_0 \mathrm{d}\mu_{eq} .$$

The result follows   □

6.4. *Proof of the Affine action Theorem 1.1.* Let σ be a Hitchin representation in the affine group whose linear part ρ is Hitchin in $\mathsf{SO}(p, p-1)$. We interpret σ as a a family of representation $(\rho_s)_{s \in [0,1]}$ in $\mathsf{SO}(p, p)$ with $\rho_0 = \rho$ as in paragraph 6.1 and proposition 6.1.

Let ψ^s be the last root flow of ρ_s and g^s be a family of reparametrisations of ψ^0 giving rise to ψ_s. Let $\dot{g} = \left.\frac{\mathrm{d} g^s}{\mathrm{d} s}\right|_{s=0}$. Since the entropy of ψ^s is constant by the Entropy Theorem 5.2, we have by lemma 6.4 that $\int_X \dot{g} \, \mathrm{d}\mu = 0$ where μ is the Bowen–Margulis measure of the last root flow of ρ_0. Thus by corollary 6.3, $M(\mu) = 0$.

Now by [10, Theorem 7.1 and Definition 4.4] if there is a measure that annihilates the Labourie–Margulis diffusion (defined in equation (10)), then the action on the affine group is not proper. This concludes the proof of Danciger and Zhang's Theorem 1.1 .

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