GHOST POLYGONS, POISSON BRACKET AND CONVEXITY

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Abstract. The moduli space of Anosov representations of a surface group in a semisimple group – an open set in the character variety – admits many more natural functions than the regular functions. We will study in particular length functions and correlation functions. Our main result is a formula that computes the Poisson bracket of those functions using some combinatorial devices called *ghost polygons* and *ghost bracket* encoded in a formal algebra called the *ghost algebra* related in some cases to the swapping algebra introduced by the second author. As a consequence of our main theorem, we show that the set of those functions – length and correlation – is stable under the Poisson bracket. We give two applications: firstly in the presence of positivity we prove the convexity of length functions, generalizing a result of Kerckhoff in Teichmüller space, secondly we exhibit subalgebras of commuting functions associated to geodesic laminations. An important tool is the study of *uniformly hyperbolic bundles* which is a generalization of Anosov representations beyond periodicity.

Introduction

The character variety of a discrete group Γ in a Lie group G admits a natural class of functions: the algebra of regular functions generated as a polynomial algebra by trace functions or *characters*. When Γ is a surface group, the character variety becomes equipped with a symplectic form generalizing the Poincaré intersection form – called the Atiyah–Bott–Goldman symplectic form [1, 11, 18] – and a fundamental theorem of Goldman [12] shows that the algebra of regular functions is stable under the Poisson bracket and more precisely that the bracket of two characters is expressed using a beautiful combinatorial structure on the ring generated by characters. The Poisson bracket associated to a surface group has been heavily studied in [12], [27]; and in the context of Hitchin representations the link between the symplectic structure, coordinates and cluster algebras discovered by Fock–Goncharov in [10] (see also Bonahon–Dreyer [5]), has generated a lot of attention: for instance see Sun–Wienhard–Zhang [25], Nie [23], Sun–Zhang [26], Choi–Jung–Kim [9] and Sun [24] for more results, and also relations with the swapping algebra [19].

On the other hand the deformation space of Anosov representations admits many other natural functions besides regular functions. *Length functions*, associated to any geodesic current, studied by Bonahon [3] in the context of Teichmüller theory, play a prominent role for Anosov representations for instance in [6] and [8]. Another class are the *correlation functions*, defined in [19] and [7].

These functions are defined as follows. For the sake of simplicity, we focus in this introduction on the case of a projective Anosov representation ρ of a hyperbolic group Γ in $\mathsf{SL}(V)$. In that case, for every non trival element g in Γ, as part of the Anosov property, $\rho(g)$ has an attractive fixed point $\xi(g)$ in $\mathbf{P}(V)$ and an attractive fixed point $\xi^*(g)$ in $\mathbf{P}(V^*)$, therefore associating to g the rank 1-projector $\mathrm{p}_\rho(g)$ whose image is $\xi(g)$ and kernel is $\xi^*(g)$. The projector only depends on the endpoints of g in $\partial_\infty\Gamma$. The assignment $g \mapsto \mathrm{p}_\rho(g)$ can then – thanks to the Anosov property again – be extended to any *geodesic* g of Γ, that is, a pair of distinct points in $\partial_\infty\Gamma$. The correlation function T_G associated to a *configuration of n-geodesics* – that is, an n-tuple $G = (g_1, \ldots, g_n)$ of geodesics up to cyclic transformation – is then

$$\mathsf{T}_G : \rho \mapsto \mathsf{T}_G(\rho) \coloneqq \mathrm{Tr}(\mathrm{p}_\rho(g_n) \ldots \mathrm{p}_\rho(g_1)) .$$

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In Teichmüller theory, the correlation function of two geodesics is the cross-ratio of the endpoints. Generally, the correlation functions of geodesics in Teichmüller theory is a rational function of cross-ratios. This is no longer the case in the higher rank.

For instance if C is a geodesic triangle given by the three oriented geodesics (g_1, g_2, g_3), the map

$$\mathsf{T}^*_C : \rho \mapsto \mathsf{T}^*_C(\rho) := \mathrm{Tr}(\mathrm{p}_\rho(g_1)\mathrm{p}_\rho(g_2)\mathrm{p}_\rho(g_3)) ,$$

is related to Goncharov triple ratio on the real projective plane.

For a geodesic current μ, its length function L_μ is defined by an averaging process – see equation (77). One can also average correlation functions: say a Γ-invariant measure μ on the set $\mathcal{C}^n$ of generic n-tuples of geodesics is an *integrable cyclic current* if it is invariant under cyclic transformations and satisfies some integrability conditions – see section 6 for precise definitions. Then the *μ-correlation function* or μ-averaged correlation function is

$$\mathsf{T}_\mu : \rho \mapsto \int_{\mathcal{C}^n/\Gamma} \mathsf{T}_G(\rho)\, \mathrm{d}\mu .$$

The corresponding functions are analytic (see [8]), but rarely algebraic.

In the case when Γ is a surface group, the algebra of functions on the deformation space of Anosov representations admits a Poisson bracket coming from the Atiyah–Bott–Goldman symplectic form.

To uniformize our notation, we write T^k_μ for T_μ when μ is supported on $\mathcal{C}^k$ and $\mathsf{T}^1_\nu = \mathsf{L}_\nu$ for the length function of a geodesic current ν. Then, one of the main results of this article, Theorem 8.3.1, gives as a corollary

Theorem A (Poisson stability)**.** *The space of length functions and correlation functions is stable under the Poisson bracket. More precisely there exists a Lie bracket on the polynomial algebra formally generated by tuples of geodesics* $(G, H) \mapsto [G, H]$ *so that*

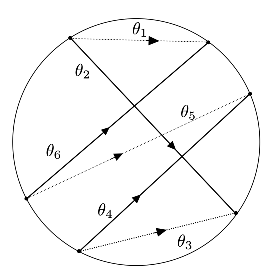
$$\{\mathsf{T}^k_\mu, \mathsf{T}^p_\nu\} = \int_{\mathcal{C}^{n+m}/\Gamma} \mathsf{T}_{[G,H]}(\rho)\ \ \mathrm{d}\mu(G)\mathrm{d}\nu(H) .$$

The full result, Theorem 8.3.1 and its Corollary 8.3.2, allows us to recursively use this formula and indeed obtain stability.

In Theorem 8.9.1 we compute explicitly what is the Hamiltonian vector field of the correlation functions. For instance in Teichmüller theory, this allows us to compute the higher derivatives of a length function along twist orbits by a combinatorial formula involving cross-ratios.

The bracket $(G, H) \mapsto [G, H]$ – that we call the *ghost bracket* – is combinatorially constructed. In this introduction, we explain the ghost bracket in the simple projective case and refer to section 3.1 for more details. Recall first that an *ideal polygon* – not necessarily embedded – is a sequence $(h_1, \ldots, h_n)$ of geodesics in $\mathbf{H}^2$ such that the endpoint of h_i is the starting point of h_{i+1}. Let then G be the configuration of n geodesics $(g_1, \ldots, g_n)$, with the endpoint of g_i not equal the starting point of g_{i+1}. The associated *ghost polygon* is given by the uniquely defined configuration $(\theta_1, \ldots \theta_{2n})$ of geodesics (see figure (1)) such that

(i) Configuration of geodesics



(ii) Ghost Polygon

Figure 1. Two ways to see a cyclically ordered tuple of geodesics

(1) $(\bar{\theta}_1, \theta_2, \bar{\theta}_3 \dots, \bar{\theta}_{2n-1}, \theta_{2n})$ is an ideal polygon,
(2) for all i, $\theta_{2i} = g_i$ and is called a *visible edge*, while θ_{2i+1} is called a *ghost edge*.

We now denote by $\lceil g, h \rceil$ the configuration of two geodesics g and h, $\varepsilon(g, h)$ their algebraic intersection, and $\bar{g}$ is the geodesic g with the opposite orientation. Then if $(\theta_i, \dots, \theta_{2n})$ and $(\zeta_i, \dots, \zeta_{2p})$ are the two ghost polygons associated to the configurations G and H, we define the *projective ghost bracket* of G and H as

$$[G, H] := G \cdot H \cdot \left(\sum_{i,j} (-1)^{i+j} \varepsilon(\zeta_j, \theta_i) \lceil \zeta_j, \theta_i \rceil \right), \tag{1}$$

which we consider as an element of the polynomial algebra formally generated by configurations of geodesics. We have similar formulas when G or H are geodesics, thus generalizing Wolpert's cosine formula [29]. In the case presented in the introduction – the study of projective Anosov representations – the ghost bracket is actually a Poisson bracket and is easily expressed in paragraph 5.3 using the swapping bracket introduced by the second author in [19]. Formula (1) is very explicit and the Poisson Stability Theorem A now becomes an efficient tool to compute recursively brackets of averaged correlations functions and length functions.

In this spirit, we give two applications of this stability theorem. Following Martone–Zhang [22], say a projective Anosov representation ρ *admits a positive cross ratio* if $0 < \mathrm{Tr}(\mathrm{p}_\rho(g)\mathrm{p}_\rho(h)) < 1$ for any two intersecting geodesics g and h. Examples come from Teichmüller spaces and Hitchin representations [17, 22]. More generally positive representations are associated to positive cross ratios [2]. Our first application is a generalisation of the convexity theorem of Kerckhoff [14] and was the initial reason for our investigation:

Theorem B (Convexity Theorem)**.** *Let μ be the geodesic current associated to a measured geodesic lamination, L_μ the associated length function. Let ρ be a projective Anosov representation which admits a positive cross ratio, then for any geodesic current ν,*

$$\{\mathsf{L}_\mu, \{\mathsf{L}_\mu, \mathsf{L}_\nu\}\} \geqslant 0 .$$

Furthermore the inequality is strict if and only if $i(\mu, \nu) \neq 0$.

Recall that in a symplectic manifold $\{f, \{f, g\}\} \geqslant 0$ is equivalent to the fact that g is convex along the Hamiltonian curves of f. This theorem involves a generalization of Wolpert's sine formula [29].

Our second, and less surprising, application is to construct commuting subalgebras in the Poisson algebra of correlation functions. Let $\mathcal{L}$ be a geodesic lamination whose complement is a union of geodesic triangles C_i. To each such triangle, we call the associated correlation function $\mathsf{T}^*_{C_i}$ an *associated triangle function*. The *subalgebra associated to the lamination* is the subalgebra generated by triangle functions and length functions for geodesic currents supported on $\mathcal{L}$.

Theorem C (Commuting subalgebra)**.** *For any geodesic lamination whose complement is a union of geodesic triangles, the associated subalgebra is commutative with respect to the Poisson bracket.*

In the context of coordinate functions and Hitchin representations, note that it is a well known fact that geodesic laminations are associated to commuting subalgebras [10, 6, 26], but observe that our results are more general: they are about any type of Anosov representations.

We now give a brief outline of the paper and of some of our constructions of independent interest, namely the construction of a theory of a (non linear) intersection and integration for ghost polygons, as well as that of uniformly hyperbolic bundles.

This paper has been structured in several parts and sections, all of them preceded by an introduction describing the content of these sections or paragraphs. We concentrate in this introduction on the projective case.

In the first and preliminary section, we recall classical constructions; first in the hyperbolic plane, we explain the dual form of a geodesic ω_g and the intersection $\varepsilon(g, h)$ of two geodesics g and h related by the following formula that we give to explain our later motivations:

$$\varepsilon(g,h) = \int_g \omega_h = -\int_h \omega_g = \int_{\mathbf{H}^2} \omega_g \wedge \omega_h \ . \tag{2}$$

We also recall the Atiyah–Bott–Goldman symplectic form.

Our first part **on the hyperbolic plane** deals with a generalization of the above formula. First we need to introduce "Anosov representations without a group", we call the corresponding notion a *uniformly hyperbolic bundle*. This is the background of the whole project. Given such a uniformly hyperbolic bundle ρ and a configuration G of geodesics – that we call a *ghost polygon* – we are then able to compute the derivatives of the correlation function T_G, for a variation $\dot{\nabla}$ of the uniformly hyperbolic bundle at ρ:

$$\mathrm{d}\mathsf{T}_G\left(\dot{\nabla}\right) = \oint_{\rho(G)} \dot{\nabla} \ ,$$

where the righthand side is a procedure called *ghost integration* which involves computing solutions of the cohomological equation of dynamical systems. Motivated by this computation, and fixing a uniformly hyperbolic bundle, we introduce the ghost intersection $\mathsf{I}_\rho(G,H)$ of two ghost polygons G and H and the ghost dual form $\Omega_{\rho(G)}$ – with values in some endomorphism bundle – of a ghost polygon so that

$$\mathsf{I}_\rho(G,H) = \oint_{\rho(G)} \Omega_H = -\oint_{\rho(H)} \Omega_G = \int_{\mathbf{H}^2} \mathrm{Tr}\left(\Omega_H \wedge \Omega_G\right) \ ,$$

a formalism reminiscent of formulae (2). All this is better encoded by introducing a *ghost bracket* on a *ghost algebra* – the polynomial algebra generated by ghost polygons – and extending correlation functions to the ghost algebra to get

$$\mathsf{T}_{[G,H]}(\rho) = \mathsf{I}_\rho(G,H) \ .$$

The constructions outlined above are the analogues of classical constructions (integration along a path, intersection of geodesics) in differential topology described in section 1, but in a non abelian setting.

The second part **on closed surfaces** then moves to averaging correlation functions and length functions, by using currents (for geodesics) and *cyclic currents* for ghost polygons. This is the part where we prove Theorem A. We prove this by carefully exchanging some integrals and using the formulae that we have obtained for the derivatives of correlation functions.

In the third part, we prove the **applications**: Theorems B and C. In the final part, **addendum**, we establish a result of independent interest on the ghost bracket: the ghost bracket satisfies the Jacobi identity except for degenerate cases. We also recall some technical points.

Let us finally comment on two points:

The general case of (non) projective Anosov representations. For the sake of simplicity, this introduction focused on the case of projective Anosov representations. More generally, one can construct correlation functions out of *geodesics decorated* with weights of the Lie group G with respect to a Θ-Anosov representations. The *Θ-decorated correlation functions* are described by configurations of Θ-decorated geodesics. The full machinery developed in this article computes more generally the brackets of these decorated correlation functions. Using that terminology, the Poisson Stability Theorem A still holds with the same statement, but the ghost bracket has to be replaced by a *decorated ghost bracket* which follows a construction given in paragraph 5.2, slightly more involved than formula (1) and not anymore related to the swapping bracket.

Beyond representations: uniformly hyperbolic bundles. One of the novelties of this paper is the introduction of a new tool allowing us to describe "universal Anosov representations" in the spirit of universal Teichmüller spaces: the definition of *uniformly hyperbolic bundles*. This new tool allows us to extend results obtained for Anosov representations, notably stability and limit curves, in a situation where no periodicity according to a discrete group is required. In particular, the (not averaged) correlation functions make sense and we are able to compute the variation of such a correlation function in proposition 4.6.1. This result follows in particular from the solution of the (dynamical) cohomological equation (proposition 2.2.1). Important constructions such as ghost integration and ghost intersection are given in the context of uniformly hyperbolic bundles.

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Contents

1. Preliminary

In this preliminary section, we recall basic and well known facts about the intersection of geodesics in the hyperbolic plane, dual forms to geodesics and the Goldman symplectic form. We also introduce one of the notions important for this paper: geodesically bounded forms.

1.1. **The hyperbolic plane, geodesics and forms.** We first recall classical results and constructions about closed geodesics in the hyperbolic plane.

1.1.1. *Geodesics and intersection.* Let us choose an orientation in $\mathbf{H}^2$. We denote in this paper by C the space of oriented geodesics of $\mathbf{H}^2$ that we identify with the space or pairwise distinct points in $\partial_\infty \mathbf{H}^2$. We denote by $\bar{g}$ the geodesic g with the opposite orientation.

Definition 1.1.1. *Let g_0 and g_1 be two oriented geodesics. The* intersection of g_0 and g_1 *is the number $\varepsilon(g_0, g_1)$ which satisfies the following rules*

$$\varepsilon(g_0, g_1) = -\varepsilon(g_1, g_0) = -\varepsilon(\bar{g}_0, g_1) ,$$

and verifying the following

- $\varepsilon(g_0, g_1) = 0$ *if g_0 and g_1 do not intersect or $g_0 = g_1$.*
- $\varepsilon(g_0, g_1) = 1$ *if g_0 and g_1 intersect and $(g_0(\infty), g_1(\infty), g_0(-\infty), g_1(-\infty)$ is oriented.*
- $\varepsilon(g_0, g_1) = 1/2$ *if $g_0(-\infty) = g_1(-\infty)$ and $(g_0(\infty), g_1(\infty), g_1(-\infty))$ is oriented.*

Observe that $\varepsilon(g_0, g_1) \in \{-1, -1/2, 0, 1/2, 1\}$ and that we have the *cocycle property*, if g_0, g_1, g_2 are the sides of an ideal triangle with the induced orientation, then for any geodesic g we have

$$\sum_{i=0}^{2} \varepsilon(g, g_i) = 0 . \tag{3}$$

We need an extra convention for coherence

Definition 1.1.2. *A* phantom geodesic *is a pair g of identical points (x, x) of $\partial_\infty \mathbf{H}^2$. If g is a phantom geodesic, h any geodesic (phantom or not), we define $\varepsilon(g, h) := 0$.*

1.1.2. *Geodesic forms.* Let us denote by $\Omega^1(\mathbf{H}^2)$ the space of 1-forms on the hyperbolic plane. A form ω in $\Omega^1(\mathbf{H}^2)$ is bounded if $|\omega_x(u)|$ is bounded uniformly for all (x, u) in $\mathsf{U}\mathbf{H}^2$ the unit tangent bundle of $\mathbf{H}^2$. We let $\mathbf{\Lambda}^\infty$ the vector space of bounded forms.

Proposition 1.1.3. *We have a $\mathsf{PSL}_2(\mathbb{R})$ equivariant mapping*

$$\begin{cases} C & \rightarrow & \Omega^1(\mathbf{H}^2) , \\ g & \mapsto & \omega_g , \end{cases}$$

which satisfies the following properties

(1) *ω_g is a closed 1-form in $\mathbf{H}^2$ supported in the tubular neighbourhood of g at distance 1, outside the tubular neighbourhood of g at distance $1/2$.*
(2) $\omega_g = -\omega_{\bar{g}}$
(3) *Let g_0 be any geodesic g, then*

$$\int_{g_0} \omega_g \quad = \quad \varepsilon(g_0, g) . \tag{4}$$

Proof. The construction runs as follows. Let us fix a function f from $\mathbb{R}^+$ to $[0, 1]$ with support in $[0, 1]$ which is constant and equal to $1/2$ on $[0, 1/2]$ neighbourhood of 0. We extend (non-continuously) f to $\mathbb{R}$ as an odd function. Let finally R_g be the "signed distance" to g, so that $R_{\bar{g}} = -R_g$. We finally set $\omega_g = -\mathrm{d}(f \circ R_g)$. Then (1) and (2) are obvious. We leave the reader check the last point in all possible cases. ◻

Remark: We extend the above map to phantom geodesics by setting $\omega_g = 0$ for a phantom geodesic and observe that the corresponding assignment still obey proposition 1.1.3.

The form ω_g is called the *geodesic form* associated to g. Such an assignment is not unique, but we fix one, once and for all. Then we have

Proposition 1.1.4. *For any pair of geodesics g_0 and g_1, $\omega_{g_1} \wedge \omega_{g_0} = f \,\mathrm{d}\,\mathrm{area}_{\mathbf{H}^2}$ with f bounded and in L^1.*

Proof. The only non-trivial case is if g_0, g_1 share an endpoint. In the upper half plane model let g_0 be the geodesic $x = 0$, while g_1 is the geodesic $x = a$. Observe that the support of $\omega_{g_1} \wedge \omega_{g_0}$ is in the sector

V defined by the inequations $y > B > 0$ and $|x/y| < C$ for some positive constants A an C. Finally as the signed distance for g_0 satisfies $\sinh(R_{g_0}) = x/y$ then

$$\omega_{g_0} = f_0 \, \mathrm{d}\left(\frac{x}{y}\right) , \, \omega_{g_1} = f \, \mathrm{d}\left(\frac{x-a}{y}\right) ,$$

where f_0 and f_1 are functions bounded by a constant D. An easy computation shows that

$$\mathrm{d}\left(\frac{x}{y}\right) \wedge \mathrm{d}\left(\frac{x-a}{y}\right) = a \, \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^3} .$$

Oberve that $|f f_0 a|$ is bounded by $D^2 a$, and

$$\int_V \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^3} \leqslant 2C \int_B^\infty \frac{1}{y^2} \mathrm{d}y = \frac{1}{B} < \infty .$$

This completes the proof. □

Remark: The above result is still true whenever g or h are phantom geodesics.

From that it follows that

Proposition 1.1.5. *For any pair of geodesics, phantom or not, g and g_0, we have*

$$\int_{g_0} \omega_g = \varepsilon(g_0, g) = \int_{\mathbf{H}^2} \omega_{g_0} \wedge \omega_g . \tag{5}$$

Moreover for any (possibly ideal) triangle T in $\mathbf{H}^2$

$$\int_{\partial T} \omega_g = 0 . \tag{6}$$

1.2. **The generic set and barycentric construction.** For any oriented geodesic g in C we denote by $\bar{g}$ the geodesic with opposite orientation, and we write $g \simeq h$, if either $g = h$ or $g = \bar{h}$. Let us also denote the extremities of g by $(\partial^- g, \partial^+ g)$ in $\partial_\infty \mathbf{H}^2 \times \partial_\infty \mathbf{H}^2$.

For $n \geqslant 2$, let us the define the *singular set* as

$$C_1^n := \{(g_1, \ldots, g_n) \mid \forall \, i, j, \;\; g_i \simeq g_j\} ,$$

and the *generic set* to be

$$C_\star^n := C^n \setminus C_1^n .$$

We define a Γ-*compact set* in $C_\star^n$ to be the preimage of a compact set in the quotient $C_\star^n/\Gamma$.

The *barycenter* of a family $G = (g_1, \ldots, g_n)$ of geodesics is the point $\mathrm{Bary}(G)$ which attains the minimum of the sum of the distances to the geodesics g_i. Choosing a uniformization, the barycentric construction yields a $\mathsf{PSL}_2(\mathbb{R})$-equivariant map from

$$\mathrm{Bary} : \begin{cases} C_\star^n & \rightarrow \quad \mathbf{H}^2 , \\ (g_1, \ldots, g_n) & \mapsto \quad \mathrm{Bary}(y) . \end{cases}$$

It follows from the existence of the barycenter map that the diagonal action of Γ on $C_\star^n$ is proper. The *barycentric section* is then the section σ of the following fibration restricted to $C_\star^n$

$$F : (\mathsf{U}\Sigma)^n \rightarrow C^n ,$$

given by

$$\sigma = (\sigma_1, \ldots, \sigma_n) ,$$

where $\sigma_i(g_1, \ldots, g_n)$ is the orthogonal projection of $\mathrm{Bary}(g_1, \ldots, g_n)$ on g_i. Obviously

Proposition 1.2.1. *The barycentric section is equivariant under the diagonal action of $\mathsf{PSL}_2(\mathbb{R})$ on $C_\star^n$ as well as the natural action of the symmetric group $\mathfrak{S}_n$.*

1.3. **Geodesically bounded forms.** We abstract the properties of geodesic forms in the following definition:

Definition 1.3.1 (Geodesically bounded forms). *Let α be a closed 1-form on $\mathbf{H}^2$. We say that α is* geodesically bounded *if*

1. *α belongs to $\mathbf{\Lambda}^\infty$, $\nabla\alpha$ is bounded.*
2. *for any geodesic g, $\alpha(\dot{g})$ is in $L^1(g, \mathrm{d}t)$, $\omega_g \wedge \alpha$ belongs to $L^1(\mathbf{H}^2)$ and*
$$\int_g \alpha = \int_{\mathbf{H}^2} \omega_g \wedge \alpha\ . \tag{7}$$
3. *Moreover for any (possibly ideal) triangle T in $\mathbf{H}^2$*
$$\int_{\partial T} \alpha = 0\ . \tag{8}$$

We denote by Ξ the vector space of geodesically bounded forms. We observe that any geodesically bounded form is closed and that any geodesic form belongs to Ξ.

1.4. **Polygonal arcs form.** We will have to consider *geodesic polygonal arcs* which are a finite union of oriented geodesic arcs
$$\boldsymbol{\gamma} = \gamma_0 \cup \cdots \cup \gamma_p\ ,$$
such that γ_i joins γ_i^- to γ_i^+ and we have $\gamma_i^- = \gamma_{i-1}^+$, while γ_0^- and γ_p^+ are distinct points at infinity. We say that $\gamma_1, \ldots, \gamma_{p-1}$ are the *interior arcs*.

We have similar to above

Proposition 1.4.1 (Dual forms to polygonal arcs). *Given a geodesic polygonal arc $\boldsymbol{\gamma} = \gamma_0 \cup \cdots \cup \gamma_p$ there exists a closed 1-form $\omega_{\boldsymbol{\gamma}}$ so that*

1. *the 1-form $\omega_{\boldsymbol{\gamma}}$ is supported on a 1-neighborhood of $\boldsymbol{\gamma}$,*
2. *Let B be a ball containing a 1-neighbourhood of the interior arcs, such that outside of B the 1-neighbourhood V_0 of γ_0 and the 1-neighbourhood V_1 of γ_p are disjoint then*
$$\omega_{\boldsymbol{\gamma}}\big|_{V_0} = \omega_{g_0}\big|_{V_0}\quad,\quad \omega_{\boldsymbol{\gamma}}\big|_{V_1} = \omega_{g_p}\big|_{V_1}\ .$$
where g_0 and g_p are the complete geodesics containing the arcs γ_0 and γ_p.
3. *For any element Φ of $\mathsf{PSL}_2(\mathbb{R})$, $\omega_{\Phi(\boldsymbol{\gamma})} = \Phi^*(\omega_{\boldsymbol{\gamma}})$.*
4. *For any geodesic g, $\int_g \omega_{\boldsymbol{\gamma}} = \varepsilon(g, [\gamma_0^-, \gamma_p^+])$.*
5. *Let $\boldsymbol{\gamma}$ be a polygonal arc with extremities at infinity x and y, then for any 1-form α in Ξ we have*
$$\int_{\mathbf{H}^2} \omega_{\boldsymbol{\gamma}} \wedge \alpha = \int_{[x,y]} \alpha\ .$$

Proof. The construction runs as the one for geodesics. Let us fix a function f from $\mathbb{R}^+$ to $[0, 1]$ with support in $[0, 1]$ which is constant and equal to $1/2$ on $[0, 1/2]$. We extend (non-continuously) f to $\mathbb{R}$ as an odd function. Let finally R_g be the "signed distance" to g, so that $R_{\bar{g}} = -R_g$. We finally set $\omega_g = -\mathrm{d}(f \circ R_g)$. Then (1), (2), (3) and (4) are obvious.

Then writing $\mathbf{H}^2 \setminus \boldsymbol{\gamma} = U \sqcup V$ where U and V are open connected sets. We have that
$$\int_U \omega_{\boldsymbol{\gamma}} \wedge \alpha = \int_U \mathrm{d}(f \circ R_g) \wedge \alpha = \frac{1}{2}\int_g \alpha\ ,$$
by carefully applying Stokes theorem. The same holds for the integral over V, giving us our desired result. ◻

The form $\omega_{\boldsymbol{\gamma}}$ is the *polygonal arc form*.

1.5. **The Goldman symplectic form.** Let S be a closed surface with Σ its universal cover that we identify with $\mathbf{H}^2$ by choosing a complete hyperbolic structure on S. Given a representation $\rho : \pi_1(S) \to G$ we let $E = \Sigma \times_\rho \mathfrak{g}$ be the bundle over S by taking the quotient of the trivial bundle $\Sigma \times \mathfrak{g} \to \Sigma$ by the action of $\pi_1(S)$ given by $\gamma(x,v) = (\gamma(x), Ad\rho(\gamma)(v))$. Let ∇ be the associated flat connection on the bundle E and denote by $\Omega^k(S) \otimes \mathrm{End}(E)$ the vector space of k-forms on S with values in $\mathrm{End}(E)$. Recall that ∇ gives rise to a differential

$$\mathrm{d}^\nabla : \Omega^k(S) \otimes \mathrm{End}(E) \to \Omega^{k+1}(S) \otimes \mathrm{End}(E) .$$

We say a 1-form α with values in $\mathrm{End}(E)$ is *closed* if $\mathrm{d}^\nabla \alpha = 0$ and *exact* if $\alpha = \mathrm{d}^\nabla \beta$. Let then consider

$$\begin{aligned} C^1_\rho(S) &:= \{\text{closed 1-forms with values in } \mathrm{End}(E)\} , \\ E^1_\rho(S) &:= \{\text{exact 1-forms with values in } \mathrm{End}(E)\} , \\ H^1_\rho(S) &:= C^1_\rho(S)/E^1_\rho(S) . \end{aligned}$$

Definition 1.5.1. *When S is closed, the* Goldman symplectic form $\mathbf{\Omega}$ *on $H^1_\rho(S)$ is given by*

$$\mathbf{\Omega}(\alpha, \beta) := \int_S \mathrm{Tr}(\alpha \wedge \beta) , \tag{9}$$

where for u and v in $\mathsf{T}S$: $\mathrm{Tr}(\alpha \wedge \beta)(u,v) := \mathrm{Tr}(\alpha(u)\beta(v)) - \mathrm{Tr}(\alpha(v)\beta(u))$.

Observe that if we consider complex bundles, the Goldman symplectic form is complex valued, while it is real valued for real bundles.

Part 1

On the hyperbolic plane

In this first part, we deal with higher rank analogues of geodesics, intersections and hyperbolic metrics on the disk, which we recalled in paragraph 1.1. We define these notions independent of the presence of a cocompact surface group and work on the hyperbolic plane and more precisely on the unit tangent bundle of the hyperbolic plane.

One of our first goals is to define (non averaged) correlation functions $\mathsf{T}_G(\rho)$ associated to a configuration of geodesics G, and uniformly hyperbolic bundle ρ (the non periodic generalisation of Anosov representations). We more precisely aim at computing the variation of $\mathsf{T}_G(\rho)$ when one varies ρ. This computation is achieved through the introduction of *ghost polygons*, their *ghost dual form* and *intersections*, as well as the *ghost integration*, generalizing step by step the classical framework of geodesics, intersection and dual forms explained in paragraph 1.1.1 .

(1) In section 2, we answer to the question: how do we generalize the hyperbolic metric on the disk in higher rank? In the periodic case, that is in the presence of a closed surface or alternatively a cocompact group acting on the disk, a good answer is given by Anosov representations. To deal with this non periodic case, we introduce *uniformly hyperbolic bundles* and their *fundamental*

projector p. Such a fundamental projector is associated to a pair of distinct points in $\partial_\infty\mathbf{H}^2$. Our main result is then given by proposition 2.2.1 that computes the variation of the fundamental projectors for a variation of a uniformly hyperbolic bundle using the cohomological equation.

(2) In section 3, we introduce some of the main players of this article: the *ghost polygons* which are associated to a cyclic configuration G of geodesics. To such a ghost polygon G and a uniformly hyperbolic bundle ∇ we associate a correlation function $\mathsf{T}_G(\nabla)$ and prove certain analytic properties.

(3) In section 4, we explain how to integrate on a ghost polygon certain 1-forms with values in the endormorphisms bundle, and among them the 1-form $\dot{\rho}$ associated to a variation of a uniformly hyperbolic bundle. This *ghost integration* is the non abelian analogue of integration along geodesics. This allows us to obtain a formula and proposition 4.6.1 which fulfills our first goal: *computing the variation of correlation functions from the variation of a hyperbolic bundle*. In this same section we introduce the dual objects to ghost integration given by certain 1-forms with values in the endomorphisms bundles, playing the role of Poincaré duality in this context.

(4) Finally in section 5, we use the dual 1-forms to define the *ghost intersection* of two ghost polygons, a procedure dual to wedging. This intersection (with respect to a uniformly hyperbolic bundle) is actually purely combinatorial: it defines a bracket on the formal polynomial algebra generated by ghost polygons. We finally explain how this combinatorial structure is related in the projective case to swapping algebras.

At each of these steps, the special case of projective uniformly hyperbolic bundles gives simpler formulae.

In the next part, we use this first part to obtain results in the periodic case after averaging. But we note that the results of this first part do not depend on the assumption of periodicity.

2. Uniformly hyperbolic bundles and projectors

We introduce the notion of *uniformly hyperbolic bundles* over the unit tangent bundle $\mathsf{U}\mathbf{H}^2$ of $\mathbf{H}^2$ – see definition 2.1.2. This notion is a universal version of Anosov representations defined in [16]. Roughly speaking an Anosov representation is given by a flat bundle satisfying some expanding/contracting features. These expanding/contracting features are measured with respect to the choice of a Euclidean metric on the bundle. However, on a compact surface this choice is irrelevant: all Euclidean metrics are uniformly equivalent. On the contrary, if we work on the hyperbolic plane, the choice of the Euclidean metric is now meaningful and has to be specified in the definition.

More specifically, we explain in the projective case, that such an object is a Euclidean vector bundle E over $\mathsf{U}\mathbf{H}^{2^2}$ satisfying some contracting properties, the metric being only considered up uniform equivalence. Associated to these bundles are

(1) A special section p of the endomorphism bundle $\mathrm{End}(E)$ given by projectors fibrewise and that we call the *fundamental projector*

(2) a flat connection ∇ on the bundle E, such that
$$\nabla_{\partial_t}\mathrm{p} = 0\ , \tag{10}$$
where ∂_t is the generator of the geodesic flow on $\mathsf{U}\mathbf{H}^2$.

(3) Special curves in flag manifolds that by extension of Anosov representations we also call *limit maps*.

(4) As ancillary definitions, we have that the notions of a
   (a) *family of uniformly hyperbolic bundles* – and the associated stability lemma –
   (b) *variation of a uniformly hyperbolic bundle*: we describe this as a 1-form $\dot{\nabla}$ with values in the endormorphism bundle of a uniformly hyperbolic bundle.
   (c) *equivalent bundles* using either gauge fixing or metric fixing.

Our main result – proposition 2.2.1 – is the description of the variation $\dot{\mathrm{p}}$ of such a projector under a variation of the data defining the uniformly hyperbolic bundles. We present the outline briefly:

differentiating $\mathrm{p} = \mathrm{p}^2$ we see that
$$\dot{\mathrm{p}} = \dot{\mathrm{p}}\mathrm{p} + \mathrm{p}\dot{\mathrm{p}}$$
and it follows that $\dot{\mathrm{p}}$ is a section of the subbundle F_0 of $\mathrm{End}(E)$ where
$$F_0 = \{B \in \mathrm{End}(E) \mid B = B\mathrm{p} + \mathrm{p}B\}.$$
As a consequence of uniform hyperbolicity this bundle splits with $F_0 = F_0^+ \oplus F_0^-$ where F_0^- contracts under forward flow an F_0^+ contracts under negative flow. Using these contraction properties, we see that $\dot{\mathrm{p}}$ is a solution of the *cohomological equation* obtained by differentiating (10),
$$\nabla\dot{\mathrm{p}} + [\dot{\nabla}, \mathrm{p}] = 0\ .$$
We then solve this equation to obtain an integral formula for $\dot{\mathrm{p}}$ under a change in connection $\dot{\nabla}$
$$\dot{\mathrm{p}}(x) = \int_{-\infty}^{0} \left(\mathrm{p}\cdot[\mathrm{p}, \dot{\nabla}_{\partial_s}]\right)(x^s)\,\mathrm{d}s - \int_{0}^{\infty}\left([\mathrm{p}, \dot{\nabla}_{\partial_s}]\cdot\mathrm{p}\right)(x^s)\,\mathrm{d}s\ ,$$
This rough outline has to be made rigorous, and has several avatars depending on whether we want to emphasize gauge fixing or metric fixing, a classical fixture in Higgs bundles.

After defining uniformly hyperbolic bundles, we consider Θ-*uniformly hyperbolic bundles*, which is just an extra decoration.

Finally, we recover Anosov representations as periodic cases of uniformly hyperbolic bundles. Also the notion of a uniformly hyperbolic bundle is the structure underlying the study of quasi-symmetric maps in [15].

This notion has a further generalization to all hyperbolic groups Γ, replacing $\mathsf{U}\mathbf{H}^2$ by a real line bundle X over
$$\partial_\infty\Gamma\times\partial_\infty\Gamma \setminus \{(x,x)\mid x\in\partial_\infty\Gamma\}\ ,$$
equipped with a Γ-action so that X/Γ is the geodesic flow of Γ. We will not discuss it in this paper, since this will needlessly burden our notation.

2.1. **Uniformly hyperbolic bundles: definitions.** Let $\mathsf{U}\mathbf{H}^2$ be the unit tangent bundle of $\mathbf{H}^2$. We denote by ∂_s the vector field on $\mathsf{U}\mathbf{H}^2$ generating the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$. This first paragraph deals with basic definitions and properties.

2.1.1. *Metrics on Grassmannian.* We first recall that given a euclidian metric g_0 on a vector space V, we define the *angle* between vector subspaces F and G of V as
$$\sphericalangle_0(F,G) = \inf\{\sphericalangle_x(u,v)\mid u\in F, v\in G\}\ ,$$
where for non zero vectors u and v
$$\sphericalangle_0(u,v) = \arccos\left(\frac{g_0(u,v)}{\sqrt{g_x(u,u)g_x(v,v)}}\right)\ .$$
Similarly, we consider the distance d_x of diameter 1 in $\mathrm{Gr}_k(E)$, associated to g_x. Finally $\sphericalangle_x(F,G)=0$ if and only if $\dim(F\cap G)\geqslant 1$.

In the next lemma and the sequel, we identify $\mathrm{Gr}_k(V^*)$ with $\mathrm{Gr}_{n-k}(V)$ where $n=\dim(V)$.

Lemma 2.1.1 (Equivalent metrics). *Let V be a vector space. Then for any positive ε there exists a constant K such that for any euclidean metric g_0 in V, following holds: Let P_0 be an element of $\mathrm{Gr}_k(V^*)$ respectively. Let then*
$$\mathcal{U} = \{\mathbf{u}\in\mathrm{Gr}_k(V)\mid \sphericalangle(\mathbf{u},P_0)\geqslant\varepsilon\}\ .$$
Le L_0 be an element of $\mathcal{U}$.

Then if B is an an element of $L_0^*\otimes P_0$ such that the graph L_B of B is $\mathcal{U}$, we have the following inequality.
$$\frac{1}{K}\,d_0(L_0,L_B)\leqslant\|B\|\leqslant K\,d_0(L_0,L_B)\ . \tag{11}$$

Proof. The easy proof relies on a compactness argument. Indeed,

$$K := \{B \in L_0^* \otimes P_0 \mid L_B \in \mathcal{U}_0\}$$

is a compact set in $L^* \otimes P$. Now, on K we have two Riemannian metrics,

(1) the one induced from the Riemannian metric d_0 on $\mathrm{Gr}_k(V)$ by the map $B \mapsto L_B$,
(2) the norm coming from the metric g_0.

It follows that these two Riemaniann metrics are equivalent by a constant $K_L = K(L, \varepsilon)$ depending only on L and ε and in particular

$$\frac{1}{K_{L_0}}\, d_0(L_0, L_B) \leqslant \|B\| \leqslant K_{L_0}\, d_0(L, L_B)\ . \tag{12}$$

It follows that

$$K := \sup_{L \in \mathcal{U}} K_L,$$

works ◻

2.1.2. *Uniformly hyperbolic bundles: definitions.* We consider the trivial bundle $E = V \times \mathsf{U}\mathbf{H}^2$. For any flat connection ∇ on E, we consider the lift $\left(\Phi_t^\nabla\right)_{t\in\mathbb{R}}$ of $(\varphi_t)_{t\in\mathbb{R}}$ given by the parallel transport along the orbits of ∂_t. When D is the trivial connection on E, we just write $(\Phi_t)_{t\in\mathbb{R}} := \left(\Phi_t^D\right)_{t\in\mathbb{R}}$ and observe that $\Phi_t(x, v) = (\varphi_t(x), v)$ where x is in $\mathsf{U}\mathbf{H}^2$ and v in V.

Definition 2.1.2 (Uniformly hyperbolic bundle)**.** *A* rank k uniformly hyperbolic projective bundle *is a pair* (∇, h) *where h is a section of the frame bundle on E,* ∇ *a trivializable connection on the bundle E, satisfying first the (standard)* bounded cocyle hypothesis*:* $\|\Phi_1^\nabla\|$ *is uniformly bounded.*

Then we assume that we have a $\left(\Phi_t^\nabla\right)_{t\in\mathbb{R}}$ *invariant decomposition at every point x*

$$E_x = L_x \oplus P_x\ ,$$

where L_x and P_x are subspaces with $\dim(L_x) = k$*, as well as a positive ε_0 such that*

(1) *There is a volume form on E, which is bounded with respect to h and ∇-parallel along orbits on the flow.*
(2) *The bundle $L \otimes P^*$ is contracting, that is there exist positive constant B and b so that for all positive real s, for all x in $\mathsf{U}\Sigma$ for all non-zero vector u and v in L_x and P_x respectively*

$$\frac{\|\Phi_s^\nabla(u)\|_{\varphi_s(x)}}{\|u\|_x} \leqslant B e^{-bs} \frac{\|\Phi_s^\nabla(v)\|_{\varphi_s(x)}}{\|v\|_x}\ . \tag{13}$$

(3) *Let $\mathcal{L}$ the closure of the image of L in* $\mathrm{Gr}_k(E)$ *and $\mathcal{P}$ the closure of the image of P in* $\mathrm{Gr}_k^*(E)$*, then for any x in* $\mathsf{U}\mathbf{H}^2$*, for any* $\mathbf{u}$ *in $\mathcal{L}_x$ and* $\mathbf{v}$ *in $\mathcal{P}_x$, we have*

$$\sphericalangle_x(\mathbf{u}, \mathbf{v}) > \varepsilon_0.$$

(4) *Let consider E as a trivial bundle* $E = V \times \mathsf{U}\mathbf{H}^2$ *over* $\mathsf{U}\mathbf{H}^2$*, where ∇ is the trivial connection. We finally assume that there exists decomposition of E, $E = L^\infty \oplus P^\infty$ – seen as a mapping in* $\mathrm{Gr}_k(V) \times \mathrm{Gr}_k^*(V)$ *– such that for all x in* $\mathsf{U}\mathbf{H}^2$, $\mathbf{u}$ *in $\mathcal{L}_x$ and* $\mathbf{v}$ *in $\mathcal{P}_x$*

$$\sphericalangle_x(L_x^\infty, \mathbf{v}) > \varepsilon_0\ , \sphericalangle_x(L_x^\infty, P_x^\infty) > \varepsilon_0 \text{ and } \quad \sphericalangle_x(P_x^\infty, \mathbf{u}) > \varepsilon_0.$$

such that furthermore there exists constant H and α such that for any x and y in $\mathsf{U}\mathbf{H}^2$

$$d_x(L_x^\infty, L_y^\infty) \leqslant H\, d(x, y)^\alpha\ ,$$

and the same holds for P^∞.[1]

The metric and scalar products considered are with respect to the metric g_x for which $h(x)$ is orthonormal.

The fundamental projector *associated to a uniformly hyperbolic bundle is the section* p *of* $\mathrm{End}(E)$ *given by the projection on L parallel to P.*

[1]This inequality is satisfied for instance when L^∞ and P^∞ are uniformly C^1. It will only be used to prove a certain subset is nonempty; see footnote 3.

We thank Tianqi Wang for pointing out to us that the first three hypothesis were not enough.

Observe that we do not require a priori any continuity on the bundles L and P. When the dimension of L_x is 1, we talk of a *projective uniformly hyperbolic bundle*, when it is k, we talk of a *rank k uniformly hyperbolic bundle*.

The hypothesis (1) is for simplification purposes. Using that hypothesis, one sees that $\det(L)$ and $\det(P)$ are respectively contracting and expanding bundles.

The bounded cocycle assumption, akin to a similar condition in Oseledets theorem, implies that there exists positive constants A, B and C so that

$$\|\Phi_s^\nabla\| \leqslant A + Be^{Cs} . \tag{14}$$

Finally let us define a notion of equivalence for uniformly hyperbolic bundles:

Definition 2.1.3 (Equivalent bundles)**.** *Two uniformly hyperbolic bundles* (∇_0, h_0) *and* (∇_1, h_1) *are* equivalent *if there is a section* B *of* $\mathsf{GL}(E)$ *so that*

(1) $\nabla_1 = B^*\nabla_0$,
(2) *The metrics* g_{h_0} *and* $B^* g_{h_1}$ *are uniformly equivalent.*

2.1.3. *Limit maps.* We start with a proposition.

Proposition 2.1.4. *Let* (∇, h) *be a uniformly hyperbolic bundle* $E = L \oplus P$*. Let us choose a trivialisation so that* ∇ *is the trivial connection. Then*

(1) *The fundamental projector* p *is parallel along the geodesic flow and a continuous bounded section of* $\mathrm{End}(E)$*:*
$$\sup_{x \in \mathsf{U}\mathbf{H}^2} \|\mathrm{p}_x\|_x < +\infty .$$
(2) *the subdundle* L *is constant along the strong stable foliation of the geodesic flow of* $\mathsf{U}\mathbf{H}^2$*, and there exist positive constants* H *and* β*, such that for all* x
$$d_x(L_x, L_y) \leqslant H\, d(x,y)^\beta .$$
(3) *Finally* P *is constant along the strong unstable foliation of* $\mathsf{U}\mathbf{H}^2$*, and there exist positive constants* H *and* β*, such that for all* x
$$d_x(P_x, P_y) \leqslant H\, d(x,y)^\beta .$$

Before giving the proof of this proposition, observe that it allows us to define the *limit maps* of the uniformly hyperbolic bundle (∇, h) as we now show

Definition 2.1.5 (Limit maps)**.** *Let us choose a trivialization* $E = V \times \mathsf{U}\mathbf{H}^2$ *so that* ∇ *is trivial.*

The limit map *of the uniformly hyperbolic bundle is the continuous map*

$$\xi : \partial_\infty \mathbf{H}^2 \to \mathrm{Gr}_k(V) ,$$

where $\xi(x) = L_y$*, for any* y *belongs to the strong stable foliation defined by* x*. Symmetrically, the* dual limit map *of the uniformly hyperbolic bundle is the continuous map*

$$\xi^* : \partial_\infty \mathbf{H}^2 \to \mathrm{Gr}_k(V^*) ,$$

where $\xi^*(x) = P_y$*, for any* y *belongs to the strong unstable foliation defined by* x*.*

We also prove

Proposition 2.1.6. *The limit maps are Hölder.*

Proof. We use proposition 2.1.4. Let S^1 in $\mathsf{U}\mathbf{H}^2$ be a fiber of the projection to $\mathbf{H}^2$. We can see ξ as a map from S^1 to $\mathrm{Gr}_k(E)$. By proposition 2.1.4, for any x in S^1,

$$d_x(\xi(x), \xi(y)) \leqslant H\, d(x,y)^\beta .$$

Since S^1 is compact, given any metric d_0 on $\mathrm{Gr}_k(V)$ coming from a euclidean metric on V, there is a constant K such that for all x in S^1

$$d_0 \leqslant K\, d_x \ .$$

It follows that for any x and y in S^1,

$$d_0(\xi(x),\xi(y)) \leqslant KH\, d(x,y)^\beta \ .$$

Hence ξ is Hölder. The same holds for ξ^*. ◻

We start the proof of proposition 2.1.4 by two lemmas. First let us write, if we have a projection π from a set F to UH^2, $F_x = \pi^{-1}(x)$ for x in UH^2.

Lemma 2.1.7. *Let (∇, h) be a uniformly hyperbolic bundle. Then there exists closed sets $\mathcal{V}$ and $\mathcal{U}$ of $\mathrm{Gr}_k(E)$ and $\mathrm{Gr}_k(E^*)$, respectively, where k is the dimension of L_x, as well as a positive real T, so that*

1. *L, P, L^∞ and P^∞ are all sections of $\mathcal{V}$ and $\mathcal{U}$*
2. *For every $\mathbf{u}$ in $\mathcal{U}_x$, $\mathbf{u}$ is transverse to P_x, for every $\mathbf{v}$ in $\mathcal{U}_x$, $\mathbf{v}$ is transverse to L_x,*
3. *Φ_T sends $\mathcal{U}$ to $\mathcal{U}$ and is $1/2$-Lipschitz: for any x in UH^2, $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{U}_x$, then*

$$d_{\varphi_T(x)}(\Phi_T(\mathbf{u}),\Phi_T(\mathbf{v})) \leqslant \frac{1}{2} d_x(\mathbf{u},\mathbf{v}) \ .$$

4. *Φ_{-T} sends $\mathcal{V}$ to $\mathcal{V}$ and is $1/2$-Lipschitz.*

Proof. Observe first that $\mathcal{L}$ and $\mathcal{P}$ are also invariant by $(\Phi_t)_{t\in\mathbb{R}}$. Thus the function on $\mathrm{Gr}_k(E)$ defined by

$$\mathbf{u} \mapsto \sphericalangle(\mathbf{u},\mathcal{P}) := \inf\{\sphericalangle_x(\mathbf{u},\mathbf{v}) \mid \mathbf{v}\in\mathcal{P}_x\} \ ,$$

is continuous and symmetrically the function

$$\mathbf{v} \mapsto \sphericalangle(\mathbf{v},\mathcal{L}) := \inf\{\sphericalangle_x(\mathbf{u},\mathbf{v}) \mid \mathbf{u}\in\mathcal{L}_x\} \ ,$$

is continuous as well. Let then ε_0 be as in hypothesis (3), and

$$\mathcal{U} := \{\mathbf{u}\in\mathrm{Gr}_k(E) \mid \sphericalangle(\mathbf{u},\mathcal{P}) \geqslant \varepsilon_0\} \ , \ \mathcal{V} := \{\mathbf{v}\in\mathrm{Gr}_k(E^*) \mid \sphericalangle(\mathbf{v},\mathcal{L}) \geqslant \varepsilon_0\} \ .$$

By the above remark $\mathcal{U}$ and $\mathcal{V}$ are closed sets. By hypothesis (3), L and P are sections of $\mathcal{U}$ and $\mathcal{V}$ and by hypothesis (4) L^∞ and P^∞ as well. Thus $\mathcal{U}$ and $\mathcal{V}$ satisfy the first condition, as well as the second.

The contraction property and the lemma 2.1.1 now implies the third and fourth properties. [2]is it ok?. ◻

Our second lemma is

Lemma 2.1.8. *Let (∇, h) be a uniformly hyperbolic bundle $E = L \oplus P$. Let us choose a trivialisation so that ∇ is the trivial connection. Then*

1. *There exist positive constants H and β, such that for all x*

$$d_x(L_x, L_y) \leqslant H\, d(x,y)^\beta \ .$$

2. *There exist positive constants H and β, such that for all x*

$$d_x(P_x, P_y) \leqslant H\, d(x,y)^\beta \ .$$

Proof. We follow, and adapt, the classical arguments of Hirsch–Pugh–Shub [13]. Let $\Gamma^0(\mathrm{Gr}_k(E))$ be the space of continuous sections of $\mathrm{Gr}_k(E)$. Using the trivialisation of ∇ we will see alternatively $\Gamma^0(\mathrm{Gr}_k(E))$ as $C^0(\mathsf{UH}^2, \mathrm{Gr}_k(V)))$ For any x in UH^2, let d_x be the associated distance on $\mathrm{Gr}_k(E_x) = \mathrm{Gr}_k(V)$ coming from the frame. For any section σ_0 and σ_1 of $\mathrm{Gr}_k(E)$, let

$$d_\infty(\sigma_0,\sigma_1) := \sup\{d_x(\sigma_0(x),\sigma_1(x)) \mid x\in\mathsf{UH}^2\} \ .$$

Let Ψ be the map from $\Gamma^0(\mathrm{Gr}_k(E))$ to itself defined by

$$[\Psi(\sigma)]x = \Phi_T(\sigma\circ\varphi_{-T}(x)) \ .$$

[2]MB: Might need expand this

Let $\mathcal{U}$, $\mathcal{V}$ and T as in the previous lemma 2.1.7. Let $\Gamma^0(\mathcal{U})$, with $\Gamma^0(\mathcal{U}) \subset \Gamma^0(\mathrm{Gr}_k(E))$ be the space of continuous sections of $\mathcal{U}$. Equipped with d_∞, $\Gamma^0(\mathcal{U})$ is a complete metric space. Lemma 2.1.7 implies that Ψ sends $\Gamma^0(\mathcal{U})$ to itself and that

$$d_\infty(\Psi(\sigma_0), \Psi(\sigma_1)) \leqslant \frac{1}{2} d_\infty(\sigma_0, \sigma_1) .$$

Step 1: A space of locally Hölder sections. Let β with $0 < \beta \leqslant \alpha$, where α is the Hölder constant appearing in hypothesis (4). For any σ a section of $\mathrm{Gr}_k(E)$, let

$$V_\beta(\sigma) := \sup_{x,y \in \mathsf{U}\mathbf{H}^2} \left\{ \frac{d_x(\sigma(x), \sigma(y))}{d(x,y)^\beta} \mid 0 < d(x,y) \leqslant 1 \right\} ,$$

and finally

$$\Gamma^{\beta,H}(\mathcal{U}) := \left\{ \sigma \in \Gamma^0(\mathcal{U}) \mid V_\beta(\sigma) \leqslant H \right\} ,$$

where $H \geqslant 1$. Observe that if σ belongs to $\Gamma^{\beta,H}$, then for all x, y with $x \neq y$, then – since $d_x(\sigma(y), \sigma(x)) \leqslant 1$ and $H \geqslant 1$– we have

$$d_x(\sigma(y), \sigma(x)) \leqslant H\, d(x,y)^\beta . \tag{15}$$

Observe that hypothesis (4) implies that L^∞ belongs to $\Gamma^{\beta,H}(\mathcal{U})$ which is therefore non empty [3].

Step 2: Closedness of the space of sections. We first prove that $\Gamma^{\beta,H}(\mathcal{U})$ is closed in $\Gamma^0(\mathcal{U})$ with respect to d_∞. Let $\{\sigma_m\}_{m\in\mathbb{N}}$ be a sequence of sections in $\Gamma^{\beta,H}(\mathcal{U})$ converging to σ. Then for all x and y in $\mathsf{U}\mathbf{H}^2$ such that $0 < d(x,y) \leqslant 1$,

$$\frac{d_x(\sigma(x), \sigma(y))}{d(x,y)^\beta} = \lim_{m\to\infty} \frac{d_x(\sigma_m(x), \sigma_m(y))}{d(x,y)^\beta} \leqslant \lim_{m\to\infty} V_\beta(\sigma_m) \leqslant H.$$

Thus $\Gamma^{\beta,H}(\mathcal{U})$ is closed in $\Gamma^0(\mathcal{U})$ for d_∞.

Step 3: Action of Ψ. We now prove that there are positive constants β_0 [4] and k_0 with $k_0 < 1$ such that for all σ,

$$V_\beta(\Psi(\sigma)) \leqslant k_0 V_\beta(\sigma) . \tag{16}$$

Let k be the Lipschitz constant of φ_{-T}. Let x and y be such that $0 < d(x,y) \leqslant 1$. Then $d(X,Y) \leqslant k$, where $X := \varphi_{-T}(x)$ and $Y := \varphi_{-T}(y)$. Moreover

$$d_x([\Psi(\sigma)](x), [\Psi(\sigma)](y]) \leqslant \frac{1}{2} d_X(\sigma(X), \sigma(Y))) \leqslant \frac{1}{2} V_\beta(\sigma)\, d(X,Y)^\beta \leqslant \frac{k^\beta}{2} V_\beta(\sigma)\, d(x,y)^\beta .$$

Thus

$$V_\beta(\Psi(\sigma)) \leqslant \frac{k^\beta}{2} V_\beta(\sigma) .$$

We now choose β, less than α, such that furthermore $k_0 := k^\beta \frac{1}{2} < 1$. This concludes the proof.

Step 4: Conclusion. We apply Banach fixed point theorem to the closed Ψ-invariant set $\Gamma^{\beta_0,H}(\mathcal{U})$. Then there is a Ψ-invariant section σ_0 in $\Gamma^{\beta_0,H}(\mathcal{U})$. By the contraction property, $\sigma_0 = L$. We have proven that L belongs to $\Gamma^{\beta_0,H}(\mathcal{U})$. The same arguments works for P. This concludes the proof. □

We can now proceed to the proof of the proposition

Proof of proposition 2.1.4. Let x and y be on the same strong stable leaf. Then, for any positive ε, there exists some non negative n, such that

$$d(x_n, y_n) \leqslant \varepsilon .$$

[3] That is the only point where hypothesis (4) is used

[4] MB: Change β to β_0 in 16 and below?

where $x_n = \varphi_{nT}(x)$ and $y_n = \varphi_{nT}(y)$. Then by lemma 2.1.8

$$d_{x_n}(L_{x_n}, L_{y_n}) \leqslant H\varepsilon^\beta .$$

It follows that

$$d_x(L_x, L_y) \leqslant \frac{1}{2^n} d_{x_n}(L_{x_n}, L_{y_n}) \leqslant \frac{1}{2^n} H\varepsilon^\beta \leqslant H\varepsilon^\beta .$$

Since this is true for any ε, we have that $L_x = L_y$ which proves the result. The same argument works for P. ◻

2.1.4. *Families of uniformly hyperbolic bundles and their variations.* In order to study families of uniformly hyperbolic bundles, we will adopt two different gauge-fixing points of view:

(1) The *fixed gauge point of view*: we allow the frame to vary but fix the connection
(2) The *fixed frame point of view*: we allow the connection to vary but fix the frame.

A natural example comes from a projective Anosov representation of a cocompact surface group. We call such an example, where the frame and the connections are invariant under the action of a cocompact surface group a *periodic bundle*. We discuss periodic bundles in 2.6.

For a vector bundle V over a topological space X, we denote by V_x the fiber at a point x in X.

Definition 2.1.9 (Bounded variation). *A* C^k-bounded variation *of a uniformly hyperbolic bundle* (∇, h) *is a family* $(\nabla^t, h_t)_{t\in]-\varepsilon,\varepsilon[}$ *of connections and frames on* E_0 *so that*

(1) $(\nabla_0, h_0) = (\nabla, h)$,
(2) *for all* t, ∇^t *is trivializable*
(3) *for all* t *close to* 0, *the* C^k*-derivatives of* $t \mapsto \nabla_{\partial_s}\dot{h}_t$ *are bounded with respect to* g_{h_t}.

We will see that any smooth family of periodic bundles is of bounded variation.

Then we have the lemma:

Lemma 2.1.10 (Stability Lemma). *Assume that* $(\nabla^t, h_t)_{t\in]-\varepsilon,\varepsilon[}$ *is a* C^k *bounded variation of a uniformly hyperbolic bundle where* $k \in \mathbb{N} \cup \{\omega\}$. *Then for* t *in some neighbourhood of zero, the bundle* (∇^t, h_t) *is uniformly hyperbolic. Let* p_t *be the associated projector, then* p_t *depends* C^k *on* t.

We prove this lemma in paragraph 2.3.

2.2. The fundamental projector and its variation.

Our goal is to compute the variation of the associated family of fundamental projectors of a bounded variation of a uniformly hyperbolic bundle. More precisely, let assume we have a uniformly hyperbolic bundle (∇_0, h_0) with decomposition

$$E_0 = L_0 \oplus P_0 .$$

We prove in this paragraph the following proposition

Proposition 2.2.1 (Variation of the fundamental projector). *Assume that we have a bounded variation* $(\nabla_t, h_t)_{t\in]-\varepsilon,\varepsilon[}$ *of the uniformly hyperbolic bundle* (∇_0, h_0) *in the fixed connection point of view, that is* ∇_t *is the trivial connection* D.

The derivative of the fundamental geodesic at a point x *in a geodesic* g, *is given by*

$$\dot{\mathrm{p}}_0 = [\dot{A}, \mathrm{p}_0] + \int_{g^+} [\mathrm{d}\dot{A}, \mathrm{p}_0] \cdot \mathrm{p}_0 + \int_{g^-} \mathrm{p}_0 \cdot [\mathrm{d}\dot{A}, \mathrm{p}_0] . \qquad (17)$$

where g^+ *is the geodesic arc from* x *to* $g(+\infty)$ *and* g^- *is the arc from* x *to* $g(-\infty)$ *(in other words with the opposite orientation to* g*), and* $\dot{A}$ *is the endormorphism so that*

$$\dot{A} \cdot h = \left.\frac{\partial}{\partial t}\right|_{t=0} h_t .$$

2.2.1. *Preliminary: subbundles of* $\mathrm{End}(E_0)$. We first adopt the fixed frame point of view. Let ∇ be a flat connection on E_0, Then p is parallel for the induced flat connection on $\mathrm{End}(E_0)$ along the flow. Let also F_0 be the subbundle of $\mathrm{End}(E_0)$ given by

$$F_0 := \{B \in \mathrm{End}(E_0) \mid B\mathrm{p} + \mathrm{p}B = B\} .$$

Observe that for any section C of $\mathrm{End}(E_0)$, $[C, \mathrm{p}]$ is a section of F_0 and that for any element A in F_0 we have $\mathrm{Tr}(A) = 0$.

Lemma 2.2.2. *The bundle F_0 decomposes as two parallel subbundles*

$$F_0 = F_0^+ \oplus F_0^- , \tag{18}$$

where we have the identification

$$F_0^+ = P^* \otimes L \quad , \quad F_0^- = L^* \otimes P . \tag{19}$$

The projection of F_0 to F_0^+ parallel to F_0^- is given by $B \mapsto \mathrm{p}B$, while the projection on F_0^- parallel to F_0^+ is given by $B \mapsto B\mathrm{p}$.

Finally there exists positive constants A and a, so that for all positive time s, endomorphisms u^+ in F_0^+ and u^- in F_0^-, we have

$$\|\Phi_{-s}(u^-)\| \leqslant Ae^{-as}\|u^-\| \ , \ \|\Phi_s(u^+)\| \leqslant Ae^{-as}\|u^+\| . \tag{20}$$

Consequently, for any section f of F_0, we write $f = f^+ + f^-$ where $f^\pm$ are sections of $F_0^\pm$ according to the decomposition (18).

Proof. Let us write

$$\mathrm{End}(E_0) = E_0^* \otimes E_0 = (L^* \otimes L) \oplus (P^* \otimes P) \oplus (L^* \otimes P) \oplus (P^* \otimes L),$$

In that decomposition, $F_0 = (P^* \otimes L) \oplus (L^* \otimes P)$. Let

$$F_0^+ = P^* \otimes L \ , \ F_0^- = L^* \otimes P .$$

Thus, we can identify F_0^+ as the set of elements whose image lie in L and F_0^- are those whose kernel is in P. Thus

$$\begin{aligned} F_0^+ &= \{B \in F_0 \mid \mathrm{p}B = B\} = \{B \in F_0 \mid B\mathrm{p} = 0\} , \\ F_0^- &= \{B \in F_0 \mid \mathrm{p}B = 0\} = \{B \in F_0 \mid B\mathrm{p} = B\} . \end{aligned}$$

Then the equation for any element B of F_0,

$$B = \mathrm{p}B + B\mathrm{p} , \tag{21}$$

corresponds to the decomposition $F_0 = F_0^+ \oplus F_0^-$. Thus the projection on F_0^+ is given by $B \mapsto \mathrm{p}B$, while the projection on F_0^- is given by $B \mapsto B\mathrm{p}$.

The definition of F_0^+ and F_0^- and the corresponding contraction properties of the definition of a uniformly hyperbolic bundles give the contraction properties on F_0^+ and F_0^-. ☐

2.2.2. *The cohomological equation.*

Proposition 2.2.3. *Let σ be a bounded section of F_0, then there exists a unique section η of F_0 so that $\nabla_{\partial_s}\eta = \sigma$. This section η is given by*

$$\eta(x) = \int_{-\infty}^{0} \mathrm{p} \cdot \sigma(\varphi_s(x)) \, \mathrm{d}s - \int_{0}^{\infty} \sigma(\varphi_s(x)) \cdot \mathrm{p} \, \mathrm{d}s . \tag{22}$$

Classically, in dynamical systems, the equation $\nabla_{\partial_s}\eta = \sigma$ is called the *cohomological equation*.

Proof. Since $\mathrm{p}\sigma$ belongs to F_0^+ while $\sigma\mathrm{p}$ belongs to F_0^-, by lemma 2.2.2, the right hand side of equation (22) makes sense using the exponential contraction properties given in the inequalities (20). Indeed, for a positive s by lemma 4.2.2 again,

$$\begin{aligned} \|\Phi_{-s}\left(\sigma(\varphi_s)\cdot \mathrm{p}\right)\| &\leqslant Ae^{-as}\|\sigma\|_\infty\ ,\\ \|\Phi_{s}\left(\mathrm{p}\cdot\sigma(\varphi_{-s})\right)\| &\leqslant Ae^{-as}\|\sigma\|_\infty\ . \end{aligned}$$

It follows that using the above equation as a definition for η we have

$$\eta(\varphi_s(x)) = \int_{-\infty}^{t} \mathrm{p}\cdot\sigma(\varphi_u(x))\mathrm{d}u - \int_{t}^{\infty} \sigma(\varphi_u(x))\cdot \mathrm{p}\ \ \mathrm{d}u\ .$$

Thus

$$\nabla_{\partial_s}\eta = \mathrm{p}\sigma + \sigma\mathrm{p} = \sigma\ ,$$

since σ is a section of F_0. Uniqueness follows from the fact that F_0 has no parallel section: indeed neither F_0^+ nor F_0^- have a parallel section. □

2.2.3. *Variation of the fundamental projector: metric gauge fixing.* We continue to adopt the variation of connection point of view and consider after gauge fixing only hyperbolic bundles where the metric is fixed.

Let $(\nabla^t, h_t)_{t\in]-\varepsilon,\varepsilon[}$ give rise to a bounded variation of the uniformly hyperbolic bundle (∇_0, h), where ∇_0 is the trivial connection D.

Our first result is

Lemma 2.2.4. *The variation of the fundamental projector* p_t *associated to* (∇^t, h) *is given by*

$$\dot{\mathrm{p}}(x) = \int_{-\infty}^{0} \left(\mathrm{p}\cdot[\mathrm{p},\dot{\nabla}_{\partial_s}]\right)(x^s)\ \mathrm{d}s - \int_{0}^{\infty} \left([\mathrm{p},\dot{\nabla}_{\partial_s}]\cdot\mathrm{p}\right)(x^s)\ \mathrm{d}s\ \ , \tag{23}$$

where $x^s = \varphi_s(x)$ *and* $\dot{\nabla}_{\partial_s}(u) = \frac{\partial}{\partial s}\big|_{t=0}\nabla^t_{\partial_s}(u)$.

Proof. Let us distinguish for the sake of this proof the following connections. Let ∇ be the flat connection on E_0 and ∇^{End} the associated flat connection on $\mathrm{End}(E_0)$. Then from the equation $\mathrm{p}^2 = \mathrm{p}$, we obtain after differentiating,

$$\dot{\mathrm{p}}\mathrm{p} + \mathrm{p}\dot{\mathrm{p}} = \dot{\mathrm{p}}\ .$$

Thus $\dot{\mathrm{p}}$ is a section of F_0. Moreover taking the variation of the equation $\nabla^{\mathrm{End}}_{\partial_s}\mathrm{p} = 0$ yields

$$\nabla^{\mathrm{End}}_{\partial_s}\dot{\mathrm{p}} = -\dot{\nabla}^{\mathrm{End}}_{\partial_s}\mathrm{p} = [\mathrm{p},\dot{\nabla}_{\partial_s}]\ .$$

In other words, the variation of the fundamental projector $\dot{\mathrm{p}}$ is a solution of the cohomological equation $\nabla^{\mathrm{End}}_{\partial_s}\eta = \sigma$, where $\sigma = [\mathrm{p},\nabla_{\partial_s}]$ and $\eta = \dot{\mathrm{p}}$. Applying proposition 2.2.3, yields the equation (23). □

2.2.4. *The fixed connection point of view and the proof of proposition 2.2.1.* We can now compute the variation of the projector in the fixed frame point of view and prove proposition 2.2.1. We first need to switch from the fixed frame point of view to the fixed connection point of view.

Let (∇^t, h) be a variation in the fixed frame point of view. Let A^t be so that $\nabla^t = A_t^{-1}\mathrm{D}A_t$ and $A_0 = \mathrm{I}d$. In particular, we have

$$\dot{\nabla}_{\partial_s} = \mathrm{D}_{\partial_s}\dot{A} = \mathrm{d}\dot{A}(\partial_s)\ . \tag{24}$$

Then the corresponding variation in the fixed connection point of view is (D, h_s) where $h_t = A_t(h)$. It follows that

$$\dot{h} = \dot{A}(h)\quad ,\quad \dot{\nabla}_{\partial_s} = \mathrm{d}\dot{A}(\partial_s) = \mathrm{D}_{\partial_s}\dot{A}\ . \tag{25}$$

Let now p_0^t be the projector – in the fixed connection point of view– associated to (D, h_t), while p^t is the projector associated to (∇^s, h). Obviously

$$\mathrm{p}_0^t = A_t\mathrm{p}^tA_t^{-1}\ ,\ \mathrm{p}_0 =: \mathrm{p}_0^0 = \mathrm{p}^0 := \mathrm{p}_0\ .$$

Thus

$$\dot{\mathrm{p}}_0 = [\dot{A}, \mathrm{p}] + \dot{\mathrm{p}} \ .$$

Using lemma 2.2.4 and equations (25), we have

$$\dot{\mathrm{p}} = \int_{-\infty}^{0} \mathrm{p} \cdot [\mathrm{p}, \dot{\nabla}_{\partial_s}] \circ \varphi_s \, \mathrm{d}s - \int_{0}^{\infty} [\mathrm{p}, \dot{\nabla}_{\partial_s}] \cdot \mathrm{p} \circ \varphi(s) \, \mathrm{d}s \ , \tag{26}$$

which yields (using the fact that $\mathrm{p}_0 = \mathrm{p}$):

$$\dot{\mathrm{p}}_0 = [\dot{A}, \mathrm{p}_0] + \int_{-\infty}^{0} \mathrm{p}_0 \cdot [\mathrm{p}_0, \dot{\nabla}_{\partial_s}] \circ \varphi_s \, \mathrm{d}s - \int_{0}^{\infty} [\mathrm{p}_0, \dot{\nabla}_{\partial_s}] \cdot \mathrm{p}_0 \circ \varphi(s) \, \mathrm{d}s \ . \tag{27}$$

From equation (24), we get that

$$\int_0^\infty [\mathrm{p}_0, \dot{\nabla}_{\partial_s}] \cdot \mathrm{p}_0 \circ \varphi(s) \, \mathrm{d}s = \int_{g^+} [\mathrm{p}_0, \mathrm{d}\dot{A}] \cdot \mathrm{p}_0 = -\int_{g^+} [\mathrm{d}\dot{A}, \mathrm{p}_0] \cdot \mathrm{p}_0 \ ,$$

while

$$\int_{-\infty}^0 \mathrm{p}_0 \cdot [\mathrm{p}_0, \dot{\nabla}_{\partial_s}] \circ \varphi(s) \, \mathrm{d}s = -\int_{g^-} \mathrm{p}_0 \cdot [\mathrm{p}_0, \mathrm{d}\dot{A}] = \int_{g^-} \mathrm{p}_0 \cdot [\mathrm{d}\dot{A}, \mathrm{p}_0] \ .$$

This concludes the proof of proposition 2.2.1.

2.3. **Proof of the stability lemma 2.1.10.** Let us first choose a continuous family of gauge transformations $(g_t)_{t \in]-\varepsilon, \varepsilon[}$ so that $g_t^* h_t = h$. The bounded variation condition implies that for a given T, for any α, there exists β so that $|s| \leqslant \beta$, implies that

$$\|\Phi_T - \Phi_T^s\| \leqslant \alpha \ ,$$

where Φ_T^s is the parallel transport at time T for ∇^s and the norm is computed with respect to h. Thus from lemma 2.1.7, for α small enough, Φ_T^s preserves $\mathcal{U}$ and is 3/4-Lipschitz, while the same holds for Φ_{-T}^s and $\mathcal{V}$. This implies that for $|s| \leqslant \beta$, (∇_s, h) is a uniformly hyperbolic bundle.

By the C^k bounded variation hypothesis, Φ_{-T}^s is a C^k-family of contracting maps, hence the fixed section is itself C^k as a function of s. This proves that the fundamental projector varies C^k in s.

2.4. **Θ-Uniformly hyperbolic bundles.** We now generalize the situation described in the previous paragraphs, using the same notational convention. Let V be a finite dimension vector space, let $\Theta = (K_1, \ldots, K_n)$ be a strictly increasing n-tuple so that

$$1 \leqslant K_1 < \ldots < K_n < \dim(V) \ .$$

Then a Θ-*uniformly hyperbolic bundle* over UH^2 is given by a pair (∇, h) for which there exists a $\left(\Phi_t^\nabla\right)_{t \in \mathbb{R}}$-invariant decomposition

$$E_0 = E_1 \oplus \ldots \oplus E_{n+1} \ ,$$

so that (∇, h) is uniformly of rank K_a for all a in $\{1, \ldots, n\}$ with invariant decomposition given by

$$E_0 = F_\mathrm{a} \oplus F_\mathrm{a}^\circ \ , \text{ with } F_\mathrm{a} = E_1 \oplus \ldots \oplus E_\mathrm{a} \ , F_\mathrm{a}^\circ = E_{\mathrm{a}+1} \oplus \ldots \oplus E_{n+1} \ .$$

The flag $(F_1, \ldots, F_n)$ will be called a Θ-*flag*.

In other words, we generalized the situation described before for Grassmannians to flag varieties.

2.5. **Projectors and notation.** In this section, we will work in the context of a Θ-uniformly hyperbolic bundle $\rho = (\nabla, h)$ associated to a decomposition of a trivializable bundle

$$E = E_1 \oplus \cdots \oplus E_{n+1} \ .$$

Let us denote $k_{\mathrm{a}} := \dim(E_{\mathrm{a}})$ and $K_{\mathrm{a}} := k_1 + \ldots k_{\mathrm{a}}$ so that $\Theta = (K_1, \ldots, K_n)$.

We then write for a geodesic g,

$$\mathrm{p}^{\mathrm{a}}(g) \ ,$$

the projection on $F_{\mathrm{a}} = E_1 \oplus \ldots \oplus E_{\mathrm{a}}$ parallel to $F^{\circ}_{\mathrm{a}} := E_{\mathrm{a}+1} \oplus \ldots \oplus E_{n+1}$.

When g is a phantom geodesic we set the convention that $\mathrm{p}^{\mathrm{a}}(g) := \mathrm{Id}$.

Observe that all $\mathrm{p}^{\mathrm{a}}(g)$ are well defined projectors in the finite dimensional vector space V which is the space of ∇-parallel sections of E. Or in other words the vector space so that in the trivialization given by ∇, $E = V \times \mathsf{U}\mathbf{H}^2$.

Finally, we will consider a Θ-geodesic g, given by a geodesic g_0 labelled by an element a of Θ and write

$$\mathrm{p}(g) := \mathrm{p}^{\mathrm{a}}(g_0) \ , \ \Theta_g = \mathrm{Tr}(\mathrm{p}^{\mathrm{a}}) = K_{\mathrm{a}} \ . \tag{28}$$

2.6. **The periodic case.** Let Σ be the universal cover of a closed surface S. We denote by π the projection from Σ to S and p the projection from $\mathsf{U}\Sigma$ to Σ.

Let Γ be the fundamental group of S and ρ be a projective Anosov representation of Γ on some vector space $\mathcal{E}$. Let E be the associated flat bundle on S with connection ∇.

We will use in the sequel the associated trivialization of the bundle $E_0 = p^*\pi^*E$ on which ∇ is trivial. Let us choose a Γ-invariant euclidean metric g on the bundle E_0. Let us finally choose a orthonormal frame h for g so that $g = g_h$.

It follows from the definition of projective Anosov representations that the corresponding bundle (∇, h) is uniformly hyperbolic. We call such a uniformly hyperbolic bundle *periodic*.

More generally, let P_Θ be the parabolic group stabilizing a Θ-flag. Then a P_Θ-Anosov representation defines a Θ-uniformly hyperbolic bundle.

Finally we observe

- Given a representation ρ, a different choice of a Γ-invariant metric yields an equivalent uniformly hyperbolic bundle.
- Similarly, two conjugate representations give equivalent uniformly hyperbolic bundles.

3. Ghost polygons

We introduce here our main tools, *ghost polygons*, and relate them to configurations of geodesics and correlation functions. This section is mainly concerned with definitions and notation.

We will consider the space C of oriented geodesics of $\mathbf{H}^2$, and an oriented geodesic g as a pair (g^-, g^+) consisting of two distinct points in $\partial_\infty \mathbf{H}^2$.

This section consists mainly of definitions: *ghost polygons, Θ-decorated ghost polygons*, and related useful notions notably in the presence of a uniformly hyperbolic bundle (*opposite endormorphisms, core diameter*). In the periodic (Anosov) case we finally show the analyticity of such correlation functions.

3.1. **Ghost polygons.** A *ghost polygon* is a cyclic collection of geodesics $\vartheta = (\theta_1, \ldots, \theta_{2p})$. The *ghost edges* are the geodesics (possibly phantom) θ_{2i+1} , and the *visible edges* are the even labelled edges θ_{2i}, such that

$$\theta^+_{2i+1} = \theta^+_{2i} \ , \ \ \theta^-_{2i-1} = \theta^-_{2i} \ .$$

Remarks:

(1) The geodesics are allowed to be phantom geodesics,
(2) It will be convenient some time to relabel the ghost edges as $\zeta_i =: \theta_{2i+1}$.
(3) It follows from our definition that $(\bar{\theta}_1, \theta_2, \bar{\theta}_3, \ldots, \theta_{2p})$ is closed ideal polygon.

We have an alternative point of view. A *configuration of geodesics of rank p* is just a finite cyclically ordered set of p-geodesics. We denote the cyclically ordered set of geodesics $(g_1,\ldots,g_p)$ by $\lceil g_1,\ldots,g_p\rceil$. The cardinality of the configuration is called the *rank* of the the configuration.

We see that the data of a ghost polygon and a configuration of geodesics is equivalent (see figure (1)):

- we can remove the ghost edges to obtain a configuration of geodesics from a ghost polygon,
- conversely, given any configuration $G=(g_1,\ldots,g_p)$, the associated ghost polygon $\vartheta=(\theta_1,\ldots,\theta_{2p})$ is given by $\theta_{2i}:=g_i$, $\theta_{2i+1}:=(g^-_{i+1},g^+_i)$

We finally say that two configurations are *non-intersecting* if their associated ghost polygons do not intersect.

Let us add some convenient definitions. Let $\vartheta=(\theta_1,\ldots,\theta_{2p})$ be a ghost polygon associated to the configuration configuration $\lceil g_1,\ldots,g_p\rceil$, We then define the opposite configurations as follows.

- For visible edge g_1 of G, the *opposite configuration* is tuple $g^*_1:=(g_1,g_2,\ldots,g_p,g_1)$.
- For ghost edge θ_1 of G, the *ghost opposite configuration* is the tuple $\theta^*_1:=(g_2,\ldots,g_p,g_1)$.

Observe that both opposite configurations are not configurations per se but actually tuples – or *ordered configurations*.

We finally define the *core diameter* $r(G)$ of a ghost polygon G to be the minimum of those R such that, if $B(R)$ is the ball of radius R centered at the barycenter $\mathrm{Bary}(G)$, then $B(R)$ intersects all visible edges. We obviously have

Proposition 3.1.1. *The map $G\mapsto r(G)$ is a continuous and proper map from $\mathcal{C}^n_\star/\mathsf{PSL}_2(\mathbb{R})$ to $\mathbb{R}$.*

3.1.1. Θ-*Ghost polygons.* We now Θ-decorate the situation. As in paragraph 2.4, let $\Theta=(K_1,\ldots,K_n)$ with $K_{\rm a}<K_{\rm a+1}$. Let G be a ghost polygon, a Θ-decoration is a map $\mathbf{a}$ from the set of visible edges to $1,\ldots,n$.

We again have the equivalent description in terms of configurations. A Θ-*configuration of geodesics of rank p* is configuration $(g_1,\ldots,g_p)$ with a map $\mathbf{a}$ – the Θ-decoration – from the collection of geodesics to $\{1,\ldots,n\}$. We think of a Θ-*decorated geodesic*, or in short a Θ-*geodesic*, as a geodesic labelled with an element of Θ.

3.2. Ghost polygons and uniformly hyperbolic bundles.

When ρ is a Θ-uniformly hyperbolic bundle and $\mathrm{p}^{\rm a}(g)$ a fundamental projector associated to a geodesic g, we will commonly use the following shorthand.

Let G be ghost polygon $(\theta_1,\theta_2,\ldots,\theta_{2p})$ be given by configuration $\lceil g_1,\ldots,g_p\rceil$.

(1) For visible edge g_i we write $\mathrm{p}_i:=\mathrm{p}^{\mathbf{a}}_i:=\mathrm{p}^{\mathbf{a}(i)}(g_i)$.
(2) For visible edge g_i, the *opposite ghost endomorphism* is
$$\mathrm{p}^{\mathbf{a}}_G(g^*_j):=\mathrm{p}_j\cdot\mathrm{p}_{j-1}\cdots\mathrm{p}_{j+1}\cdot\mathrm{p}_j\,. \tag{29}$$
(3) For ghost edge ζ_i, the *opposite ghost endormorphism* is
$$\mathrm{p}^{\mathbf{a}}_G(\zeta^*_i):=\mathrm{p}_i\cdot\mathrm{p}_{i-1}\ldots\mathrm{p}_{i+1}\,. \tag{30}$$
The reader should notice that in the product above, the indices are decreasing.

The opposite ghost endomorphisms have a simple structure in the context of projective uniformly hyperbolic bundles (that is when $\Theta=\{1\}$).

Lemma 3.2.1 (Opposite endormorphisms)**.** *When $\Theta=\{1\}$, $\mathrm{p}_G(\theta^*_i)=\mathsf{T}_G(\rho)\,\mathrm{p}(\theta_i)$.*

Proof. Let $G=(\theta_1,\ldots,\theta_{2p})$ be a ghost polygon with configuration $\lceil g_1,\ldots,g_p\rceil$. If $g^+_i=g^-_{i+1}$ then $p_{i+1}p_i=0$ and the equality holds trivially with both sides zero. We thus can assume there is a ghost edge $\zeta_i=\theta_{2i+1}$ for each $i\in\{1,\ldots,p\}$.

When $\Theta=\{1\}$ all projectors have rank 1. Thus for visible edge g_i
$$\mathrm{p}_G(g^*_i)=\mathrm{p}_i\mathrm{p}_{i-1}\ldots\mathrm{p}_{i+1}\mathrm{p}_i \quad=\quad \mathrm{Tr}(\mathrm{p}_i\ldots\mathrm{p}_{i+1})\,\mathrm{p}_i=\mathsf{T}_G(\rho)\,\mathrm{p}(g_i)\,.$$

For a ghost edge ζ_i as $\mathrm{Tr}(\mathrm{p}_{i+1}\mathrm{p}_i) \neq 0$

$$\mathrm{p}_G(\zeta_i^*) = \mathrm{p}_i\mathrm{p}_{i-1}\dots\mathrm{p}_{i+1} \quad = \quad \mathrm{p}_i\mathrm{p}_{i+1}\frac{\mathrm{Tr}(\mathrm{p}_n\dots\mathrm{p}_1)}{\mathrm{Tr}(\mathrm{p}_i\mathrm{p}_{i+1})} = \mathsf{T}_G(\rho)\,\mathrm{q}\,,$$

where $\mathrm{q} =: \frac{1}{\mathrm{Tr}(\mathrm{p}_i\mathrm{p}_{i+1})}\mathrm{p}_i\mathrm{p}_{i+1}$. Then we see that q has trace 1, its image is the image of p_i, and its kernel is the kernel of p_{i+1}. Thus q is the rank 1 projector on the image of p_i, parallel to the kernel of p_{i+1}. Hence $\mathrm{q} = \mathrm{p}(\zeta_i)$. The result follows. ◻

3.3. **Correlation function.** Given a Θ-configuration of geodesics $G = \lceil g_1, \dots, g_p \rceil$ given by a p-uple of geodesics $(g_1^0, \dots, g_p^0)$, with a Θ-decoration **a** the *correlation function* associated to G is

$$\mathsf{T}_G : \rho \mapsto \mathsf{T}_{\lceil g_1,\dots,g_p \rceil}(\rho) := \mathrm{Tr}\left(\mathrm{p}^{\mathrm{a}(p)}(g_p^0)\cdots\mathrm{p}^{\mathrm{a}(1)}(g_1^0)\right) = \mathrm{Tr}\left(\mathrm{p}(g_p)\cdots\mathrm{p}(g_1)\right)\,, \tag{31}$$

where p is the projector associated to the uniformly hyperbolic bundle ρ. The reader should notice (again) that the geodesics and projectors are ordered reversely.

3.4. **Analyticity in the periodic case.** In this subsection we will treat first the case of complex bundles, that is representation in $\mathsf{SL}(n,\mathbb{C})$ of the (complex) parabolic group $\mathsf{P}_\Theta^\mathbb{C}$ associated to Θ. We now have, as a consequence of [8, Theorem 6.1], the following

Proposition 3.4.1. *Let G be a ghost polygon. Let $(\rho_u)_{u\in\mathrm{D}}$ be an analytic family of $\mathsf{P}_\Theta^\mathbb{C}$-Anosov representations parametrized by the unit disk D. Then, the function $u \mapsto \mathsf{T}_G(\rho_u)$ is analytic. Moreover the map $G \mapsto \mathsf{T}_G$ is a continuous function with values in the analytic functions.*

Proof. Indeed the correlation functions only depends on the limit curve of the representation and thus the analyticity of the limit curve proved in [8, Theorem 6.1] gives the result. ◻

We deduce the general analyticity result from this proposition by complexifying the representation.

4. Ghost integration

In this section, given a Θ-uniformly hyperbolic bundle ρ, a Θ-ghost polygon G and a 1-form α on $\mathbf{H}^2$ with values in the endormorphism bundle of a uniformly hyperbolic bundle , we produce a complex (or real) number denoted

$$\oint_{\rho(G)} \alpha\,.$$

This procedure is called *ghost integration*. The construction is motivated by the following formula that allows us to compute the variation of a correlation function with respect to a variation of uniformly hyperbolic bundles:

$$\mathrm{d}\mathsf{T}_G\left(\dot{\nabla}\right) = \oint_{\rho(G)} \dot{\nabla}\,.$$

(1) In order to define ghost integration, we first have to consider which type of forms we wish to integrate. This is done in paragraph 4.1.
(2) In paragraph 4.2 we define *line integration*, a procedure reminiscent of how one gets the solution of the cohomogical solution: integrating a “contracting part" towards the future and a “dilating part" towards the past. We, in particular show some crucial convergence properties of the line integration in lemma 4.2.2.
(3) We then define the ghost integration in paragraph 4.1, using the line integration as a building block.
(4) In paragraph 4.4, we obtain other formulae depending on the type of forms considered, or whether we are in the projective case or not.

(5) In paragraph 4.5, we introduce the dual cohomology object $\Omega_{\rho(G)}$, the *ghost dual form to a ghost polygon*, which is a 1-form with values in the endomorphism bundle so that

$$\int_{\mathbf{H}^2} \operatorname{Tr}\left(\alpha \wedge \Omega_{\rho(G)}\right) = \oint_{\rho(G)} \alpha \ .$$

(6) We finally achieve one of our goals by relating ghost integration to the derivative of correlation functions in paragraph 4.6.

4.1. **Bounded and geodesically bounded forms.** In this paragraph, we define a certain type of 1-forms with values in $\operatorname{End}(E)$, where E is a uniformly hyperbolic bundle (∇, h). All norms and metrics will be using the Euclidean metric g_h on E associated to a framing h.

Definition 4.1.1 (Bounded forms). *A* bounded 1-form *ω on $\mathbf{H}^2$ with values in* $\operatorname{End}(E)$ *is a form so that $\|\omega_x(u)\|_x$ is bounded uniformly for all (x,u) in $\mathsf{U}\mathbf{H}^2$. Let us denote $\mathbf{\Lambda}^\infty(E)$ the vector spaces of those forms and*

$$\|\omega\|_\infty = \sup_{(x,u)\in\mathsf{U}\mathbf{H}^2} \|\omega_x(u)\|_x \ .$$

As an example of such forms, we have

(1) Given a Θ-geodesic g, given by a (possibly phantom) geodesic g_0, and an element a of Θ, the *projector form* is
$$\beta_{\rho(g)} := \omega_g \ \mathrm{p}(g) = \omega_g \ \mathrm{p}^{\mathrm{a}}(g_0) \ . \tag{32}$$
where we used the notation (28).
(2) Any Γ-equivariant continuous form in the case of a periodic bundle.
(3) Given $(A_t)_{t\in]-1,1[}$ a bounded variation of a uniformly hyperbolic bundle (see definition 2.1.9, the form
$$\dot{A} := \left.\frac{\partial A_t}{\partial t}\right|_{t=0} ,$$
is by definition a bounded 1-form.

We do not require forms in $\mathbf{\Lambda}^\infty(E)$ to be closed.

Definition 4.1.2 (Geodesically bounded forms). *A form α is* geodesically bounded *if for any parallel section A of* $\operatorname{End}(E)$, $\operatorname{Tr}(\alpha A)$ *is geodesically bounded as in definition 1.3.1. We denote by $\Xi(E)$ the set of 1-forms which are geodesically bounded.*

Again for any geodesic, the projector form $\beta_{\rho(g)}$ is geodesically bounded. However Γ-equivariant forms are never geodesically bounded unless they vanish everywhere.

4.2. **Line integration.** Let ω be a 1-form in $\mathbf{\Lambda}^\infty(E)$. Let x be a point on the oriented geodesic g and Q a parallel section of $\operatorname{End}(E)$ along g. The *line integration* of ω – with respect to the uniformly hyperbolic bundle ρ – is given by

$$\boldsymbol{S}_{x,g,\mathrm{Q}}(\omega) \quad := \quad \int_{g^+} \operatorname{Tr}\left(\mathrm{Q}\,[\omega,\mathrm{p}]\,\mathrm{p}\right) + \int_{g^-} \operatorname{Tr}\left(\mathrm{Q}\,\mathrm{p}\,[\omega,\mathrm{p}]\right) \ . \tag{33}$$

Observe that since for a projector p, we have

$$\operatorname{Tr}\left(A\,\mathrm{p}\,[B,\mathrm{p}]\right) = \operatorname{Tr}\left([\mathrm{p},A]\,\mathrm{p}\,B\right) \ ,$$

we have the equivalent formulation

$$\boldsymbol{S}_{x,g,\mathrm{Q}}(\omega) \quad = \quad \int_{g^+} \operatorname{Tr}\left(\omega\,\mathrm{p}\,[\mathrm{p},\mathrm{Q}]\right) + \int_{g^-} \operatorname{Tr}\left(\omega\,[\mathrm{p},\mathrm{Q}]\,\mathrm{p}\right) \ . \tag{34}$$

Now let α be a section of $\operatorname{End}(E)$ so that $\mathrm{d}\alpha$ belongs to $\mathbf{\Lambda}^\infty(E)$. We also define the *primitive line integration* of α by

$$\begin{aligned} \boldsymbol{J}_{x,g,\mathrm{Q}}(\alpha) \quad &:= \quad \operatorname{Tr}\left(\alpha(x)\,[\mathrm{p},\mathrm{Q}]\right) + \boldsymbol{S}_{x,g,\mathrm{Q}}(\mathrm{d}\alpha) \\ &= \quad \operatorname{Tr}\left([\alpha(x),\mathrm{p}]\,\mathrm{Q}\right) + \boldsymbol{S}_{x,g,\mathrm{Q}}(\mathrm{d}\alpha) \ . \end{aligned}$$

4.2.1. *Bounded linear forms and continuity.*

Proposition 4.2.1 (Continuity). *The line integration operator*

$$\omega \mapsto \boldsymbol{S}_{x,g,Q}(\omega)\ ,$$

is a continuous linear form on $\boldsymbol{\Lambda}^\infty(E)$.

This proposition is an immediate consequence of the following lemma

Lemma 4.2.2 (Exponential decay). *There exist positive constants* B *and* b*, only depending on* Q *and* x*, so that for any* ω *in* $\boldsymbol{\Lambda}^\infty(E)$ *if* y *is a point in* g^+*,* z *a point in* g^- *and denoting* ∂_t *the tangent vector to the geodesic* g*, then*

$$\begin{aligned} \left|\mathrm{Tr}\left(\mathrm{Q}\,[\omega_y(\partial_t),\mathrm{p}]\,\mathrm{p}\right)\right| &\leqslant Be^{-bd(x,y)}\|\omega\|_\infty\ , & (35)\\ \left\|[\mathrm{Q},\mathrm{p}]\,\mathrm{p}\right\|_z &\leqslant Be^{-bd(x,z)}\ , & (36)\\ \left\|\mathrm{p}\,[\mathrm{Q},\mathrm{p}]\right\|_y &\leqslant Be^{-bd(x,y)}\ . & (37)\end{aligned}$$

Proof. Let us choose a trivialization of E so that ∇ is trivial. By hypothesis ω is in $\boldsymbol{\Lambda}^\infty(E)$ and thus

$$\|\omega_y(\partial_t)\|_y \leqslant \|\omega\|_\infty\ . \tag{38}$$

Then

$$\sigma : y \mapsto \sigma(y) := [\omega_y(\partial_t),\mathrm{p}]\,\mathrm{p}\ ,$$

is a section of F_0^-. Since p is bounded – see proposition 2.1.4 – there exists k_1 such that for all y

$$\|\sigma(y)\|_y \leqslant k_1\|\omega\|_\infty\ .$$

By lemma 2.2.2, F_0^- is a contracting bundle in the negative direction, which means there exists positive constants A and a so that if $y = \varphi_t(x)$ with $t > 0$, then

$$\|\Phi^\nabla_{-t}(\sigma(y))\|_x \leqslant Ae^{-at}\|\sigma(y)\|_y\ ,$$

where ∇ is the connection. However in our context, since we have trivialized the bundle, Φ^∇_{-t} is the identity fiberwise, and thus combining the previous remarks we get that if y is in g^+, then

$$\left\|[\omega_y(\partial_t),\mathrm{p}]\,\mathrm{p}\right\|_x \leqslant Ae^{-a(d(y,x)}\|\omega\|_\infty\ . \tag{39}$$

By Cauchy–Schwarz, for all endomorphisms U and V, we have

$$|\,\mathrm{Tr}(U\,V)| \leqslant \|U\|_x\|V\|_x\ . \tag{40}$$

Thus combining equations (40) and (39) we obtain

$$\left|\mathrm{Tr}\left([\omega_y(\partial_t),\mathrm{p}]\,\mathrm{p}\,\mathrm{Q}\right)\right| \leqslant \|\mathrm{Q}\|_x\ \|[\omega_y(\partial_t),\mathrm{p}]\,\mathrm{p}\|_x \leqslant Ae^{-a(d(y,x)}\|\mathrm{Q}\|_x\ \ \|\omega\|_\infty\ ,$$

and the inequality (35) follows. Similarly, $[\mathrm{Q},\mathrm{p}]\mathrm{p}$ is a parallel section of F_0^-, thus the inequality (35) is an immediate consequence of inequality 20. ☐

4.2.2. *Properties of the primitive line integration.* We explain now two properties of the primitive line integration that will be useful in the definition of the ghost intergration;

Proposition 4.2.3. *The primitive line integration* $\boldsymbol{J}_{x,g,Q}(\alpha)$ *does not depend on the choice of* x *on* g.

Proof. Let us write for the sake of this proof $\boldsymbol{J}_x := \boldsymbol{J}_{x,g,Q}(\alpha)$. Let μ be the geodesic arc from y to x. Let us consider a parametrization of g so that $x = g(s_0)$ and $y = g(t_0)$. Then letting $\omega = \mathrm{d}\alpha$

$$\begin{aligned} \boldsymbol{J}_y - \boldsymbol{J}_x &= \mathrm{Tr}\left((\alpha(y)-\alpha(x))\,[\mathrm{p},\mathrm{Q}]\right)\\ &+ \int_{t_0}^{\infty}\mathrm{Tr}\left(\omega(\dot{g})\,\mathrm{p}\,[\mathrm{p},Q]\right)\mathrm{d}t + \int_{t_0}^{-\infty}\mathrm{Tr}\left(\omega(\dot{g})\,[\mathrm{p},Q]\,\mathrm{p}\right)\mathrm{d}t\\ &- \int_{s_0}^{\infty}\mathrm{Tr}\left(\omega(\dot{g})\,\mathrm{p}\,[\mathrm{p},Q]\right)\mathrm{d}t - \int_{s_0}^{-\infty}\mathrm{Tr}\left(\omega(\dot{g})\,[\mathrm{p},Q]\,\mathrm{p}\right)\mathrm{d}t\end{aligned}$$

$$= \int_{s_0}^{t_0} \mathrm{Tr}\left(\omega(\dot{g})\ ([\mathrm{p}, Q] - \mathrm{p}\,[\mathrm{p}, Q] - [\mathrm{p}, Q]\,\mathrm{p})\right)\,\mathrm{d}s = 0\ ,$$

where the last equality comes form the fact that, since p is a projector

$$[\mathrm{p}, Q]\,\mathrm{p} + \mathrm{p}\,[\mathrm{p}, Q] = [\mathrm{p}, Q]\ . \quad \square$$

Finally we have,

Proposition 4.2.4. *Assume that β is bounded. Then $\boldsymbol{J}_{m,Q}(\beta) = 0$.*

Proof. Let $\varpi = \mathrm{d}\beta$. It follows that

$$\mathrm{Tr}\,(\varpi(\partial_t)\,\mathrm{p}\,[\mathrm{p},\mathrm{Q}]) = \frac{\partial}{\partial t}\,\mathrm{Tr}\,(\beta\,\mathrm{p}\,[\mathrm{p},\mathrm{Q}])\ .$$

Thus by the exponential decay lemma 4.2.2, we have

$$\int_{g^+} \mathrm{Tr}(\varpi\,\mathrm{p}\,[\mathrm{p},\mathrm{Q}]) = -\,\mathrm{Tr}(\beta(x)\,\mathrm{p}\,[\mathrm{p},\mathrm{Q}])\ .$$

Similarly

$$\int_{g^-} \mathrm{Tr}(\varpi\,\mathrm{p}\,[\mathrm{p},\mathrm{Q}]) = -\,\mathrm{Tr}(\beta(x)\,[\mathrm{p},\mathrm{Q}]\,\mathrm{p})\ .$$

It follows that

$$\boldsymbol{S}_{x,g_0,Q}(\varpi) = -\,\mathrm{Tr}(\beta(x)\,\mathrm{p}\,[\mathrm{p},\mathrm{Q}]) - \mathrm{Tr}(\beta(x)\,[\mathrm{p},\mathrm{Q}]\,\mathrm{p}) = -\,\mathrm{Tr}(\beta(x)\,[\mathrm{p},\mathrm{Q}])\ .$$

This concludes the proof. $\square$

4.3. **Ghost integration: the construction.** Let now G be a configuration of geodesics with a Θ-decoration $\mathbf{a}$. Let ρ be a Θ-uniformly hyperbolic bundle, where $G = \lceil g_1, \ldots, g_p \rceil$. Let $\mathrm{p}_i = \mathrm{p}^{\mathbf{a}(i)}(g_i)$ and

$$\mathrm{P}_i = \mathrm{p}_{i-1} \ldots \mathrm{p}_{i+1}\ .$$

Let α be a closed 1-form with values in $\mathrm{End}(E)$. Assume that α belongs to $\boldsymbol{\Lambda}^\infty(E)$. Let β be a primitive of α – that is a section of $\mathrm{End}(E)$ so that $\mathrm{d}\beta = \alpha$ – let

$$\boldsymbol{J}_{\rho(G)}(\beta) \coloneqq \sum_{i=1}^{n} \boldsymbol{J}_{g_i,\mathrm{P}_i}(\beta)\ ,$$

Proposition 4.3.1. *The quantity $\boldsymbol{J}_{\rho(G)}(\beta)$ only depends on the choice of α and not of its primitive.*

Proof. Let β_0 and β_1 two primitives of α. Observe that $B \coloneqq \beta_1 - \beta_0$ is constant, then

$$\boldsymbol{J}_G(\beta_1) - \boldsymbol{J}_G(\beta_0) = \sum_{i=1}^{p} \mathrm{Tr}\,(B[\mathrm{p}_i, \mathrm{P}_i]) = \sum_{i=1}^{p} \mathrm{Tr}\,(B\mathrm{p}_i\mathrm{P}_i) - \sum_{i=1}^{p} \mathrm{Tr}\,(B\mathrm{P}_i\mathrm{p}_i) = 0\ ,$$

since $\mathrm{P}_i\mathrm{p}_i = \mathrm{p}_{i-1}\mathrm{P}_{i-1}$. $\square$

Definition 4.3.2 (Ghost integration)**.** *We define the* ghost integration *of a 1-form α in $\boldsymbol{\Lambda}^\infty(E)$ with respect to a Θ-ghost polygon G and a uniformly hyperbolic bundle ρ to be the quantity*

$$\oint_{\rho(G)} \alpha \coloneqq \boldsymbol{J}_{\rho(G)}(\beta)\ ,$$

where β is a primitive of α.

Gathering our previous results, we summarize the important properties of ghost integration:

Proposition 4.3.3. *The ghost integration enjoys the following properties:*

(1) *The map $\alpha \mapsto \oint_{\rho(G)} \alpha$ is a continuous linear form on $\boldsymbol{\Lambda}^\infty(E)$.*

(2) *Assume $\alpha = \mathrm{d}\beta$, where β is a bounded section of* $\mathrm{End}(E)$*. Then*

$$\oint_{\rho(G)} \alpha = 0 .$$

We remark that the second item implies that ghost integration is naturally an element of the dual of the first bounded cohomology with coefficients associated to the bundle.

Proof. These are consequences of the corresponding properties for $J_{x,g,Q}$ proved respectively in propositions 4.2.1, 4.2.4 and 7.1.1. ◻

4.4. **Ghost integration of geodesic forms.** Recall that we denoted by $\Xi(E)$ the space of geodesically bounded forms, and observe that for any geodesic g, the projector form $\beta_{\rho(g)}$ belongs to $\Xi(E)$.

Proposition 4.4.1 (Alternative formula)**.** *Let ρ be a Θ-uniformly hyperbolic bundle. Let G be configuration of geodesics of rank p associated to a ghost polygon $\vartheta := (\theta_1, \ldots \theta_{2p})$ and a Θ-decoration. Assume that α is in $\Xi(E)$. Then*

$$\oint_{\rho(G)} \alpha \quad = \quad -\left(\sum_{i=1}^{2p} (-1)^i \int_{\theta_i} \mathrm{Tr}\left(\alpha \, \mathrm{p}_G(\theta_i^*)\right)\right) ,$$

where $\mathrm{p}^{\mathbf{a}}_G(\theta_i^*)$ denotes the opposite ghost endomorphism to θ_i.

In the context of projective uniformly hyperbolic bundle, that is $\Theta = \{1\}$, then the previous formula is much simpler as an immediate consequence of lemma 3.2.1.

Proposition 4.4.2 (Projective formula)**.** *Let G be configuration of geodesics of rank p associated to a ghost polygon $\vartheta := (\theta_1, \ldots \theta_{2p})$ and a Θ-decoration. Let ρ be a projective uniformly hyperbolic bundle. Assume that α is in $\Xi(E)$. Then*

$$\oint_{\rho(G)} \alpha \quad = \quad -\mathsf{T}_G(\rho)\left(\sum_{i=1}^{2p} (-1)^i \int_{\theta_i} \mathrm{Tr}\left(\alpha \, \mathrm{p}(\theta_i)\right)\right) .$$

Observe that both formulae above do not make sense for a general bounded form. Observe also that

Proposition 4.4.3. *Let G be a ghost polygon, and α a 1-form with values in the center of* $\mathrm{End}(E)$ *then*

$$\oint_{\rho(G)} \alpha = 0 .$$

4.4.1. *An alternative construction: a first step.*

Proposition 4.4.4. *Let x be a point in $\mathbf{H}^2$, $\gamma_i^\pm$ the geodesic from x to $g_i^\pm$. Assume that α is in $\Xi(E)$ then*

$$\oint_{\rho(G)} \alpha \quad = \quad \sum_{i=1}^{p}\left(\int_{\gamma_i^+} \mathrm{Tr}\left(\alpha \, \mathrm{p}_i \, [\mathrm{p}_i, \mathrm{P}_i]\right) + \int_{\gamma_i^-} \mathrm{Tr}\left(\alpha \, [\mathrm{p}_i, \mathrm{P}_i] \, \mathrm{p}_i \right)\right) .$$

Proof. Fix a point x_i in each of the g_i. Let β be a primitive of α so that $\beta(x) = 0$. Let η_i be the geodesic from x to x_i. It follows that, since α is geodesically bounded, we have by the cocycle formula (8)

$$\int_{\gamma_i^+} \mathrm{Tr}(\mathrm{P}_i \, [\alpha, \mathrm{p}_i] \, \mathrm{p}_i) = \int_{\eta_i} \mathrm{Tr}(\mathrm{P}_i \, [\alpha, \mathrm{p}_i] \, \mathrm{p}_i) + \int_{g_i^+} \mathrm{Tr}(\mathrm{P}_i \, [\alpha, \mathrm{p}_i] \, \mathrm{p}_i) .$$

Similarly

$$\int_{\gamma_i^-} \mathrm{Tr}(\mathrm{P}_i \, \mathrm{p}_i \, [\alpha, \mathrm{p}_i]) = \int_{\eta_i} \mathrm{Tr}(\mathrm{P}_i \, \mathrm{p}_i \, [\alpha, \mathrm{p}_i] + \int_{g_i^-} \mathrm{Tr}(\mathrm{P}_i \, \mathrm{p}_i \, [\alpha, \mathrm{p}_i]) .$$

Observe now that, using the relation $[\mathrm{p}, Q] \, \mathrm{p} + \mathrm{p} \, [\mathrm{p}, Q] = [\mathrm{p}, Q]$, we have

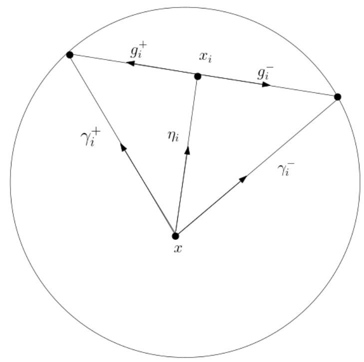


FIGURE 2. Curves $\gamma_i^\pm$ and η_i

$$\int_{\eta_i} \mathrm{Tr}(\mathrm{P}_i\,[\alpha,\mathrm{p}_i]\,\mathrm{p}_i) + \int_{\eta_i} \mathrm{Tr}(\mathrm{P}_i\,\mathrm{p}_i\,[\alpha,\mathrm{p}_i]) = \int_{\eta_i} \mathrm{Tr}(\mathrm{P}_i\,[\alpha,\mathrm{p}_i]) = \mathrm{Tr}\left(\mathrm{P}_i\,[\beta(x_i),\mathrm{p}_i]\right)\ .$$

Thus, we can now conclude the proof:

$$\begin{aligned} \boldsymbol{J}_{\rho(G)}(\beta) &= \sum_{i=1}^{p}\left(\int_{\gamma_i^+} \mathrm{Tr}\left(\mathrm{P}_i\,[\alpha,\mathrm{p}_i]\,\mathrm{p}_i\right) + \int_{\gamma_i^-} \mathrm{Tr}\left(\mathrm{P}_i\,\mathrm{p}_i\,[\alpha,\mathrm{p}_i]\right)\right) \\ &= \sum_{i=1}^{p}\left(\int_{\gamma_i^+} \mathrm{Tr}(\mathrm{p}_i\,[\mathrm{p}_i,\mathrm{P}_i]\,\alpha) + \int_{\gamma_i^-} \mathrm{Tr}([\mathrm{p}_i,\mathrm{P}_i]\,\mathrm{p}_i\,\alpha)\right)\ . \end{aligned}$$

□

Proof of proposition 4.4.1. Let us assume we have a ghost polygon $\vartheta = (\theta_1,\ldots,\theta_{2p})$ given by a configuration of geodesics $G = \lceil g_1,\ldots,g_p\rceil$. Let $\mathrm{p}_i = \mathrm{p}(g_i)$ and α an element of $\Xi(E)$. We have

$$\begin{aligned} \mathrm{p}_i\,[\mathrm{p}_i,\mathrm{P}_i] &= \mathrm{p}_i\mathrm{P}_i - \mathrm{p}_i\mathrm{P}_i\mathrm{p}_i\ , \\ [\mathrm{p}_i,\mathrm{P}_i]\,\mathrm{p}_i &= \mathrm{p}_i\mathrm{P}_i\mathrm{p}_i - \mathrm{P}_i\mathrm{p}_i = \mathrm{p}_i\mathrm{P}_i\mathrm{p}_i - \mathrm{p}_{i-1}\mathrm{P}_{i-1}\ . \end{aligned}$$

Since α is geodesically bounded we have

$$\oint_{\rho(G)} \alpha = \sum_{i=1}^{p}\left(\int_{\gamma_i^+} \mathrm{Tr}(\alpha\,\mathrm{p}_i\,\mathrm{P}_i\,) - \int_{\gamma_{i+1}^-} \mathrm{Tr}(\alpha\,\mathrm{p}_i\,\mathrm{P}_i)\right) - \sum_{i=1}^{p}\left(\int_{\gamma_i^+} \mathrm{Tr}(\alpha\,\mathrm{p}_i\,\mathrm{P}_i\,\mathrm{p}_i) - \int_{\gamma_i^-} \mathrm{Tr}(\alpha\,\mathrm{p}_i\,\mathrm{P}_i\,\mathrm{p}_i)\right)\ .$$

For $i \in \{1,\ldots,p\}$, let ζ_i be the ghost edge joining g_{i+1}^- to g_i^+, that is $\zeta_i = \theta_{2i+1}$. For a closed form β which is geodesically bounded the cocycle formula (8) yields

$$\int_{\gamma_i^+} \beta - \int_{\gamma_i^-} \beta = \int_{g_i} \beta\ \ ,\quad \int_{\gamma_{i+1}^-} \beta - \int_{\gamma_i^+} \beta = -\int_{\zeta_i} \beta\ .$$

Thus

$$\boldsymbol{J}_{\rho(G)}(\alpha) = \sum_{i=1}^{p}\left(\int_{\zeta_i} \mathrm{Tr}(\alpha\mathrm{p}_i\mathrm{P}_i) - \int_{g_i} \mathrm{Tr}(\alpha\mathrm{p}_i\mathrm{P}_i\mathrm{p}_i)\right)\ .$$

To conclude we need first to observe that as g_i is a visible geodesic then $\mathrm{p}_i\mathrm{P}_i\mathrm{p}_i$ is the opposite ghost endomorphism $\mathrm{p}_G(g_i^*)$. On the other hand as ζ_j is a ghost edge then $\mathrm{p}_j\mathrm{P}_j$ is the opposite ghost endomorphism $\mathrm{p}_G(\zeta_j^*)$. Thus

$$\boldsymbol{J}_{\rho(G)}(\alpha) = -\left(\sum_{i=1}^{2p}(-1)^i \int_{\theta_i} \mathrm{Tr}\left(\alpha\ \mathrm{p}_G(\theta_i^*)\right)\right)\ .$$

□

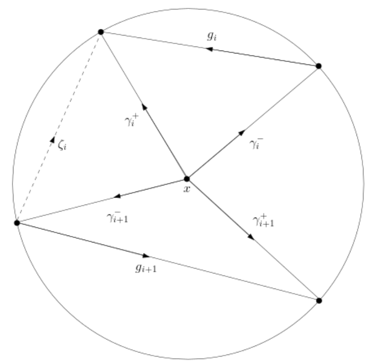


FIGURE 3. Appearance of ghosts

4.4.2. *Another altenative form with polygonal arcs.* Let $G = (\theta_1, \ldots, \theta_{2p})$ be a Θ ghost polygon given by configuration $\lceil g_1, \ldots, g_p \rceil$ with $g_i = \theta_{2i}$. Let x be the barycenter of G. Let x_i be the projection of x on g_i. For a ghost edge $\zeta_i = \theta_{2i+1}$, let us consider the polygonal arc η_i given by

$$\eta_i = a_i \cup b_i \cup c_i \cup d_i \ ,$$

where

- the geodesic arc a_i is the arc (along g_{i+1}) from g_{i+1}^- to x_{i+1},
- the geodesic arc b_i joins x_{i+1} to x,
- the geodesic arc c_i joins x to x_i,
- the geodesic arc d_i joins x_i to g_i^+.

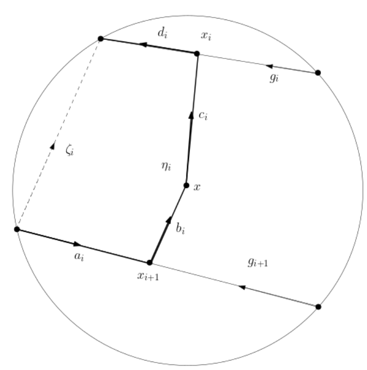


FIGURE 4. Polygonal arc η_i for ghost edge ζ_i

We then have, using the same notation as in proposition 4.4.1

Proposition 4.4.5 (ALTERNATIVE FORMULA II). *We have for α in $\Xi(E)$*

$$\oint_{\rho(G)} \alpha \;=\; -\sum_i \int_{g_i} \mathrm{Tr}\Big(\alpha\ \mathrm{p}_G(\theta_i^*)\Big) + \int_{\eta_i} \mathrm{Tr}\Big(\alpha\ \mathrm{p}_G(\theta_i^*)\Big)\ . \tag{41}$$

Proof. The proof relies on the fact that for α in $\Xi(E)$, and ζ_i a ghost edge we have

$$\int_{\eta_i} \alpha = \int_{\zeta_i} \alpha\ .$$

Then the formula follows from proposition 4.4.1. □

REMARK: GHOST INTEGRATION AND RHOMBUS INTEGRATION. The process described for the ghost integration is a generalization of the Rhombus integration described in [21].

4.5. **A dual cohomology class.** Let ρ be a Θ-uniformly hyperbolic bundle. Now let G be a Θ-ghost polygon with configuration $\lceil g_1,\ldots,g_p\rceil$ and Θ-decoration $\mathbf{a}$. Let $\vartheta=(\theta_1,\ldots,\theta_{2p})$ be the associated ghost polygon and denote by $\zeta_i=\theta_{2i+1}$ the ghost edges. Let η_i be the associated polygonal arc associated to the ghost edge ζ_i as in paragraph 1.4.

Definition 4.5.1. *The* ghost dual form to $\rho(G)$ *is*

$$\Omega_{\rho(G)}\coloneqq\sum_{i=1}^{p}\left(\omega_{g_i}\mathrm{p}_G(g_i^*)-\omega_{\eta_i}\mathrm{p}_G(\zeta_i^*)\right)\ .$$

Observe that $\rho(G)$ incorporates a Θ-decoration and so $\Omega_{\rho(G)}$ depends on the Θ-decoration. Actually $\Omega_\rho(G)$ depends on a choice of the mapping $g\mapsto\omega_g$.

Proposition 4.5.2 (ghost dual form). *We have the following properties*

(1) *The ghost dual form belongs to* $\Xi(E)$.
(2) *Assume that* α *belongs to* $\Xi(E)$. *Then*

$$\oint_{\rho(G)}\alpha=\int_{\mathbf{H}^2}\mathrm{Tr}\left(\alpha\wedge\Omega_{\rho(G)}\right)\ . \tag{42}$$

(3) (exponential decay inequality) *Finally, there exist positive constants* K *and* a *only depending on* ρ *and* R_0 *so that if the core diameter of* G *is less than* R_0, *then*

$$\|\Omega_{\rho(G)}(y)\|_y\leqslant Ke^{-a\ d(y,\mathrm{Bary}(G))}\ , \tag{43}$$

and, moreover, $\Omega_{\rho(G)}(y)$ *vanishes when* $d(y,\mathrm{Bary}(G))\geqslant R_0+2$ *and* $d(y,g)>2$ *for all visible edges* g *of* G.

Observe that even though $\Omega_{\rho(G)}$ depends on some choice, the left-hand side of the formula (42) does not depend on any choice. Later we will need the following corollary which we prove right after we give the proof of the proposition.

Corollary 4.5.3. *We have the following bounds: The map*

$$\varphi_G\ :\ y\ \ \mapsto\ \ \|\Omega_{\rho(G)}(y)\|_y\ ,$$

belongs to $L^1(\mathbf{H}^2)$, *and* $\|\varphi_G\|_{L^1(\mathbf{H}^2)}$ *is bounded by a continuous function of the core diameter of* G. *The map*

$$\psi_{G,y}\ :\ \gamma\ \ \mapsto\ \ \|\Omega_{\rho(G)}(\gamma y)\|_y\ ,$$

belongs to $\ell^1(\Gamma)$, *and* $\|\psi_{G,y}\|_{\ell^1(\Gamma)}$ *is bounded by a continuous function of the core diameter of* G. *Finally the map*

$$\varphi\ :H\ \ \mapsto\ \ \|\Omega_{\rho(H)}\|_\infty=\sup_{y\in\mathbf{H}^2}\|\Omega_{\rho(H)}(y)\|_y\ ,$$

is bounded on every compact set of $\mathcal{G}^p_\star$

Proof of proposition 4.5.2. We first prove the exponential decay inequality (43) which implies in particular that $\Omega_\rho(G)$ belongs to $\mathbf{\Lambda}^\infty(E)$.

Let $r(G)$ be the core diameter of G. As usual, let g_i be a visible edges, x be the barycenter of all g_i and x_i be the projection of x on g_i. By the construction of the polygonal arc η_i, it follows that outside of the ball of radius $r(G)+2$ centered at x, that

$$\Omega_{\rho(G)}=\sum_i\omega^-_{g_i}[\mathrm{P}_i,\mathrm{p}_i]\mathrm{p}_i+\sum_i\omega^+_{g_i}\mathrm{p}_i[\mathrm{P}_i,\mathrm{p}_i]\ ,$$

where $\omega^\pm_{g_i}=f_i^\pm\omega_{g_i}$ where $f_i^\pm$ is a function with values in $[0,1]$ with support in the 2-neighbourhood of the arc $[x_i,g_i^\pm]$. Then the decay given in equation (43) is an immediate consequence of the exponential decay given in inequality (36).

Observe now that $\Omega_{\rho(G)}$ is closed. Let A be a parallel section of $\mathrm{End}(E)$, then it is easily seen that $\mathrm{Tr}(\Omega_{\rho(G)}A)$ is geodesically bounded. It follows that $\Omega_{\rho(G)}$ is in $\Xi(E)$.

Then the result follows from the alternative formula for ghost integration in proposition 4.4.5. □

Proof of corollary 4.5.3. Given a ghost polygon H whose set of visible edges is g_H, and core diameter less than R_0. Let

$$V_H \leqslant \{y \in \mathbf{H}^2 \mid d(y, \mathrm{Bary}(H)) \leqslant R_0 + 2 \text{ or } d(y, g) \leqslant 2 \text{ for some } g \in g_H\}$$

Observe that the volume of $V_H(R) := V_H \cap B(\mathrm{Bary}(H), R)$ has some linear growth as a function of R, and moreover this growth is controlled as a function of R_0. This, and the exponential decay inequality (43), implies that φ_G, whose support is in V_H, is in $L^1(\mathbf{H}^2)$ and that its norm is bounded by a constant that only depends on R_0. Similarly consider

$$F_{H,y} := \{\gamma \in \Gamma \mid d(\gamma(y), \mathrm{Bary}(H)) \leqslant R_0 + 2 \text{ or } d(\gamma(y), g) \leqslant 2 \text{ for } g \text{ in } g_H\} .$$

and

$$F_{H,y}(R) := \{\gamma \in F_{H,y} \mid d(\gamma(y), \mathrm{Bary}(H)) \leqslant R\} .$$

Then the cardinality of the subset $F_{H,y}(R)$ has linear growth depending only on R_0. Hence, for every y,

$$\gamma \mapsto \sum_{\gamma \in F_{H,y}} K_0 e^{-a(d(\gamma y, \mathrm{Bary}(H))} ,$$

seen as a function of H is in $\ell^1(\Gamma)$ and its ℓ^1 norm is bounded as a function of R_0.

Hence – as a consequence of the exponential decay inequality (43) – for every y, the map

$$\gamma \mapsto \|\Omega_{\rho(G)}(\gamma y)\|_y ,$$

is in $\ell^1(\Gamma)$ and its ℓ^1 norm is bounded by a function of R_0.

Finally from inequality (43), we have obtained that there is a constant R_1 only depending on R_0 such that

$$\sup_{y \in \mathbf{H}^2} \|\Omega_{\rho(H)}(y)\|_y \leqslant \sup_{y \in B(\mathrm{Bary}(H), R_1)} \|\Omega_{\rho(H)}(y)\|_y + 1 .$$

The bounded cocycle hypothesis, equation (14), implies that $\sup_{y \in B(\mathrm{Bary}(H), R_1)} \|\Omega_{\rho(H)}(y)\|_y$ is bounded by a function only depending on R_1, and thus $\sup_{y \in \mathbf{H}^2} \|\Omega_{\rho(H)}(y)\|_y$ is bounded by a function of R_0. This completes the proof of the corollary. ◻

4.6. **Derivative of correlation functions.** In this paragraph, as a conclusion of this section, we relate the process of ghost integration with the derivative of correlation functions.

Proposition 4.6.1. *Let $(\nabla_t, h_t)_{t \in]-\varepsilon, \varepsilon[}$ be a bounded variation of a uniformly hyperbolic bundle $\rho = (\nabla, h)$. Assume that G is a Θ-ghost polygon, then*

$$\mathrm{d}\mathsf{T}_G(\dot{\nabla}) = \oint_{\rho(G)} \dot{\nabla} . \tag{44}$$

This proposition is an immediate consequence of the following lemma, which is itself an immediate consequence of the definition of the line integration in paragraph 4.2 and lemma 2.2.4:

Lemma 4.6.2. *Let (∇_t, h_t) be a family of uniformly hyperbolic bundles with bounded variation – see definition 2.1.9 – associated to a family of fundamental projectors $(\mathrm{p}_t)_{t \in]-\varepsilon, \varepsilon[}$. Then for a decorated geodesic g,*

$$\mathrm{Tr}\left(\dot{\mathrm{p}}_0(g) \cdot Q\right) = \boldsymbol{J}_{\rho(g), Q}\left(\dot{\nabla}\right) . \tag{45}$$

4.7. **Integration along geodesics.** For completeness, let us introduce *ghost integration for geodesics*: we define for any geodesically bounded 1-form α in $\Xi(E)$ and a Θ-geodesic g,

$$\oint_{\rho(g)} \alpha := \int_g \mathrm{Tr}\left(\alpha \mathrm{p}_g\right) . \tag{46}$$

Remark: It is important to observe that, contrary to a general ghost polygon, we only integrate geodesically bounded forms, not bounded ones. In particular, we cannot integrate variations of uniformly hyperbolic bundles.

5. Ghost intersection and the ghost algebra

In this section we will effectively define and compute the *ghost intersections* of ghost polygons or geodesics. This is the objective of propositions 5.1.1 and 5.1.2.

The *ghost intersection* is a generalization of the intersection of geodesics. However this procedure is non abelian in nature: it requires the choice of a uniformly hyperbolic bundle ρ.

The relation between ghost integration and ghost intersection is akin to the relation between integrating along a geodesic and intersections of geodesic. Thanks to the ghost dual form Ω_g of a ghost polygon, we will define the ghost intersection with respect to a uniformly hyperbolic bundle ρ as

$$\mathsf{I}_\rho(G,H) = \oint_{\rho(G)} \Omega_H = -\oint_{\rho(H)} \Omega_G = \int_{\mathbf{H}^2} \mathrm{Tr}\left(\Omega_H \wedge \Omega_G\right) .$$

Observe that the above formulae mimic the formulae of the first section about intersection of geodesics in the hyperbolic plane. We then compute effectively the value of the ghost intersection, and as a consequence show that it only depends on G and H.

We then move to a more formal point of view in order to perform some computations more efficiently in the future.

We define the associated *ghost algebra* in paragraph 5.2, the ghost algebra is a vector space generated by ghost polygons equipped with a bracket,

$$(G,H) \mapsto [G,H] ,$$

we naturally extend the correlation function to the ghost algebra and reinterpret the intersection in the crucial proposition 5.2.3 as

$$I_\rho(G,H) = \mathsf{T}_{[G,H]}(\rho) .$$

In paragraph 5.3, we relate the corresponding ghost bracket for the projective case to the swapping bracket defined in [19] by the second author.

In the somewhat independent paragraph 5.4, we define and study *natural maps* from the ghost algebra to itself. The point of this construction is that the swapping bracket is a Lie bracket, while this is not true of the ghost bracket, and we will use the Jacobi identity later on in our applications.

We will freely use the definitions given in section 3 for ghost polygons.

5.1. **Ghost intersection: definitions and computations.**

The intersection will be complex (or real) valued and always with respect to a uniformly hyperbolic bundle. We proceed step by step with the definitions: intersections of two geodesics, of one geodesic with a ghost polygons and finally of two polygons. In all cases, the intersection is with respect to a representation.

5.1.1. *Intersecting two geodesics.* Let g and h two Θ-geodesics (in other words, geodesics labelled with an element of Θ). Let us define

$$\mathsf{I}_\rho(g,h) := \oint_{\rho(g)} \beta^0_h , \tag{47}$$

where $\beta^0_h := \beta_h - \frac{\Theta_h}{\dim(E)}\,\mathrm{Id}$ is the trace free part of β_h and Θ_h is defined in equation (28). A straightforward computation using equation (7) and (46) then gives

$$\mathsf{I}_\rho(g,h) = \varepsilon(h,g)\left(\mathsf{T}_{\lceil g,h\rceil}(\rho) - \frac{1}{\dim(E)}\Theta_g\Theta_h\right) . \tag{48}$$

By convention, the quantity $\varepsilon(g,h)$ for two Θ-decorated geodesics g and h is the same as the intersection of the underlying geodesics.

5.1.2. *Intersecting a ghost polygon with a geodesic.* Let ρ be a Θ-uniformly hyperbolic bundle. Let G be a Θ-ghost polygon and h a Θ-geodesic. The *ghost intersection of G and g* is

$$\mathsf{I}_\rho(G,h) := -\oint_{\rho(G)} \beta_{\rho(h)} = -\int_{\mathbf{H}^2} \mathrm{Tr}\left(\beta_{\rho(h)} \wedge \Omega_{\rho(G)}\right) = -\int_h \mathrm{Tr}\left(\Omega_{\rho(G)}\ \mathrm{p}(h)\right) =: -\mathsf{I}_\rho(h,G)\ . \tag{49}$$

By convention we set $\mathsf{I}_\rho(h,G) := -\mathsf{I}_\rho(G,h)$. We will prove that we can effectively compute the ghost intersection, and in particular show that the ghost intersection only depends on G, h and ρ. Then we have

Proposition 5.1.1 (Computation of ghost intersection I)**.** *Let G be a configuration of geodesics, associated to a ghost polygon $\vartheta = (\theta_1, \ldots, \theta_{2p})$. The ghost intersection of h and G satisfies*

$$\mathsf{I}_\rho(G,h) = \sum_{i=1}^{2p} (-1)^{i+1} \varepsilon(h,\theta_j)\ \mathsf{T}_{\lceil h,\theta_i^* \rceil}(\rho)\ , \tag{50}$$

where θ_i^* is the opposite configuration as in paragraph 3.1. In the projective case, that is $\Theta = \{1\}$ we have

$$\mathsf{I}_\rho(G,h) = \mathsf{T}_G(\rho) \left(\sum_{i=1}^{2p} (-1)^{i+1} \varepsilon(h,\theta_j)\ \mathsf{T}_{\lceil h,\theta_i \rceil}(\rho) \right)\ . \tag{51}$$

5.1.3. *Intersecting two ghost polygons.* We define the *ghost intersection of two ghost polygons* or equivalently of two configuration of geodesics G and H to be

$$\mathsf{I}_\rho(G,H) := \oint_{\rho(G)} \Omega_{\rho(H)} = \int_{\mathbf{H}^2} \mathrm{Tr}\left(\Omega_{\rho(H)} \wedge \Omega_{\rho(G)}\right)\ . \tag{52}$$

We can again compute this intersection effectively and in particular show that the ghost intersection only depends on G, H and ρ:

Proposition 5.1.2 (Computation of ghost intersection II)**.** *The ghost intersection of the two configuration G and H, associated respectively to the ghost polygons $\vartheta = (\theta_i)_{i\in I}$, with $I = [1,2p]$, and $\varsigma = (\sigma_j)_{j\in J}$, with $J = [1,2m]$, respectively, is given by*

$$\mathsf{I}_\rho(G,H) = \sum_{i\in I, j\in J} (-1)^{i+j} \varepsilon(\sigma_j,\theta_i)\ \mathsf{T}_{\lceil \sigma_j^*,\theta_i^* \rceil}(\rho)\ .$$

In the projective case, this simplifies as

$$\mathsf{I}_\rho(G,H) = \mathsf{T}_G(\rho)\mathsf{T}_H(\rho) \left(\sum_{i\in I, j\in J} (-1)^{i+j} \varepsilon(\sigma_j,\theta_i)\ \mathsf{T}_{\lceil \sigma_j,\theta_i \rceil}(\rho) \right)\ .$$

5.2. **Θ-Ghost bracket and the ghost space.** We develop a more formal point of view. Our goal is proposition 5.2.3 that identifies the intersection as a correlation function. Let $\mathcal{A}$ be the vector space generated by Θ-ghost polygons (or equivalently configurations of Θ-geodesics) and Θ-geodesics. We add as a generator the element $\mathbf{1}$, and call it the *Casimir element*. By definition, we say $\mathbf{1}$ has rank 0. We will see that the Casimir element will generate the center.

Recall also that we can reverse the orientation on geodesics. The corresponding *reverse orientation* on configuration is given by $\overline{\lceil g_1, \ldots, g_p \rceil} := \lceil \bar{g}_p, \ldots, \bar{g}_1 \rceil$.

Definition 5.2.1 (Θ-ghost bracket on $\mathcal{A}$)**.** *We define the bracket on the basis of $\mathcal{A}$ and extend it by linearity.*

(1) *The bracket of $\mathbf{1}$ with all elements is 0.*
(2) *Let G and H be two configurations of Θ-geodesics, associated respectively to the ghost polygons $\vartheta = (\theta_i)_{i\in I}$, with $I = [1,2p]$ and $\varsigma = (\sigma_j)_{j\in J}$, with $J = [1,2m]$ respectively. Their* Θ-ghost bracket *is given by*

$$[G,H] := \sum_{i\in I, j\in J} \varepsilon(\sigma_j,\theta_i)(-1)^{i+j} \lceil \theta_i^*, \sigma_j^* \rceil\ ,$$

where we recall that θ_j^* is the opposite ghost configuration defined in paragraph 3.1.

(3) *Let g and h be two Θ-geodesics and G a ghost polygon as above. Then we define*

$$\begin{aligned}[g,h] &:= \varepsilon(h,g)\left(\lceil h,g\rceil - \Theta_h\Theta_g\cdot\mathbf{1}\right) ,\\ [G,h] &:= \sum_{j\in J}(-1)^{j+1}\varepsilon(h,\theta_j)\lceil h,\theta_j^*\rceil =: -[h,G] ,\end{aligned}$$

Finally $\mathcal{A}$ equipped with the ghost bracket is called the ghost algebra*.*

We observe that the ghost bracket is antisymmetric. However, the Θ-ghost bracket does not always satisfy the Jacobi identity: there are some singular cases. We actually prove in the Appendix C, as Theorem C.3.1 the following result

Theorem 5.2.2. *Assume A, B, and C are ghost polygons and that*

$$V_A\cap V_B\cap V_C=\emptyset ,$$

where V_A, V_B and V_C are the set of vertices of A, B and C respectively, then

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 .$$

Finally we now extend the map T by linearity to $\mathcal{A}$ so as to define $\mathsf{T}_G(\rho)$ for G an element of $\mathcal{A}$, while defining

$$\mathsf{T}_{\mathbf{1}}(\rho):=\frac{1}{\dim(E)} .$$

The purpose of this formal point of view is to obtain the simple formula:

Proposition 5.2.3. *We have for G,H ghost polygons then*

$$\mathsf{I}_\rho(G,H)=\mathsf{T}_{[G,H]}(\rho) . \tag{53}$$

This formula will allow us to compute recursively Poisson brackets of correlation functions.

Proof. This is a simple rewriting of Propositions 5.1.1 and 5.1.2. □

5.3. **The projective case: swapping and ghost algebras.** Throughout this section, we will restrict ourselves to the projective case, that is $\Theta=\{1\}$.

5.3.1. *Ghost polygons and multifractions.* In [19], the second author introduced the *swapping algebra* $\mathcal{L}$ consisting of polynomials in variables (X,x), where (X,x) are points in S^1, together with the relation $(x,x)=0$. We introduced the *swapping bracket* defined on the generators by

$$[(X,x),(Y,y)]=\varepsilon\left((Y,y),(X,x)\right)\left((X,y)\cdot(Y,x)\right) .$$

We proved that the swapping bracket gives to the swapping algebra the structure of a Poisson algebra.

We also introduced the *multifraction algebra* $\mathcal{B}$ which is the vector space in the fraction algebra of $\mathcal{L}$ generated by the *multifractions* which are elements defined, when X and x are a n tuples of points in the circle and σ an element of the symmetric group $\mathfrak{S}(n)$ by

$$[X,x;\sigma]:=\frac{\prod_{i=1}^n(X_i,x_{\sigma(i)})}{\prod_{i=1}^n(X_i,x_i)} .$$

We proved that the multifraction algebra is stable by the Poisson bracket, while it is obviously stable by multiplication.

Definition 5.3.1 (Extended swapping algebra)**.** *Let us consider the algebra $\mathcal{B}_0$ which is generated as a vector space by the multifraction algebra to which we add extra generators denoted ℓ_g for any geodesic g [5] as well as a central element $\mathbf{1}$; we finally extend the swapping bracket to $\mathcal{B}_0$ by adding*

$$[\ell_g,\ell_h]:=\frac{1}{g\,h}[g,h]+\varepsilon(g,h)\mathbf{1} \ , \quad [G,\ell_h]:=\frac{1}{h}[G,h]=:-[\ell_h,G] \ . \tag{54}$$

[5] The generator ℓ_g is formally a logarithm $\log(g)$ of the geodesic g.

We call $\mathcal{B}_0$ with the extended swapping bracket, the extended swapping algebra*.*

The reversing orientation is defined on generators by $\overline{\ell_g} = \ell_{\overline{g}}$,

We then have

Proposition 5.3.2. *The extended swapping algebra is a Poisson algebra. The reversing isomorphisms antipreserves the Poisson structure: $\left[\overline{G}, \overline{H}\right] = -\overline{[G, H]}$.*

Proof. This is just a standard check that adding "logarithmic derivatives" to a Poisson algebra still gives a Poisson algebra. We first see that that

$$\partial_g : z \mapsto [\ell_g, z] = \frac{1}{g}[g, z] \ ,$$

is a derivation on the fraction algebra of the swapping bracket. Indeed,

$$\begin{aligned}\partial_g([z, w]) &= \frac{1}{g}[g, [z, w]] = \frac{1}{g}\left([z, [g, w]] - [w, [g, z]]\right) \\ &= \left(\left[z, \frac{[g, w]}{g}\right] - \left[w, \frac{[z, w]}{g}\right]\right) = [z, \partial_g(w)] + [\partial_g(z), w] \ .\end{aligned}$$

Moreover, the bracket of derivation gives $[\partial_g, \partial_h](z) = [[\ell_g, \ell_h], z]$ Let us check this last point:

$$\partial_g\left(\partial_h(z)\right) = \frac{1}{g}\left[g, \frac{1}{h}[h, z]\right] = -\frac{1}{gh^2}[g, h][h, z] + \frac{1}{gh}[g, [h, z]] \ .$$

Thus, we complete the proof of the proposition

$$[\partial_g, \partial_h](z) = [g, h]\left(-\frac{[h, z]}{gh^2} - \frac{[g, z]}{hg^2}\right) + \frac{1}{gh}[[g, h], z]] = \left[\frac{[g, h]}{gh}, z\right] \ . \quad \square$$

5.3.2. *Ghost algebra and the extended swapping algebra.* In the projective case, it is convenient to consider the free polynomial algebra $\mathcal{A}_P$ generated by the ghost polygons, and extend the *ghost bracket* by the Leibnitz rule to $\mathcal{A}_P$.

In this paragraph, we will relate the algebras $\mathcal{A}_P$ and $\mathcal{B}_0$, more precisely we will show:

Theorem 5.3.3. *There exists a homomorphisms of commutative algebra map*

$$\pi : \mathcal{A}_P \to \mathcal{B}_0 \ ,$$

which is surjective, preserves the bracket and and the reversing the orientation isomorphism:

$$[\pi(A), \pi(B)] = \pi[A, B] \quad , \quad \overline{\pi(A)} = \pi\left(\overline{A}\right) \ .$$

Finally if A belongs to the kernel of π, then for any uniformly hyperbolic bundle ρ, $\mathsf{T}_A(\rho) = 0$.

Thus, $\mathcal{A}_P/\ker(\pi)$ is identified as an algebra with bracket with $\mathcal{B}_0$; in particular $\mathcal{A}_P/\ker(\pi)$ is a Poisson algebra.

This will allow in the applications to reduce our computations to calculations in the extended swapping algebra, making use of the fact that the extended swapping algebra is a Poisson algebra by proposition 5.3.2.

Unfortunately, we do not have the analogue of the swapping bracket in the general Θ-case, although the construction and result above suggests to find a combinatorially defined ideal $\mathcal{I}$ in the kernel of $\mathsf{T}(\rho)$ for any ρ, so that $\mathcal{A}/\mathcal{I}$ satisfies the Jacobi identity.

5.3.3. *From the ghost algebra to the extended swapping algebra.* In this paragraph, we define the map π of Theorem 5.3.3. The map π is defined on the generators by

$$\begin{aligned} g &\longmapsto \pi(g) := \ell_g \ , \\ G = \lceil g_1, \ldots, g_p \rceil &\longmapsto \pi(G) := [X, x; \sigma] = \frac{\prod_{i=1}^{n}(g_i^+, g_{i+1}^-)}{\prod_{i=1}^{n}(g_i^+, g_i^-)} \ . \end{aligned}$$

where $X = (g_i^+)$, $x = (g_i^-)$, $\sigma(i) = i + 1$. Cyclicity is reflected by

$$\pi(\lceil g_1, \ldots, g_p \rceil) = \pi(\lceil g_2, \ldots, g_p, g_1 \rceil) \ . \tag{55}$$

Conversely, we then have the following easy construction.

Let $X = (X_1, \ldots, X_k)$, $x = (x_1, \ldots, x_k)$, and g_i the geodesic (X_i, x_i). Let σ be a permutation of $\{1, \ldots, k\}$ and let us write $\sigma = \sigma_1, \ldots, \sigma_q$ be the decomposition of σ into commuting cycles σ_i or order k_i with support I_i. For every i, let m_i be in I_i and let us define

$$h^i_j = g_{\sigma_i^{j-1}(m_i)} \ , \ G_i = \lceil h^i_1, \ldots h^i_{k_i} \rceil \ .$$

We then have with the above notation

$$[X, x; \sigma] = \pi(G_1 \ldots G_q) \ . \tag{56}$$

Corollary 5.3.4. *The map π is surjective.*

In the sequel, the decomposition (56) will be referred as the *polygonal decomposition* of the multifraction $[X, x; \sigma]$. We also obviously have

Lemma 5.3.5. *Any tuples of ghost polygons is the polygonal decomposition of a multifraction.*

5.3.4. *The map π and the evaluation T.* For any multifraction $B = [X, x; \sigma]$ and uniformly hyperbolic bundle ρ associated to limit curves ξ and dual limit curves ξ^*, we define

$$\mathsf{T}^P_B(\rho) := \frac{\prod_i \langle V_i, v_{\sigma(i)} \rangle}{\prod_i \langle V_i, v_i \rangle} \ ,$$

where V_i is a non-zero vector in $\xi^*(X_i)$ while v_i is a non-zero vector in $\xi(x^i)$.

Given ρ, we now extend $G \mapsto \mathsf{T}_G(\rho)$ and $\mathsf{T}^P_B(\rho)$ to homomorphisms of commutative free algebras to $\mathcal{A}_p$ and $\mathcal{B}_0$. We then have the following result which follows at once since we are only considering rank 1 projectors.

Lemma 5.3.6. *We have, for all uniformly hyperbolic bundles ρ*

$$\mathsf{T}^P_{\pi(G)}(\rho) = \mathsf{T}_G(\rho) \ , \ \mathsf{T}_G(\rho) = \mathsf{T}_{\bar{G}}(\rho^*) \ ,$$

This proposition implies that for every G in the kernel of π, for every ρ, $\mathsf{T}_G(\rho) = 0$.

5.3.5. *Swapping bracket.* We now compute the brackets of multifractions. We shall use the notation of paragraph 3.1 where the opposite configuration g^* of a ghost or visible edge g is defined. Observe that g^* is an ordered configuration. Then we have

Proposition 5.3.7 (Computation of swapping brackets)**.** *Let G and H be two multifractions that are images of ghost polygons: $G = \pi(\theta_1, \ldots, \theta_{2p})$ and $H = \pi(\zeta_1, \ldots, \zeta_{2q})$. Then their swapping bracket is given by*

$$[G, H] = \left(G\, H \left(\sum_{i,j} \varepsilon(\zeta_j, \theta_i)(-1)^{i+j} \pi(\lceil \theta_i, \zeta_j \rceil) \right) \right) \ . \tag{57}$$

Moreover, for $g = (X, x)$ and $h = (Y, y)$ geodesics, we have in the fraction algebra of the swapping algebra.

$$[\ell_h, \ell_g] = (\varepsilon(g, h)\, \pi(\lceil g, h \rceil)) \ . \tag{58}$$

$$[G, \ell_h] = \left(G \left(\sum_i \varepsilon(h, \theta_i)(-1)^{i+1} \pi(\lceil \theta_i, h \rceil) \right) \right) \ . \tag{59}$$

Using the notation θ_i^* for the opposite edge, we have, for every i and j

$$\pi(\lceil\theta_i^*, \zeta_j^*\rceil) = G\, H\, \pi(\lceil\theta_i, \zeta_j\rceil)\ . \tag{60}$$

Proof. In this proof, we will omit to write π and confuse a ghost polygon and its image under π. Equation (58) follows at once from the definition. Now let $G = \lceil g_1, \ldots, g_p\rceil$, let η_i be the ghost edges joining g_{i+1}^- to g_i^+. Then we may write in the fraction algebra of the swapping algebra

$$\lceil g_1, \ldots, g_p\rceil = \frac{\prod_{i=1}^p \eta_i}{\prod_{i=1}^p g_i}\ .$$

Using logarithmic derivatives we then have

$$\frac{1}{G}[G, \ell_h] = \sum_{i=1}^p \left(\frac{1}{h\,\eta_i}[\eta_i, h] - \frac{1}{h\, g_i}[g_i, h]\right) = \sum_{i=1}^p \left(\varepsilon(h, \eta_i)\lceil\eta_i, h\rceil - \varepsilon(h, g_i)\lceil g_i, h\rceil\right)\ ,$$

which gives equation (59). Writing now

$$G = \lceil g_1, \ldots, g_p\rceil = \frac{\prod_{i=1}^p \eta_i}{\prod_{i=1}^q g_i}\ \ ,\ \ H = \lceil h_1, \ldots, h_q\rceil = \frac{\prod_{i=1}^q \nu_i}{\prod_{i=1}^q h_i}\ ,$$

where η_i and ν_i are ghost edges of G and H respectively, we get

$$\begin{aligned}\frac{[G,H]}{G\,H} &= \sum_{(i,j)}\left(\frac{1}{g_i\, h_j}[g_i, h_j] - \frac{1}{g_i\,\nu_j}[g_i, \nu_j] + \frac{1}{\eta_i\,\nu_j}[\eta_i, \nu_j] - \frac{1}{\eta_i\, h_j}[\eta_i, h_j]\right)\\ &= \sum_{(i,j)}\left(\varepsilon(h_j, g_i)\lceil g_i, h_j\rceil - \varepsilon(\nu_j, g_i)\lceil g_i, \nu_j\rceil + \varepsilon(\nu_j, \eta_i)\lceil\eta_i, \nu_j\rceil - \varepsilon(h_j, \eta_i)\lceil\eta_i, h_j\rceil\right)\ ,\end{aligned}$$

which is what we wanted to prove. The equation (60) follows from the definition of the map π. ◻

As a corollary we obtain

Corollary 5.3.8. *The map π preserves the bracket.*

Proof. The proof follows at once from proposition 5.1.1 and 5.1.2 which computes the ghost intersection and recognizing each term as the correlation functions of a term obtained in the corresponding ghost bracket in proposition 5.3.7. ◻

5.3.6. *Proof of Theorem 5.3.3.* We have proved all that we needed to prove: the theorem follows from corollary 5.3.8 and 5.3.4, as well as lemma 5.3.6.

5.4. **Natural maps into the ghost algebra.**

Definition 5.4.1 (Natural maps)**.** *Let w be a p-multilinear map from the ghost algebra to itself. We say w is* natural, *if for tuples of integers $(n_1, \ldots, n_p)$ there exists an integer q, a real number A such that given a tuple of ghost polygons $G = (G_1, \ldots, G_p)$ with G_i in $\mathcal{G}^{n_i}$, then*

$$w(G_1, \ldots, G_p) = \sum_{i=1}^q \lambda_i H_i\ ,$$

where H_i are ghost polygons, λ_i are real numbers less than A and, moreover, every visible edge of H_i is a visible edge of one of the G_i.[6]

[6]The existence of q is actually a consequence of the definition: there only finitely many polygons with a given set of visible edges

We will extend the definition of the core diameter to any element of the ghost algebra by writing, whenever H_i are distinct ghost polygons ghost polygons

$$r\left(\sum_{i=1}^{q} \lambda_i H_i\right) := \sup_{i=1,\ldots,q} \left(|\lambda_i| r(H_i)\right) ,$$

We also recall that the core diameter of a ghost polygon, only depends on the set of its visible edges. We then define the core diameter of a tuple of polygons $G = (G_1, \ldots, G_n)$, as the core diameter of the union of the set of edges of the G_i's.

We then have the following inequality of core diameters for a natural map w, $G = (G_1, \ldots, G_p)$ and q and A as in the definition

$$r(w(G)) \leqslant A\, r(G) . \tag{61}$$

We now give an example of a natural map

Proposition 5.4.2. *The map* $(G_1, \ldots, G_n) \mapsto [G_1, [G_2, [\ldots [G_{n-1}, G_n] \ldots]]]$ *is a natural map.*

Proof. This follows at once from the definition of the ghost bracket and a simple induction argument. □

Part 2

On closed surfaces

We now move to closed surfaces, or in other words we only consider periodic uniformly hyperbolic bundles which are exactly Anosov representations. In an analogue of the way that one can define length for currents, we define an *average correlation functions* using *cyclic currents* and a careful study of integrals allows us to prove the main results announced in the introduction.

(1) In section 6, we introduce *cyclic currents* which are certain types of measures on the space of ghost polygons. This allows us to define what are averaged correlations function T_μ when μ is a cyclic current.
(2) In section 7, we deal with the following crucial technical and dry issue: how to exchange ghost integration and integration with respect to a cyclic current? The results are necessary to proceed.
(3) In section 8, using the combinatorial work of the first part and the previous sections, we are finally able to prove the main result of the article a *combinatorial formula to compute the Poisson bracket of correlation functions*. This is obtained by a see-saw argument: we first compute the derivatives of length functions and correlations functions, then this allows us to recognize hamiltonian functions using the ghost dual forms we defined for ghost polygons, and finally we can compute the Poisson bracket of correlations and length functions.

6. Geodesic and cyclic currents

In this section, building on the classical notion of geodesic currents introduced by Bonahon in [3], we define the notion of higher order geodesic currents, called *cyclic currents*. Among them we identify *integrable currents*, show how they can average correlation functions and produce examples of them.

Recall that C is the set of oriented geodesics in $\mathbf{H}^2$. The set of Θ-geodesics is then denoted $\mathcal{G} := C\times\Theta$.

6.1. **Cyclic current.** First recall that a *signed measure* is a linear combination of finitely many *positive measures*. Any signed measure is the difference of two positive measures. A *cyclic current* is a Γ-invariant signed measure on C^n invariant under cyclic permutation. As a first example let us consider for μ and ν geodesic current, the signed measure $\mu\wedge\nu$ given by

$$\mu\wedge\nu := \frac{1}{2}\varepsilon\left(\mu\otimes\nu-\nu\otimes\mu\right), \tag{62}$$

where we recall that $\varepsilon(g,h)$ is the intersection number of the two geodesics g and h.

Lemma 6.1.1. *The signed measure $\mu\wedge\nu$ is a cyclic current supported on intersecting geodesics. Moreover $\mu\wedge\nu=-\nu\wedge\mu$.*

Proof. We have

$$\begin{aligned}
2\int_{\mathcal{G}^2/\Gamma} f(g,h)\,\mathrm{d}\mu\wedge\nu(g,h) &= \int_{\mathcal{G}^2/\Gamma} f(g,h)\;\varepsilon(g,h)\left(\mathrm{d}\mu(g)\mathrm{d}\nu(h)-\mathrm{d}\nu(g)\mathrm{d}\mu(h)\right)\\
&= \int_{\mathcal{G}^2/\Gamma} f(h,g)\;\varepsilon(h,g)\left(\mathrm{d}\mu(h)\mathrm{d}\nu(g)-\mathrm{d}\nu(h)\mathrm{d}\mu(g)\right)\\
&= \int_{\mathcal{G}^2/\Gamma} f(h,g)\;\varepsilon(g,h)\left(\mathrm{d}\nu(h)\mathrm{d}\mu(g)-\mathrm{d}\mu(h)\mathrm{d}\nu(g)\right)\\
&= 2\int_{\mathcal{G}^2/\Gamma} f(h,g)\,\mathrm{d}\mu\wedge\nu(g,h)\,.
\end{aligned}$$

Hence $\mu\wedge\nu$ is cyclic. The last assertions are obvious. □

Our main definition is the following, let ρ be a Θ-Anosov representation of Γ, the fundamental group of a closed surface.

Definition 6.1.2 (Integrable currents). *We give several definitions, let w be a natural map from $C^{p_1}\times\cdots\times C^{p_q}$ to C^m*

1. *a w-*cyclic current *is a Γ-invariant measure $\mu=\mu_1\otimes\cdots\otimes\mu_q$ where μ_i are Γ-invariant cyclic currents on C^{n_i},*
2. *the w-cyclic current μ is a (ρ,w)-*integrable *current if there exists a neighborhood U of ρ in the moduli space of (complexified) Θ-Anosov representations of Γ, and a positive function F in $L^1(\mathcal{G}^k_\star/\Gamma,\eta)$ so that for all σ in U, and G in $\mathcal{G}^k_\star$;*
$$|\mathsf{T}_{w(G)}(\sigma)|\leqslant F(G)\,,$$
where F_0 is the lift of F to $C^k_\star$.
3. *When w is the identity map* Id, *we just say a current is ρ-*integrable, *instead of (ρ,Id)-integrable.*
4. *A current of order k, is w-*integrable *or* integrable *if it is (ρ,w)-integrable or ρ-integrable for all representations ρ.*

6.1.1. Γ-*compact currents.*

Definition 6.1.3. *A Γ-invariant w-cyclic current μ is* Γ-compact *if it is supported on a Γ-compact set of $\mathcal{G}^p_\star$. Obviously a Γ-compact cyclic current is integrable for any natural map w.*

Here is an important example of a Γ-compact cyclic current: Let $\mathcal{L}$ be a geodesic lamination on S with a component of its complement C being a geodesic triangle. Let $\pi:\mathbf{H}^2\mapsto S$ be the universal covering of S and x a point in C

Then

$$\pi^{-1}C = \bigsqcup_{i\in\pi^{-1}(x)} C_i \ .$$

The closure of each C_i is an ideal triangle with cyclically ordered edges (g_i^1, g_i^2, g_i^3). We consider the opposite cyclic ordering (g_i^3, g_i^2, g_i^1). The notation δ_x denotes the Dirac measure on X supported on a point x of X. Then we obviously have

Proposition 6.1.4. *The measure defined on $\mathcal{G}^p$ by*

$$\mu_C^* = \frac{1}{3}\sum_{i\in\pi^{-1}(x)} \left(\delta_{(g_i^1,g_i^3,g_i^2)} + \delta_{(g_i^2,g_i^1,g_i^3)} + \delta_{(g_i^3,g_i^2,g_i^1)}\right) \ ,$$

is a Γ-compact cyclic current.

6.1.2. *Intersecting geodesics.* Let us give an example of an integrable current.

Proposition 6.1.5. *Let μ be Γ-invariant cyclic current supported on pairs of intersecting geodesics. Assume furthermore that $\mu(\mathcal{G}^2/\Gamma)$ is finite. Then μ is integrable.*

This follows at once from the following lemma.

Lemma 6.1.6. *Let ρ_0 be a Θ-Anosov representation. Then there exists a constant K_ρ in an neighborhood U of ρ_0 in the moduli space of Anosov representations, such that for any ρ in U and any pair of intersecting geodesics*

$$|\mathsf{T}_{\lceil g,h\rceil}(\rho)| \leqslant K_\rho \ .$$

Proof. Given any pair of geodesics (g_1, g_0) intersecting at a point x, then we can find an element γ in Γ, so that γx belongs to a fundamental domain V of Γ. In particular, there exists a pair of geodesics h_0 and h_1 passing though V so that

$$\mathsf{T}_{\lceil g_0,g_1\rceil}(\eta) = \mathsf{T}_{\lceil h_0,h_1\rceil}(\eta) = \operatorname{Tr}\left(\mathrm{p}_\eta(h_0)\ \mathrm{p}_\eta(h_1)\right) \ ,$$

where p_η is the fundamental projector for η. Since the set of geodesics passing through V is relatively compact, the result follows by the continuity of the fundamental projector $\mathrm{p}_\eta(h)$ on h and η. □

Corollary 6.1.7. *Given μ and ν, then $\mu\wedge\nu$ is integrable.*

Proof. Let

$$A := \{(g,h) \mid \varepsilon(g,h) = \pm 1\} \ , \ B := \left\{(g,h) \mid \varepsilon(g,h) = \pm\frac{1}{2}\right\} \ .$$

Observe first that denoting the Bonahon intersection by i, we have

$$\left|\mu\wedge\nu\ \ (A/\Gamma)\right| \leqslant i(\bar{\mu},\bar{\nu}) < \infty \ ,$$

where the last inequality is due to Bonahon [3], and $\bar{\lambda}$ is the symmetrised current of λ.

As Γ acts with compact quotient on the set of triples of points on $\partial\mathbf{H}^2$, it follows that Γ acts on B with compact quotient and therefore $\mu\wedge\nu(B)$ is finite. Therefore taking the sum we have that $\mu\wedge\nu(C^2/\Gamma)$ is finite. □

6.1.3. *A side remark.* Here is an example of (ρ, w)-integrable current. First we have the following inequality: given a representation ρ_0, there is a constant K_0, a neighborhood U of ρ_0, such that for every k-configuration G of geodesics and ρ in U then

$$|\mathsf{T}_G(\rho)| \leqslant e^{kK_0\ r(G)} \ .$$

Since this is just a pedagogical remark that we shall not use, we do not fill the details of the proof. From that inequality we see that if $G \mapsto e^{kK_0\ r(G)}$ is in $L^1(\mathcal{G}^k_\star/\Gamma, \mu)$ then μ is (ρ, w)-integrable.

7. Exchanging integrals

To use ghost integration to compute the Hamiltonian of the average of correlation functions with respect to an integrable current, we will need to exchange integrals.

This section is concerned with proving the two Fubini-type exchange theorems we will need. Recall that the form $\beta_{\rho(g)}$ is defined in equation 32.

Theorem 7.0.1 (Exchanging integrals I)**.** *Let μ be a Γ-invariant geodesic current. Let G be a Θ-ghost polygon. Then*

(1) $\int_{\mathcal{G}} \beta_g \mathrm{d}\mu(g)$ *— defined pointwise — is an element of $\mathbf{\Lambda}^\infty(E)$,*
(2) *the map $g \mapsto \oint_{\rho(G)} \beta_g$ is in $L^1(\mathcal{G}, \mu)$,*
(3) *finally, we have the exchange formula*

$$\oint_{\rho(G)} \left(\int_{\mathcal{G}} \beta_{\rho(g)} \, \mathrm{d}\mu(g) \right) = \int_{\mathcal{G}} \left(\oint_{\rho(G)} \beta_{\rho(g)} \right) \mathrm{d}\mu(g) \, . \tag{63}$$

Similarly, we have a result concerning ghost intersection forms. We have to state it independently in order to clarify the statement. Let us first extend the assigement $G \mapsto \Omega_G$ by linearity to the whole ghost algebra, and observe that if we have distinct ghost polygons G_i and

$$H = \sum_{i=1}^{q} \lambda_i G_i \, , \text{ with } \sup_{i \in \{1,\dots,q} |\lambda_i| = A \, ,$$

Then

$$\|\Omega_H(y)\| \leqslant qA \sup_{i \in \{1,\dots,q} \|\Omega_{G_i}(y)\| \, .$$

Theorem 7.0.2 (Exchanging integrals II)**.** *Let μ be a w-cyclic and Γ-compact current of rank p. Let G be a ghost polygon. Let w be a natural map. Then*

(1) $\int_{\mathcal{G}^p} \Omega_{\rho(w(H))} \mathrm{d}\mu(H)$ *— defined pointwise — is an element of $\mathbf{\Lambda}^\infty(E)$,*
(2) *the map $H \mapsto \oint_{\rho(G)} \Omega_{\rho(w(H))}$ is in $L^1(\mathcal{G}^p, \mu)$,*
(3) *finally, we have the exchange formula*

$$\oint_{\rho(G)} \left(\int_{\mathcal{G}^p} \Omega_{\rho(w(H))} \, \mathrm{d}\mu(H) \right) = \int_{\mathcal{G}^p} \left(\oint_{\rho(G)} \Omega_{\rho(w(H))} \right) \mathrm{d}\mu(H) \, . \tag{64}$$

We first concentrate on Theorem 7.0.1, then prove Theorem 7.0.2 in paragraph 7.6.

7.1. **Exchanging line integrals.**

Theorem 7.0.1 is an immediate consequence of a similar result involving line integrals.

Proposition 7.1.1. *Let μ be a Γ-invariant geodesic current on $\mathcal{G}$, then*

(1) $\int_{\mathcal{G}} \beta_g \mathrm{d}\mu(g)$ *— defined pointwise — is an element of $\mathbf{\Lambda}^\infty(E)$,*
(2) *Let g_0 be a geodesic, x a point on g_0 and Q a parallel section of* $\mathrm{End}(E)$ *along g_0, then the map*

$$g \mapsto \boldsymbol{S}_{x,g_0,Q}\left(\beta_g\right) \, ,$$

is in $L^1(\mathcal{G}, \mu)$.
(3) *We have the exchange formula*

$$\boldsymbol{S}_{x,g_0,Q}\left(\int_{\mathcal{G}} \beta_{\rho(g)} \, \mathrm{d}\mu(g) \right) = \int_{\mathcal{G}} \boldsymbol{S}_{x,g_0,Q}\left(\beta_{\rho(g)}\right) \, \mathrm{d}\mu(g) \, . \tag{65}$$

We prove the first item in proposition 7.2, the second item in 7.4 and the third in 7.5.

7.2. **Average of geodesic forms and the first item.** Let μ be a Γ-invariant measure on $\mathcal{G}$. Let y be a point in $\mathbf{H}^2$, and

$$G(y,R) := \{g \in \mathcal{G} \mid d(g,y) \leqslant R\} .$$

As an immediate consequence of the Γ-invariance we have

Proposition 7.2.1. *For every positive R, there is a constant $K(R)$ so that for every y in $\mathbf{H}^2$*

$$\mu(G(y,R)) \leqslant K(R) . \tag{66}$$

Observe now that if g is not in $G(y) := G(y,2)$, then y is not in the support of ω_g and thus $\beta_{\rho(g)}(y) = 0$. We then define

Definition 7.2.2. *The* μ-integral of geodesic forms *is the form α so that at a point y in $\mathbf{H}^2$*

$$\alpha_y := \int_{G(y)} \beta_g(y)\, \mathrm{d}\mu(g) = \int_{\mathcal{G}} \beta_g(y)\, \mathrm{d}\mu(g) .$$

We use some abuse of language and write

$$\alpha =: \int_{\mathcal{G}} \beta_{\rho(g)} \mathrm{d}\mu(g) .$$

The form α_y is well defined since $G(y)$ is compact. Moreover, the next lemma gives the proof of the first item of proposition 7.1.1

Lemma 7.2.3. *The μ-integral of geodesic forms belongs to $\mathbf{\Lambda}^\infty(E)$ and we have a constant K_5 only depending on ρ and μ so that*

$$\left\| \int_{\mathcal{G}} \beta_{\rho(g)} \mathrm{d}\mu(g) \right\|_\infty \leqslant K_5 . \tag{67}$$

Proof. We have

$$\left| \int_{\mathcal{G}} \beta_{\rho(g)}(y)\, \mathrm{d}\mu(g) \right| = \left| \int_{G(y)} \beta_{\rho(g)}(y)\, \mathrm{d}\mu(g) \right| \leqslant \mu(G(y)) \sup_{g \in G(y)} \left\| \beta_{\rho(g)} \right\|_\infty . \tag{68}$$

Then by proposition 7.2.1, there is a constant k_1 so that $\mu(G(y)) \leqslant k_1$. Recall that $\beta_{\rho(g)} = \omega_g \mathrm{p}(g)$. Then by the $\mathsf{PSL}_2(\mathbb{R})$ equivariance, ω_g is bounded independently of g, while by lemma 2.1.4, p is a bounded section of $\mathrm{End}(E)$. The result follows. ◻

7.3. **Decay of line integrals.** We now recall the following definition.

$$\boldsymbol{S}_{x,g,\mathrm{Q}}(\omega) \quad = \quad \int_{g^+} \mathrm{Tr}\left(\omega\, \mathrm{p}\, [\mathrm{p}, \mathrm{Q}]\right) + \int_{g^-} \mathrm{Tr}\left(\omega\, [\mathrm{p}, \mathrm{Q}]\, \mathrm{p}\right) .$$

We prove in this paragraph the following two lemmas.

Lemma 7.3.1 (Vanishing)**.** *Let g_0 be a geodesic and x a point in g_0, Let g be a geodesic such that $d(g,g_0) > 1$, then for any function ψ on $\mathbf{H}^2$ with values in $[0,1]$:*

$$\boldsymbol{S}_{x,g_0,Q}(\psi \beta_{\rho(g)}) = 0 . \tag{69}$$

Proof. This follows at once from the fact that under the stated hypothesis, the support of ω_g does not intersect g_0. ◻

Lemma 7.3.2 (Decay)**.** *For any endomorphism Q and representation ρ, there exist positive constants K and k, so that for all g so that $d(g,x) > R$, for any function ψ on $\mathbf{H}^2$ with values in $[0,1]$:*

$$\left| \boldsymbol{S}_{x,g_0,Q}(\psi \beta_{\rho(g)}) \right| \leqslant K e^{-kR}.$$

Proof. We assume x and g are so that $d(g, x) > R$. It is enough (using a symmetric argument for g_0^-) to show that

$$\left|\int_{g_0^+} \mathrm{Tr}(\psi\beta_{\rho(g)} \cdot \mathrm{p} \cdot [\mathrm{p}, \mathrm{Q}])\right| \leqslant Ke^{-kR} ,$$

where g_0^+ is the arc on g_0 from x to $+\infty$. Let us denote by $g_0^+(R)$ the set of points of g_0^+ at distance at least R from x:

$$g_0^+(R) := \{y \in g_0^+ \mid d(y, x) \geqslant R\} .$$

Then if y belongs to g_0^+ and does not belong to $g_0^+(R-1)$, then $d(y, x) < R - 1$. Thus $d(y, g) > 1$. Thus, by lemma 7.3.1, $\beta_{\rho(g)}(y)$ vanishes for y in g_0^+ and not in $g_0^+(R-1)$. Thus

$$\left|\int_{g_0^+} \mathrm{Tr}(\psi\beta_{\rho(g)} \cdot \mathrm{p} \cdot [\mathrm{p}, \mathrm{Q}])\right| \leqslant \int_{g_0^+(R-1)} \left|\mathrm{Tr}(\beta_{\rho(g)}(\partial_t) \cdot \mathrm{p} \cdot [\mathrm{p}, \mathrm{Q}])\right| \, \mathrm{d}t .$$

Then the result follows from the exponential decay lemma 4.2.2. ◻

7.4. **Cutting in pieces and dominating: the second item.** We need to decompose $\mathcal{G}$ into pieces. Let g_0 be an element of $\mathcal{G}$ and x a point on g_0. Let $x^+(n)$ – respectively $x^-(n)$ – the point in g_0^+ – respectively g_0^- – at distance n from x. Let us consider

$$\begin{aligned} U_0 &:= \{g \in \mathcal{G} \mid d(g, g_0) > 1\} ,\\ V_n^+ &:= \{g \in \mathcal{G} \mid d(g, x^+(n)) < 2 \text{ and for all } 0 \leqslant p < n ,\ d(g, x^+(n)) \geqslant 2\} ,\\ V_n^- &:= \{g \in \mathcal{G} \mid d(g, x^-(n)) < 2 \text{ and for all } 0 \leqslant p < n,\ d(g, x^-(n)) \geqslant 2\} . \end{aligned}$$

This gives a covering of $\mathcal{G}$:

Lemma 7.4.1. *We have the decomposition*

$$\mathcal{G} = U_0 \cup \bigcup_{n\in\mathbb{N}} V_n^{\pm} ,$$

Proof. When g does not belong to U_0, there is some y in g_0 so that $d(g, y) \leqslant 1$, hence some n so that either $d(y, g^+(n)) \leqslant 2$, while for all $0 \leqslant p < n$ we have $d(y, g^+(p)) > 2$, or $d(y, g^-(n)) \leqslant 2$, while for all $0 \leqslant p < n$ we have $d(y, g^-(p)) > 2$. ◻

Let now

$$n(g) = \sup\{m \in \mathbb{N} \mid g \in V_m^+ \cup V_m^-\} .$$

By convention, we write $n(g) = +\infty$, whenever g does not belong to $\bigcup_{n\in\mathbb{N}} V_n^{\pm}$.

The non-negative *control function* F_0 on $\mathcal{G}$ is defined by $F_0(g) = e^{-n(g)}$. We now prove

Lemma 7.4.2 (Dominating lemma). *For any positive k, the function $(F_0)^k$ is in $L^1(\mathcal{G}, \mu)$.*

Moreover, there exist positive constants K_9 and k_9 so that for all functions ψ on $\mathbf{H}^2$ with values in $[0, 1]$ we have

$$\left|\boldsymbol{S}_{x,g_0,Q}(\psi\beta_g)\right| \leqslant K_9(F_0(g))^{k_9} . \tag{70}$$

We now observe that the second item of proposition 7.1.1 is an immediate consequence of this lemma.

Proof. We first prove that F_0 and all its powers are in $L^1(\mathcal{G}, \mu)$. Observe that $V_n^{\pm} \subset G(x^{\pm}(n), 2)$. It follows from proposition 7.2.1 that $\mu(V_n^{\pm}) \leqslant K(2)$. Moreover, for any g in $V_n^{\pm}$, $F_0(g)^k \leqslant e^{-kn}$. The decomposition of lemma 7.4.1 implies that F_0^k is in $L^1(\mathcal{G}, \mu)$.

Let g be an element of $\mathcal{G}$.

(1) When g belongs to U_0, then by lemma 7.3.1, $\boldsymbol{S}_{x,g_0,Q}(\beta_{\rho(g)}) = 0$. Hence $\left|\boldsymbol{S}_{x,g_0,Q}(\beta_{\rho(g)})\right| \leqslant A(F_0(g))^a$, for any positive A and a.

(2) When g does not belong to U_0, then g belongs to $V_{n(g)}^{\pm}$ with $n(g) < \infty$. By lemma B.0.1, we have $d(x, g) \geqslant n(g)$. It follows from lemma 7.3.2 that for any positive function ψ, we have

$$\left|\boldsymbol{S}_{x,g_0,Q}(\psi\beta_{\rho(g)})\right| \leqslant Ke^{-kn(g)} = KF_0(g)^k .$$

The last inequality concludes the proof. □

7.5. **Proof of the exchange formula of proposition 7.1.1.** Let us choose, for any positive real R, a *cut-off function* ψ_R, namely a function on $\mathbf{H}^2$ with values in $[0,1]$, with support in the ball with center x and radius $R+1$, and equal to 1 on the ball of radius x and radius R. We write

$$\left|\boldsymbol{S}_{x,g_0,Q}\left(\int_{\mathcal{G}} \beta_{\rho(g)}\, \mathrm{d}\mu(g)\right) - \int_{\mathcal{G}} \boldsymbol{S}_{x,g_0,Q}\left(\beta_{\rho(g)}\right) \mathrm{d}\mu(g)\right| \leqslant A(R) + B(R) + C(R)\ , \qquad (71)$$

where

$$\begin{aligned} A(R) &= \left|\boldsymbol{S}_{x,g_0,Q}\left(\int_{\mathcal{G}} \beta_{\rho(g)}\, \mathrm{d}\mu(g)\right) - \boldsymbol{S}_{x,g_0,Q}\left(\psi_R \int_{\mathcal{G}} \beta_{\rho(g)} \mathrm{d}\mu(g)\right)\right|\ ,\\ B(R) &= \left|\boldsymbol{S}_{x,g_0,Q}\left(\psi_R \int_{\mathcal{G}} \beta_{\rho(g)}\, \mathrm{d}\mu(g)\right) - \int_{\mathcal{G}} \boldsymbol{S}_{x,g_0,Q}\left(\psi_R \beta_{\rho(g)}\right)\, \mathrm{d}\mu(g)\right|\ ,\\ C(R) &= \left|\int_{\mathcal{G}} \boldsymbol{S}_{x,g_0,Q}\left(\psi_R \beta_{\rho(g)}\right)\, \mathrm{d}\mu(g) - \int_{\mathcal{G}} \boldsymbol{S}_{x,g_0,Q}\left(\beta_{\rho(g)}\right)\, \mathrm{d}\mu(g)\right|\ . \end{aligned}$$

We will prove the exchange formula (the third item of proposition 7.1.1) as an immediate consequence of the following three steps

Step 1: By lemma 7.2.3, $\alpha = \int_{\mathcal{G}} \beta_{\rho(g)} \mathrm{d}\mu_g$ is in $\boldsymbol{\Lambda}^\infty(E)$. By definition of a cutoff function, the support of $(1-\psi(R))\ \alpha$ vanishes at any point y so that $d(x,y) < R$. Thus the exponential decay lemma 4.2.2 guarantees that

$$A(R) = |\boldsymbol{S}_{x,g_0,Q}\left((1-\psi(R))\ \alpha\right)| \leqslant K_4 e^{-k_4 R} \|\alpha\|_\infty\ .$$

Hence $\lim_{R\to\infty} A(R) = 0$.

Step 2: Observe that

$$\psi_R \int_{\mathcal{G}} \beta_{\rho(g)}\, \mathrm{d}\mu(g) = \int_{\mathcal{G}} \psi_R \beta_{\rho(g)}\, \mathrm{d}\mu(g)\ .$$

Moreover the function $g \mapsto \psi_R \beta_g$ is continuous from $\mathcal{G}$ to $\boldsymbol{\Lambda}^\infty(E)$. Thus follows from the continuity of $\boldsymbol{S}_{x,g_0,Q}$ proved in proposition 4.2.1 implies that $B(R) = 0$.

Final Step: As a consequence of Lebesgue's dominated convergence theorem and the domination proved in lemma 7.4.2, we have that $\lim_{R\to\infty} C(R) = 0$.

Combining all steps

$$\lim_{R\to\infty} (A(R) + B(R) + C(R)) = 0\ .$$

Hence thanks to equation (71), we have

$$\boldsymbol{S}_{x,g_0,Q}\left(\int_{\mathcal{G}} \beta_{\rho(g)}\, \mathrm{d}\mu(g)\right) = \int_{\mathcal{G}} \boldsymbol{S}_{x,g_0,Q}\left(\beta_{\rho(g)}\right)\, \mathrm{d}\mu(g)\ .$$

7.6. **Proof of Theorem 7.0.2.** We assume now that μ is a Γ-compact current of order $k > 1$. We may also assume – by decomposing into the positive and negative part, that μ is a positive current.

Proof of the first item. We want to show that $\int_{\mathcal{G}^p} \Omega_{\rho(w(H))} \mathrm{d}\mu(H)$ — defined pointwise — is an element of $\boldsymbol{\Lambda}^\infty(E)$.

Since μ is Γ-compact, it follows that the core diameter of any H in the support of μ is bounded by some constant R_0 by proposition 3.1.1.

It will be enough to prove that

$$\int_{\mathcal{G}^p} \|\Omega_{\rho(w(H))}(y)\|_y\, \mathrm{d}\mu(H) \quad \leqslant \quad K_0\ ,$$

for some constant K_0 that depends on μ. Let $\mathcal{K}$ be a fundamental domain for the action of Γ on $\mathcal{G}^p$. Observe now that

$$\int_{\mathcal{G}^p} \|\Omega_{\rho(w(H))}(y)\|_y \,\mathrm{d}\mu(H) = \sum_{\gamma\in\Gamma} \int_{\gamma\mathcal{K}} \|\Omega_{\rho(w(H))}(y)\|_y \,\mathrm{d}\mu(H)$$

$$= \int_{\mathcal{K}} \left(\sum_{\gamma\in\Gamma} \|\Omega_{\rho(w(H))}(\gamma(y))\|_y \right) \mathrm{d}\mu(H) = \int_{\mathcal{K}} \|\psi_{w(H),y}\|_{\ell^1(\Gamma)} \,\mathrm{d}\mu(H) ,$$

where

$$\psi_{w(H),y} \,:\, \gamma \mapsto \|\Omega_{\rho(w(H))}(\gamma(y))\|_y .$$

By the second assertion of corollary 4.5.3, the map $\psi_{H,y}$ is in $\ell^1(\Gamma)$ and its norm is bounded by a continuous function of the core diameter $r(w(H))$ of $w(H)$, hence by a continuous function of $r(H)$ by inequality (61), hence by a constant on the support of μ, since r is Γ-invariant and continuous by proposition 3.1.1 and μ is Γ-compact.

Since $r(H)$ is bounded on the support of μ, the first item of the theorem follows. □

Proof of the second item. Let us consider the map

$$\Psi : H \mapsto \oint_{\rho(G)} \Omega_{\rho(w(H))} = \int_{\mathbf{H}^2} \mathrm{Tr}\left(\Omega_{\rho(w(H))} \wedge \Omega_{\rho(G)}\right) ,$$

where we used formula (42) in the last equality. Our goal is to prove Ψ is in $L^1(\mathcal{G}^p, \mu)$. We have that

$$\|\Omega_{\rho(w(H))} \wedge \Omega_{\rho(G)}(y)\| \leqslant \|\Omega_{\rho(w(H))}(y)\| \, \|\Omega_{\rho(G)}(y)\| .$$

It follows that

$$\begin{aligned}
\int_{\mathcal{G}^p} |\Psi(H)| \,\mathrm{d}\mu(H) &\leqslant \int_{\mathcal{G}^p} \int_{\mathbf{H}^2} \|\Omega_{\rho(G)}(y)\| \, \|\Omega_{\rho(w(H))}(y)\| \,\mathrm{d}y \,\mathrm{d}\mu(H) \\
&\leqslant \int_{\mathbf{H}^2} \|\Omega_{\rho(G)}(y)\| \left(\int_{\mathcal{G}^p} \|\Omega_{\rho(w(H))}(y)\| \;\mathrm{d}\mu(H) \right) \mathrm{d}y \\
&\leqslant K_0 \int_{\mathbf{H}^2} \|\Omega_{\rho(G)}(y)\| \,\mathrm{d}y = K_0 \;\left\|\Omega_{\rho(G)}\right\|_{L^1(\mathbf{H}^2)} ,
\end{aligned}$$

where we used the inequality in the first item in the third inequality. We can now conclude by using the first assertion, corollary 4.5.3. □

Proof of the exchange formula. We use again a family of cutoff functions $\{\psi_n\}_{n\in\mathbb{N}}$ defined on $\mathcal{G}^p$ with values in $[0, 1]$ so that each ψ_n has a compact support, and ψ_n converges to 1 uniformly on every compact set.

It follows from the Lebesgue's dominated convergence theorem and the second item that

$$\lim_{n\to\infty} \int_{\mathcal{G}^p} \left(\oint_{\rho(G)} \Omega_{\rho(w(H))} \right) \psi_n \,\mathrm{d}\mu(H) = \int_{\mathcal{G}^p} \left(\oint_{\rho(G)} \Omega_{\rho(w(H))} \right) \mathrm{d}\mu(H) . \tag{72}$$

Recall now that by the last assertion of corollary 4.5.3, $\|\Omega_{\rho(H)}\|_\infty$ is bounded on every compact set and Γ-invariant, hence bounded on the support of μ. Thus we have the following convergence in $\mathbf{\Lambda}^\infty(E)$

$$\lim_{n\to\infty} \int_{\mathcal{G}^p} \Omega_{\rho(w(H))} \;\psi_n \,\mathrm{d}\mu(H) = \int_{\mathcal{G}^p} \Omega_{\rho(w(H))} \,\mathrm{d}\mu(H) , \tag{73}$$

From the continuity obtained in proposition 4.3.3, we then have that

$$\lim_{n\to\infty} \oint_{\rho(G)} \int_{\mathcal{G}^p} \Omega_{\rho(w(H))} \;\psi_n \,\mathrm{d}\mu(H) = \oint_{\rho(G)} \int_{\mathcal{G}^p} \Omega_{\rho(w(H))} \,\mathrm{d}\mu(H) . \tag{74}$$

Finally, for every n, since ψ_n has compact support the following formula holds

$$\oint_{\rho(G)} \left(\int_{\mathcal{G}^p} \Omega_{\rho(w(H))} \;\psi_n \,\mathrm{d}\mu(w(H)) \right) = \int_{\mathcal{G}^p} \left(\oint_{\rho(G)} \Omega_{\rho(w(H))} \right) \psi_n \,\mathrm{d}\mu(H) .$$

The exchange formula now follows from both assertions (72) and (74). □

8. Hamiltonian and brackets: average of correlation and length functions

Recall that in this part, we have left the realm of uniformly hyperbolic bundles in general and focus only on periodic ones. This corresponds to the study of Anosov representations of the fundamental group of a closed surface.

The fact that S is closed allows us to introduce a new structure: the smooth part of the representation variety of projective representations carries the Goldman symplectic form, defined in paragraph 1.5, see also [18]. Hence we have a Poisson bracket on functions on the character variety.

In this section, we recall the definition of *length functions* in paragraph 8.1 and introduce *averaged correlation functions* in paragraph 8.2. The main results are then described in paragraph 8.3.

The proof of these results occupy the rest of the section.

(1) In paragraph 8.5, we prove the regularity of the functions that we consider.
(2) In paragrah 8.6, we use the result about derivatives of length functions to identify the Hamiltonian vector field of these length functions, and compute their Poisson brackets.
(3) In paragraph 8.7, we compute the bracket of a length function with a (non averaged) correlation function, using the formula obtained in proposition 4.6.1 relating the derivative of a correlation function with the ghost integration.
(4) In paragraph 8.8, we move to compute the Poisson bracket of a length function with an averaged correlation function.
(5) In paragraph 8.9 we use the previous computation to identify the Hamiltonian of a correlation function.
(6) Finally in paragraph 8.10, we compute the Poisson bracket of correlation functions.

All these computations rely on the technical results dealt with in the previous section: exchanging integrals.

8.1. **Averaged length function: definition.** As a first step in the construction, let us consider a Θ-decorated current μ^{a} supported on $C \times \{\mathrm{a}\}$ where a is in Θ. The associated *length function* on the character variety of Anosov representations is the function $\mathsf{L}^{\mathrm{a}}_{\mu^{\mathrm{a}}}$ defined by

$$\mathsf{L}^{\mathrm{a}}_{\mu^{\mathrm{a}}}(\rho) := \int_{\mathsf{U}\Sigma/\Gamma} R^{\sigma}_{\mathrm{a}}\, \mathrm{d}\mu^{\mathrm{a}} \ , \tag{75}$$

where R^{σ}_{a} is the (complex valued in the case of complex bundles) 1-form associated to a section σ of $\det(F_{\mathrm{a}})$ by $\nabla_u \sigma = R^{\sigma}(u)\cdot\sigma$. Although R^{σ} depends on the choice of the section σ, the integrand over $\mathsf{U}\Sigma$ does not in the real case. In the complex case, we see the length functions as taking values in $\mathbb{C}/2\pi i\mathbb{Z}$, since $2\pi i\mathbb{Z}$ is the the group of periods of the form $\frac{\mathrm{d}z}{z}$.

Recall that in our convention $\det(F_{\mathrm{a}})$ is a contracting bundle and thus the real part of L_{μ} is positive. Moreover for a closed geodesic γ, the associated geodesic current, supported on $C \times \{\mathrm{a}\}$ is also denoted by γ^{a}.

$$\exp\left(-\mathsf{L}^{\mathrm{a}}_{\gamma}(\rho)\right) = \det\left(\mathrm{Hol}(\gamma)\big|_{F_{\mathrm{a}}}\right) \ , \tag{76}$$

where $\mathrm{Hol}(\gamma)$ is the holonomy of γ. For a geodesic current δ supported on a closed geodesic, the length function L_{δ} is analytic. This extends to all geodesic currents by density and Morera's Theorem (See [8] for a related discussion in the real case). The notion extends naturally – by additivity – to a general Θ-geodesic current.

We can now extend the length function to any Θ-geodesic current. Let μ be a Θ-geodesic current on $C \times \Theta$, we can then write uniquely

$$\mu = \sum_{\mathrm{a}\in\Theta} \mu^{\mathrm{a}} \ ,$$

where μ^{a} is supported on $C \times \{\mathrm{a}\}$, then by definition the μ-averaged length function[7] is

$$\mathsf{L}_\mu(\rho) := \sum_{\mathrm{a}\in\Theta} \mathsf{L}^{\mathrm{a}}_{\mu^{\mathrm{a}}}(\rho)\ . \tag{77}$$

8.2. **Averaged correlation function: definition.** When w is a natural map, μ a (ρ, w)-integrable cyclic current, the associated *averaged correlation function of order n* $\mathsf{T}_{w(\mu)}$ on the moduli space of Θ-Anosov representations is defined by

$$\mathsf{T}_{w(\mu)}(\rho) := \int_{\mathcal{G}^n/\Gamma} \mathsf{T}_{w(G)}(\rho)\ \mathrm{d}\mu(G)\ , \tag{78}$$

where $G = (G_1, \ldots, G_p)$ with and T_G is the correlation function associated to a Θ-configuration of geodesics defined in paragraph 3.3. As we shall see in proposition 8.5.1, the function $\mathsf{T}_{w(\mu)}$ is analytic .

8.3. **The main result.** Our main result is a formula for the Poisson bracket of those functions. We use a slightly different convention, writing T^k for a correlation function of order k and $\mathsf{T}^1_\mu = \mathsf{L}_\mu$.

Theorem 8.3.1 (Ghost representation)**.** *Let μ be either a w-integrable Θ-cyclic currents at ρ_0 or a Θ-geodesic current. Similarly, let ν be either a v-integrable Θ-cyclic currents at ρ_0 or a Θ-geodesic current.*

Then the measure $\mu \otimes \nu$ is z-integrable at ρ_0, where $z(G, H) = [w(G), v(H)]$ and moreover

$$\begin{aligned}\{\mathsf{T}^p_{w(\mu)}, \mathsf{T}^n_{v(\nu)}\}(\rho) &= \int_{\mathcal{G}^{p+n}/\Gamma} \mathsf{I}_\rho\left(w(G), v(H)\right)\ \mathrm{d}\mu(G)\mathrm{d}\nu(H)\\ &= \int_{\mathcal{G}^{p+n}/\Gamma} \mathsf{T}_{[w(G),v(H)]}(\rho)\ \mathrm{d}\mu(G)\mathrm{d}\nu(H)\ .\end{aligned}$$

As a corollary, generalizing Theorem A given in the introduction, using a simple induction and proposition 5.4.2 we get

Corollary 8.3.2 (Poisson stability)**.** *The vector space generated by length functions, averaged correlations functions and constants is stable under Poisson bracket. More precisely, let $\mu_1, \ldots \mu_p$ cyclic currents of order n_i, and $N = n_1 + \ldots n_p$ then*

$$\{\mathsf{T}^{n_1}_{\mu_1}, \{\mathsf{T}^{n_2}_{\mu_2}, \ldots \{\mathsf{T}^{n_{p-1}}_{\mu_{p-1}}, \mathsf{T}^{n_p}_{\mu_p}\} \ldots\}\}(\rho) = \int_{\mathcal{G}^N/\Gamma} \mathsf{T}^N_{[G_1,[G_2,[\ldots,[G_{p-1},G_p]\ldots]]]}(\rho)\ \mathrm{d}\mu_1(G_1)\ldots\mathrm{d}\mu_1(G_p)\ .$$

In the course of the proof, we will also compute the Hamiltonians of the corresponding functions.

Theorem 8.3.3 (Hamiltonian)**.** *Let μ be a Θ-geodesic current. The Hamiltonian of the length function L_μ is H^0_μ the trace free part of H_μ, where*

$$H_\mu := -\int_{\mathcal{G}} \beta_{\rho(g)}\ \mathrm{d}\mu(g)\ ,$$

Let w be a natural function. Let ν be a (ρ, w) integrable cyclic current. The Hamiltonian of the correlation function $\mathsf{T}_w(\nu)$ of order n, with $n > 1$ is

$$\Omega_{w(\nu)} := \int_{\mathcal{G}^n} \Omega_{\rho(w(G))}\ \mathrm{d}\nu(G)\ ,$$

Both H_μ and $\Omega_{w(\nu)}$ are in $\mathbf{\Lambda}^\infty(E)$.

[7]In the complex case, since the logarithm, hence the length, is defined up to an additive constant, the Hamiltonian is well defined and the bracket of a length function and any other function makes sense.

8.4. **Preliminary and convention in symplectic geometry.** Our convention is that if f is a smooth function and $\mathbf{\Omega}$ a symplectic form, the *Hamiltonian vector field* X_f of f and the *Poisson bracket* $\{f,g\}$ of f and g are defined by

$$\mathrm{d}f(Y) = \mathbf{\Omega}(Y, X_f)\,, \tag{79}$$

$$\{f,g\} = \mathrm{d}f(X_g) = \mathbf{\Omega}(X_g, X_f) = -\mathrm{d}g(X_f)\,. \tag{80}$$

Observe that if Ω is a complex valued symplectic form – which naturally take entries in the complexified vector bundle – and f a complex valued function then the Hamiltonian vector field is a complexified vector field. The bracket of two complex valued functions is then a complex valued function.

In the sequel, we will not write different results in the complex case (complex valued symplectic form and functions) and the (usual) real case.

8.5. **Regularity of averaged correlations functions.** We prove here

Proposition 8.5.1. *Let w be a natural function. Let μ be a (ρ, w)-integrable current, then*

1. $\mathsf{T}_{w(\mu)}$ *is an analytic function in a neighborhood of* ρ*,*
2. *For any tangent vector v at ρ, then $\mathrm{d}\mathsf{T}_{w(G)}(v)$ is in $L^1(\mu)$ and*

$$\mathrm{d}\mathsf{T}_{w(\mu)}(v) = \int_{\mathcal{G}^n_\star/\Gamma} \mathrm{d}\mathsf{T}_{w(G)}(v)\mathrm{d}\mu(G)\,.$$

As in proposition 3.4.1, we work in the context of complex uniform hyperbolic bundles, possibly after complexification of the whole situtation.

Proof. Let us first treat the case when μ is Γ-compact. In that case, the functions $\mathsf{T}_G : \rho \mapsto T_{w(G)}$ are all complex analytic by proposition 3.4.1, uniformly bounded with uniformly bounded derivatives in the support of μ. Thus the result follows from classical results.

We now treat the non Γ-compact case. Let now consider an exhaustion of $\mathcal{G}^n_\star/\Gamma$ by compacts K_n and write $\mu_n = 1_{K_n}\mu$. Let then

$$\mathsf{T}_n = \int_{K_n} \mathsf{T}_{w(\mu_n)}\mathrm{d}\mu\,.$$

Then by our integrability hypothesis and Lebesgue dominated convergence Theorem, T_n converges uniformly to $\mathsf{T}_{w(\mu)}$. Since all T_n are complex analytic, by Morera Theorem, $\mathsf{T}_{w(\mu)}$ is complex analytic and T_n converges C^∞ to $\mathsf{T}_{w(\mu)}$. It thus follows that

$$\mathrm{d}\mathsf{T}_{w(\mu)}(v) = \lim_{n\to\infty} \mathrm{d}\mathsf{T}_n(v) = \lim_{n\to\infty}\int_{K_n} \mathrm{d}\mathsf{T}_{w(G)}(v)\mathrm{d}\mu(G)\,.$$

We now conclude the proof using Lemma A.0.3. ◻

8.6. **Length functions: their Hamiltonians and brackets.** We first start by computing the bracket and Hamiltonian of length functions. The first step in our proof is to understand the variation of length.

Proposition 8.6.1. *The derivatives of a length function with respect to a variation $\dot\nabla$ is given by*

$$\mathrm{d}\mathsf{L}_\mu(\dot\nabla) = \int_{\Theta\times\mathsf{U}\Sigma/\Gamma} \mathrm{Tr}\left(\mathrm{p}\dot\nabla\right)\,\mathrm{d}\mu(x)\,. \tag{81}$$

Proof. By the linearity of the definition, see equation (77), it is enough to consider a Θ-geodesic current μ^{a} supported on $C \times \{\mathrm{a}\}$.

Let $E^{\mathrm{a}} := \bigwedge^{\dim(F_{\mathrm{a}})} E$, and Λ^{a} the natural exterior representation from $\mathfrak{sl}(E)$ to $\mathfrak{sl}(E^{\mathrm{a}})$. Then by [21, Lemma 4.1.1] and formula (76) we have

$$\mathrm{d}\mathsf{L}_\mu(\dot\nabla) = \int_{\{\mathrm{a}\}\times\mathsf{U}\Sigma/\Gamma} \mathrm{Tr}\left(\mathrm{p}^1_{\mathrm{a}}\Lambda^{\mathrm{a}}(\dot\nabla)\right)\,\mathrm{d}\mu^{\mathrm{a}}(x)\,, \tag{82}$$

where p^1_a is the section of $\mathrm{End}(E_a)$ given by the projection on the line $\det(F_a)$ induced by the projection on F_a parallel to F°_a – see section 2.4 for notation.

We now conclude by observing–using just a litle bit of linear algebra– that for any element in $\mathfrak{sl}(E)$

$$\mathrm{Tr}\left(p^1_a\Lambda^a(A)\right) = \mathrm{Tr}\left(p_a A\right) .$$

Indeed let us choose a basis $(e_1,\ldots,e_p)$ of F_a completed by a basis $(f_1,\ldots,f_m)$ of F°_a and choose a metric so that this basis is orthonormal. Then

$$\Lambda^a(A)(e_1\wedge\ldots\wedge e_p) = \sum_{i=1^p} e_1\wedge\ldots e_{i-1}\wedge A(e_i)\wedge e_{i+1}\wedge\ldots e_p ,$$

$$\mathrm{Tr}\left(p^1_a\Lambda^a(A)\right) = \langle e_1\wedge\ldots\wedge e_p, \Lambda^a(A)(e_1\wedge\ldots\wedge e_p)\rangle = \sum_{i=1}^{p}\langle e_i\ ,\ A(e_i)\rangle = \mathrm{Tr}(p_a A) . \qquad \square$$

Let then

$$H_\mu = -\int_{\mathcal{G}} \beta_{\rho(g)}\ \mathrm{d}\mu(g) .$$

We proved that H_μ lies in $\mathbf{\Lambda}^\infty(E)$ in lemma 7.2.3. We now prove the following proposition.

Proposition 8.6.2 (Length functions). *The Hamiltonian vector field of L_μ is given by H^0_μ, which is the trace free part of H^μ. Then*

$$\{\mathsf{L}_\nu,\mathsf{L}_\mu\} = \mathbf{\Omega}(H^0_\mu,H^0_\nu) = \int_{\mathcal{G}^2_\star/\Gamma} \mathsf{I}_\rho(g,h)\ \mathrm{d}\nu(g)\otimes\mathrm{d}\mu(h) , \tag{83}$$

Observe that if μ and ν are both supported on finitely many geodesics, then the support of $\mu\otimes\nu$ is finite in $\mathcal{G}^2$ and its cardinality is the geometric intersection number of the support of μ, with the support of ν. This is a generalization of Wolpert cosine formula, see [28].

Remark that $\varepsilon\mu\otimes\nu$ is supported in $\mathcal{G}^2$ on a set on which Γ acts properly.

Proof. Let us first consider the computation of $\mathbf{\Omega}(H_\mu,H_\nu)$. Let Δ_0 be a fundamental domain for the action of Γ on $\mathbf{H}^2$ and Δ_1 be a fundamental domain for the action of Γ on $\mathcal{G}^2$. Then denoting p^0_g the traceless part of p_g

$$\begin{aligned}
\mathbf{\Omega}(H^0_\mu,H^0_\nu) &= \int_{\Delta_0}\mathrm{Tr}\left(\left(\int_{\mathcal{G}}\beta^0_h\mathrm{d}\mu(h)\right)\wedge\left(\int_{\mathcal{G}}\beta^0_g\mathrm{d}\nu(g)\right)\right)\\
&= \int_{\Delta_0}\int_{\mathcal{G}\times\mathcal{G}}\omega_h\wedge\omega_g\,\mathrm{Tr}\left(p^0(g)p^0(h)\right)\ \mathrm{d}\mu(h)\mathrm{d}\nu(g)\\
&= \int_{\Delta_1}\int_{\mathbf{H}^2}\omega_h\wedge\omega_g\,\mathrm{Tr}\left(p^0(g)p^0(h)\right)\ \mathrm{d}\mu(h)\mathrm{d}\nu(g)\\
&= \int_{\Delta_1}\varepsilon(h,g)\,\mathrm{Tr}(p^0(g)p^0(h))\ \mathrm{d}\mu(h)\mathrm{d}\nu(g) .
\end{aligned}$$

Let us comment on this series of equalities: the first one is the definition of the symplectic form and that of H_μ and H_ν, for the second one, we use the pointwise definition of H_μ and H_ν, for the third one we use proposition 1.1.5. Observe that the final equality gives formula (83).

From the third equality we also have

$$\mathbf{\Omega}(H^0_\mu,H^0_\nu) = \int_{\Delta_1}\left(\int_{g}\omega_h\,\mathrm{Tr}\left(p^0(g)p^0(h)\right)\right)\mathrm{d}\mu(g)\mathrm{d}\nu(h) .$$

Let now consider the fibration $z : \mathsf{U}\Sigma\times\mathcal{G}\to\mathcal{G}^2$ and observe that $z^{-1}(\Delta_1)$ is a fundamental domain for the action of Γ in $\mathsf{U}\Sigma\times\mathcal{G}$. Let Δ_2 be a fundamental domain for the action of Γ on $\mathsf{U}\Sigma$ and observe that $\Delta_2\times\mathcal{G}$ is a fundamental domain for the action on Γ on $\mathsf{U}\Sigma\times\mathcal{G}$. Then the above equation leads to

$$\mathbf{\Omega}(H^0_\mu,H^0_\nu) = \int_{z^{-1}(\Delta_1)}\omega_h\,\mathrm{Tr}\left(p^0(g)p^0(h)\right)\ \mathrm{d}\mu(g)\mathrm{d}\nu(h)$$

$$= \int_{\Delta_2\times\mathcal{G}} \omega_h \operatorname{Tr}\left(\mathrm{p}^0(g)\mathrm{p}^0(h)\right)\, \mathrm{d}\mu(g)\mathrm{d}\nu(h) \quad = \quad \int_{\Delta_2} \operatorname{Tr}\left(\mathrm{p}^0(g)\int_{\mathcal{G}}\beta^0_{\rho(h)}\mathrm{d}\nu(h)\right)\mathrm{d}\mu(g)$$
$$= -\int_{\Delta_2}\operatorname{Tr}\left(\mathrm{p}^0(g)H^0_\nu\right)\mathrm{d}\mu(g) \quad = \quad -\mathrm{d}\mathsf{L}_\mu(H^0_\nu) = \mathrm{d}\mathsf{L}_\nu(H^0_\mu)\ .$$

As a conclusion, it remains to prove that H^0_μ is the Hamiltonian of L_μ. For simplicity, let us prove it first when μ is the current δ_g supported on the closed geodesic g of $\mathbf{H}^2/\Gamma$. Let X be a closed Γ equivariant 1-form with values in $\operatorname{End}_0(E)$, where $\operatorname{End}_0(E)$ is the vector space of traceless endormorphisms, on $\mathbf{H}^2$.Then, by proposition 8.6.1

$$\mathrm{d}L_{\delta(g)}(X) = \int_g \operatorname{Tr}(\mathrm{p}^0(g)X^*)\ .$$

Let $\bar{g}$ be a lift of g in $\mathbf{H}^2$, Γ_0 the subgroup of Γ fixing $\bar{g}$. Then we can rewrite the previous equation as

$$\mathrm{d}L_{\delta(g)}(X) \quad = \quad \sum_{\gamma\in\Gamma/\Gamma_0}\int_{\gamma\bar{g}\cap\Delta_0}\operatorname{Tr}\left(\mathrm{p}^0(\gamma\bar{g})X\right)\ .$$

Let us now use χ_0 the characteristic function of Δ_0. We now have the string of equalities that we explain after

$$\begin{aligned}
\mathrm{d}L_{\delta(g)}(X) \quad &= \quad \sum_{\gamma\in\Gamma/\Gamma_0}\int_{\gamma\bar{g}}\operatorname{Tr}\left(\mathrm{p}^0(\gamma\bar{g})\chi_0X\right)\\
&= \quad \sum_{\gamma\in\Gamma/\Gamma_0}\int_{\mathbf{H}^2}\omega_{\gamma\bar{g}}\wedge\operatorname{Tr}\left(\mathrm{p}^0(\gamma\bar{g})\chi_0X\right)\\
&= \quad \int_{\mathbf{H}^2}\operatorname{Tr}\left(\left(\sum_{\gamma\in\Gamma/\Gamma_0}\omega_{\gamma\bar{g}}\mathrm{p}^0(\gamma\bar{g})\right)\wedge\chi_0X\right)\\
&= \quad \int_{\mathbf{H}^2}\operatorname{Tr}\left(\left(\int_C\omega_h\mathrm{p}^0(h)\,\mathrm{d}\delta_g(h)\right)\wedge\chi_0X\right)\\
&= \quad \int_{\mathbf{H}^2}\operatorname{Tr}(H^0_{\delta_g}\wedge\chi_0X) = \int_{\Delta_0}\operatorname{Tr}(H^0_{\delta_g}\wedge X) = \mathbf{\Omega}(H^0_{\delta_g},X)\ .
\end{aligned}$$

The second equality comes from the fact that χ_0X is with compact support, hence geodesically bounded and we can use proposition 1.1.5. The third equality uses the fact that we have a sum over only finitely many γ in Γ/Γ_0. The fourth uses the definition of δ_g, and the last ones comes from the definitions.

We have proved that H^0_μ is the Hamiltonian of L_μ for all μ, geodesic currents for a closed geodesic. The general result follows by density of this type of currents.

This completes the proof. ◻

As noted, the above gives a generalization of Wolpert's cosine formula. Explicitly we have for two Θ-geodesic currents μ,ν then

$$\{\mathsf{L}_\nu,\mathsf{L}_\mu\} = \int_{(\mathcal{G}^2)_\star/\Gamma}\varepsilon(g,h)\left(\operatorname{Tr}\left(\mathrm{p}(g)\mathrm{p}(h)\right) - \frac{\Theta(g)\Theta(h)}{\dim(E)}\right)\,\mathrm{d}\mu(g)\mathrm{d}\nu(h)\ . \tag{84}$$

8.7. **Bracket of length function and discrete correlation function.** We have

Proposition 8.7.1. *Let G be a Θ-configuration and μ a Θ-geodesic current, then*

$$\{\mathsf{T}_G,\mathsf{L}_\mu\} = -\int_{\mathcal{G}}\left(\oint_{\rho(G)}\beta_{\rho(g)}\right)\,\mathrm{d}\mu(g) = \int_{\mathcal{G}}\mathsf{I}_\rho(G,g)\mathrm{d}\mu(g)\ .$$

Proof. By proposition 44, we have

$$\mathrm{d}\mathsf{T}_G(H_\mu) = -\oint_{\rho(G)}\left(\int_{\mathcal{G}}\beta_{\rho(g)}\mathrm{d}\mu\right)\ .$$

Thus by the exchange formula (65), we have

$$\mathrm{dT}_G(H_\mu) = -\int_{\mathcal{G}}\left(\oint_{\rho(G)}\beta_{\rho(g)}\right)\,\mathrm{d}\mu(g)\ .$$

Thus we conclude by using equation (49)

$$\{\mathsf{T}_G,\mathsf{L}_\mu\} = \mathrm{dT}_G(H_\mu) = -\int_{\mathcal{G}}\left(\oint_{\rho(G)}\beta_{\rho(g)}\right)\,\mathrm{d}\mu(g) = \int_{\mathcal{G}}\mathsf{I}_\rho(G,g)\,\mathrm{d}\mu(g)\ .$$

□

8.8. **Bracket of length functions and correlation functions.** Our first objective is, given a family of flat connections $(\nabla_t)_{t\in]-\varepsilon,\varepsilon[}$ whose variation at zero is $\dot{\nabla}$, to compute $\mathrm{dT}_\mu(\dot{\nabla})$.

Proposition 8.8.1 (Bracket of length and correlation functions)**.** *Assume that the Θ-cyclic current μ is (ρ,w)-integrable. Then*

$$\{\mathsf{T}_{w(\mu)},\mathsf{L}_\nu\}(\rho) \quad = \quad \int_{\mathcal{G}^{n+1}/\Gamma}\mathsf{I}_\rho(w(G),g)\,\mathrm{d}\nu(g)\,\mathrm{d}\mu(G)\ .$$

Proof. By Theorem 8.6.2, the Hamiltonian vector field of L_ν is given by

$$H^0_\nu = -\int_{\mathcal{G}}\beta^0_{\rho(g)}\,\mathrm{d}\nu(g)\ .$$

Let Δ be a fundamental domain for the action of Γ on C^n, and observe that $\Delta\times\mathcal{G}$ is a fundamental domain for the action of Γ on $\mathcal{G}^{n+1}$. It follows since H_ν is ρ-equivariant and proposition 4.6.1 that

$$\begin{aligned}\{\mathsf{T}_{w(\mu)},\mathsf{L}_\nu\} = \mathrm{dT}_{w(\mu)}(H^0_\nu) \quad &= \quad \int_\Delta \mathrm{dT}_{w(G)}(H^0_\nu)\,\mathrm{d}\mu(G)\\ = \int_\Delta\left(\oint_{\rho(w(G))}H^0_\nu\right)\mathrm{d}\mu(G) \quad &= \quad -\int_\Delta\int_{\mathcal{G}}\left(\oint_{\rho(w(G))}\beta_{\rho(g)}\right)\ \mathrm{d}\nu(g)\mathrm{d}\mu(G)\\ = \int_\Delta\int_{\mathcal{G}}\left(\mathsf{I}_\rho(w(G),g)\right)\ \mathrm{d}\nu(g)\mathrm{d}\mu(G) \quad &= \quad \int_{\mathcal{G}^{n+1}/\Gamma}\mathsf{I}_\rho(w(G),g)\,\mathrm{d}\mu(G)\mathrm{d}\nu(g)\ .\end{aligned}$$

For the second equality we used proposition 8.5.1 and that integrating a 1-form with values in the center gives a trivial result by proposition 4.4.3. □

8.9. **Hamiltonian of correlation functions.** We now prove the following result.

Proposition 8.9.1 (Hamiltonian of correlation functions)**.** *Let w be a natural function. Let μ be a (ρ,w)-integrable Θ-current. Then for every y in $\mathbf{H}^2$, $\Omega_{\rho(G)}$ belongs to $L^1(\mathcal{G}^p,\mu)$. Moreover*

$$\Omega_{w(\mu)}(\rho) := \int_{\mathcal{G}^p}\Omega_{\rho(w(G))}\mathrm{d}\mu(G) \tag{85}$$

seen as vector field on the character variety, is the Hamiltonian of the correlation function $\mathsf{T}_{w(\mu)}$.

We first prove proposition 8.9.1 under the additional hypothesis that μ is a Γ-compact current, then move to the general case by approximation.

Proof for a Γ-compact current. Assume μ is a Γ-compact current. By the density of derivatives of length functions, it is enough to prove that for any geodesic current ν associated to a length function L_ν whose Hamiltonian is H_ν we have

$$\{\mathsf{L}_\nu,\mathsf{T}_{w(\mu)}\} = \mathbf{\Omega}(\Omega_{w(\mu)},H_\nu) = \mathrm{dL}_\nu(\Omega_{w(\mu)})\ .$$

Then using a fundamental domain Δ_0 for the action of Γ on $\mathsf{U}\Sigma$, and Δ_1 a fundamental domain for the action of Γ on $\mathcal{G}^n$, and finally denoting ν_0 the flow invariant measure in $\mathsf{U}\Sigma$ associated to the current ν

$$\mathrm{dL}_\nu(\Omega_{w(\mu)}) = \int_{\Delta_0}\mathrm{Tr}(\mathrm{p}\Omega_{w(\mu)})\mathrm{d}\nu_0(g) \quad = \quad \int_{\Delta_0}\left(\int_{\mathcal{G}^n}\mathrm{Tr}(\mathrm{p}(g)\Omega_{\rho(w(G))})\,\mathrm{d}\mu(G)\right)\mathrm{d}\nu_0(g)$$

$$\begin{aligned}
&= \int_{\mathcal{G}^n}\left(\int_{\Delta_0}\mathrm{Tr}(\mathsf{p}(g)\Omega_{\rho(w(G))})\,\mathrm{d}\nu_0(g)\right)\mathrm{d}\mu(G) &=& \int_{\Delta_1}\left(\int_{\mathsf{U}\Sigma}\mathrm{Tr}(\mathsf{p}(g)\Omega_{\rho(w(G))})\,\mathrm{d}\nu_0(g)\right)\mathrm{d}\mu(G)\\
&= \int_{\Delta_1}\int_{\mathcal{G}}\int_{g}\mathrm{Tr}(\mathsf{p}(g)\Omega_{\rho(w(G))}))\,\mathrm{d}\nu(g)\mathrm{d}\mu(G) &=& \int_{\mathcal{G}^n/\Gamma}\left(\int_{\mathcal{G}}\int_{\mathbf{H}^2}\mathrm{Tr}(\omega_g\mathsf{p}(g)\wedge\Omega_{\rho(w(G))})\right)\mathrm{d}\nu(g)\mathrm{d}\mu(G)\\
&= -\int_{(\mathcal{G}^n/\Gamma)\times\mathcal{G}}\mathsf{I}_\rho(w(G),g)\,\mathrm{d}\mu(G)\mathrm{d}\nu(g) &=& \{\mathsf{L}_\nu,\mathsf{T}_\mu\}\ .
\end{aligned}$$

The first equality uses equation (81), the second uses the definition of Ω_μ, the third one comes from Fubini's theorem, the fourth one from lemma A.0.2, the fifth one from the fibration from $\mathsf{U}\Sigma$ to C, the sixth one from formula (7), the seventh one definition (49). ☐

Proof for an integrable μ. Let us now prove the general case when μ is a ρ-integrable current. Let us consider an exhaustion $\{K_m\}_{m\in\mathbb{N}}$ of $\mathcal{G}^p/\Gamma$ by compact sets. Assume that the interior of K_{m+1} contains K_m. Let $\mathcal{K}$ be a fundamental domain of the action Γ on $\mathcal{G}^p_\star$. Let

$$\mathsf{T}_m(\rho) := \int_{K_m}\mathsf{T}_{w(G)}(\rho)\,\mathrm{d}\mu(G)\ .$$

The functions T_m are analytic and converges C^0 on every compact set to T_μ by the integrability of μ. Thus, by Morera's Theorem, T_μ is analytic and $\{\mathsf{T}_m\}_{m\in\mathbb{N}}$ converges C^∞ on every compact . Let us call X the Hamiltonian vector field of T_μ and X_m the Hamiltonian vector field of T_m. It follows that $\{X_m\}_{m\in\mathbb{N}}$ converges to X.

We let C_m be the preimage of K_m in $\mathcal{G}^p$. We have just proven in the previous paragraph that the Hamiltonian of T_m is

$$X_m = \int_{C_m}\Omega_{\rho(H)}\,\mathrm{d}\mu\ .$$

From corollary 4.5.3, for every y and H, the function $\gamma\mapsto\|\Omega_{\rho(\gamma w(H))}(y)\|$, is in $\ell^1(\gamma)$. It follows that

$$X_m(y) = \int_{C_m\cap\mathcal{K}}\left(\sum_{\gamma\in\Gamma}\Omega_{\rho(\gamma H)}(y)\,\mathrm{d}\mu(H)\right)\ .$$

Since $\{X_m(y)\}_{m\in\mathbb{N}}$ converges for any exhaustion of $\mathcal{K}$ to $X(y)$. It follows by lemma A.0.3 that

$$H\mapsto\sum_{\gamma\in\Gamma}\Omega_{\rho(\gamma w(H))}(y)\,\mathrm{d}\mu(H)\ ,$$

is in $L^1(\mathcal{K},\mu)$ and that

$$X(y) = \int_{\mathcal{K}}\sum_{\gamma\in\Gamma}\Omega_{\rho(\gamma w(H))}(y)\,\mathrm{d}\mu(H) = \int_{\mathcal{G}^p}\Omega_{\rho(w(H))}(y)\,\mathrm{d}\mu(H)\ ,$$

where we applied Fubini again in the last equality. This is what we wanted to prove. ☐

8.10. **Bracket of correlation functions.** We have

Proposition 8.10.1 (Bracket of correlation functions)**.** *Let μ and ν be two integrable Θ-currents of rank m and n respectively. Let $p = m+n$, then*

$$\{\mathsf{T}_{w(\nu)},\mathsf{T}_{v(\mu)}\} = \int_{\mathcal{G}^p/\Gamma}\mathsf{I}_\rho(w(H),v(G))\,\mathrm{d}\nu\otimes\mathrm{d}\mu(H,G)\ . \tag{86}$$

Proof. We have

$$\begin{aligned}
\{\mathsf{T}_{w(\nu)},\mathsf{T}_{v(\mu)}\} = \mathrm{d}\mathsf{T}_{w(\nu)}(\Omega_{v(\mu)}) &=& \int_{\mathcal{G}^n/\Gamma}\mathrm{d}\mathsf{T}_{w(H)}(\Omega_{v(\mu)})\,\mathrm{d}\nu(H)\\
= \int_{\mathcal{G}^n/\Gamma}\left(\oint_{\rho(w(H))}\Omega_{v(\mu)}\right)\mathrm{d}\nu(H) &=& \int_{\mathcal{G}^n/\Gamma}\left(\oint_{\rho(w(H))}\int_{\mathcal{G}^m}\Omega_{\rho(v(G))}\mathrm{d}\mu(G)\right)\mathrm{d}\nu(H)
\end{aligned}$$

$$= \int_{\mathcal{G}^n/\Gamma} \left(\int_{\mathcal{G}^m} \oint_{\rho(w(H))} \Omega_{\rho(v(G))} \mathrm{d}\mu(G) \right) \mathrm{d}\nu(H) \quad = \quad \int_{\mathcal{G}^p/\Gamma} \mathsf{I}_\rho(w(H), v(G)) \, \mathrm{d}\nu(H) \mathrm{d}\mu(G) \, .$$

The crucial point in this series of equalities is the exchange formula for the fifth equality which comes from Theorem 7.0.2. ☐

With the above, we have completed the proof of the ghost representation Theorem 8.3.1.

Part 3
Applications

We prove in this last part, two applications of the main result. First the convexity of length functions (in the projective case), then we explain that geodesic laminations define commuting subalgebras on the character varieties, a feature which is well known for Hitchin components by Bonahon–Dreyer [6], Zhe Sun and Tengren Zhang [26], Sun–Wienhard–Zhang [25], and and goes back in Teichmüller theory to Bonahon [4], but that we extend to any deformation space of Anosov representations.

9. Convexity of length functions

Our goal is a generalization of Kerckhoff theorem [14] of the convexity of length functions, as well as a generalization of Wolpert's Sine Formula for the second derivatives along twist orbits [29]. Both results will follow from computations in the ghost algebra combined with the Ghost Representation Theorem 8.3.1.

Our first theorem is a generalization of Wolpert's Sine Formula.

Theorem 9.0.1. *Let μ be an oriented geodesic current supported on non-intersecting geodesics. Then for any geodesic current ν and any projective representation ρ, we have*

$$\{\mathsf{L}_\mu, \{\mathsf{L}_\mu, \mathsf{L}_\nu\}\}(\rho) = 2 \int_{C^{3,+}/\Gamma} \varepsilon(g_0, h)\varepsilon(g_1, h) \left(\mathsf{T}_{\lceil g_1, h, g_0 \rceil - \lceil g_1, h \rceil \lceil g_0, h \rceil} \right)(\rho) \ \ \mathrm{d}\mu^2(g_1, g_0) \mathrm{d}\nu(h) \, .$$

where $C^{3,+}$ is the set of (g_1, h, g_0) so that h intersects both g_1 and g_0, as well as h intersecting g_1 before g_0.

Observe that in this formula, the representation is just assumed to be Anosov.

Let us first say, following Martone–Zhang [22] that a representation has *a positive cross ratio* if for all intersecting geodesics g and h

$$0 < \mathsf{T}_{\lceil g,h \rceil}(\rho) < 1 \, .$$

We now restate the Convexity Theorem B.

Theorem 9.0.2 (Convexity)**.** *Let μ be an oriented geodesic current supported on non-intersecting geodesics. Then for any geodesic current ν and any projective representation ρ with a positive cross ratio, we have*

$$\{\mathsf{L}_\mu, \{\mathsf{L}_\mu, \mathsf{L}_\nu\}\}(\rho) \geqslant 0 \, . \tag{87}$$

Furthermore the inequality is strict if and only if $i(\mu, \nu) \neq 0$.

We start by computing double brackets in the Ghost Algebra.

9.1. **Double derivatives of length functions in the swapping algebra.** In order to prove our convexity result, we will need to calculate double brackets. By Theorem 5.3.3, as the map $A \to \mathsf{T}_A$ on the ghost algebra factors through the extended swapping bracket $\mathcal{B}_0$, it suffices to do our calculations in $\mathcal{B}_0$. For simplicity, we will further denote the elements ℓ_g in $\mathcal{B}_0$ by g.

Lemma 9.1.1. *Let h be an oriented geodesic and g_0, g_1 be two geodesics so that $\varepsilon(g_0, g_1) = 0$. Let $\varepsilon_i = \varepsilon(g_i, h)$.*

1. *Assume first that $\varepsilon_0\varepsilon_1 = 0$, then $[g_1, [g_0, h]] = 0$.*
2. *Assume otherwise that h intersect g_1 before g_0 or that $g_1 = g_0$. Then*

$$\begin{aligned}[g_1, [g_0, h]] &= \varepsilon_1\varepsilon_0 \left(\lceil g_1, h, g_0\rceil - \lceil g_1, h\rceil \lceil g_0, h\rceil\right) \\ &= \varepsilon_1\varepsilon_0 \lceil g_1, h\rceil \lceil g_0, h\rceil \left(\lceil \gamma_0, \gamma_1\rceil - 1\right) ,\end{aligned}$$

where $\gamma_0 := (g_0^+, h^-)$ and $\gamma_1 := (h^+, g_1^-)$. [8]

Proof. From equation (54) of paragraph 5.3,

$$[g_0, h] = \varepsilon(h, g_0)\lceil g_0, h\rceil \ + \varepsilon(g_0, h)\mathbf{1} .$$

It follows that if $\varepsilon(g_0, h) = 0$, then

$$[g_1, [g_0, h]] = 0 . \tag{88}$$

The same holds whenever $\varepsilon(g_1, h) = 0$ by the symmetry given the Jacobi identity for the extended swapping bracket, which gives, since $[g_0, g_1] = 0$,

$$[g_1, [g_0, h]] = [g_0, [g_1, h]] .$$

Assume now that $\varepsilon_0\varepsilon_1 \neq 0$. Then let $(g_0, \zeta_0, h, \eta_0)$ be the associated ghost polygon to $\lceil g_0, h\rceil$ with ghost edges $\zeta_0 = (g_0^+, h^-)$ and $\eta_0 = (h^+, g_0^-)$. Thus from the hypothesis $\varepsilon(g_0, g_1) = 0$, and using the notation $\varepsilon_i = \varepsilon(g_i, h)$ we get from equation (59)

$$[g_1, [g_0, h]] = -\varepsilon_0\lceil g_0, h\rceil \left(\varepsilon_1\lceil g_1, h\rceil - \varepsilon(g_1, \zeta_0)\lceil g_1, \zeta_0\rceil - \varepsilon(g_1, \eta_0)\lceil g_1, \eta_0\rceil\right) .$$

Since h intersects g_1 before g_0, we have $\varepsilon(g_1, \eta_0) = 0$ and $\varepsilon(g_1, \zeta_0) = \varepsilon(g_1, h)$. Thus

$$[g_1, [g_0, h]] \quad = \quad \varepsilon_1\varepsilon_0 \left(\lceil g_1, \zeta_0\rceil\lceil g_0, h\rceil - \lceil g_1, h\rceil\lceil g_0, h\rceil\right) . \tag{89}$$

As $\zeta_0 = (g_0^+, h^-)$, formulating the computations the swapping algebra, we get

$$\begin{aligned}\lceil g_1, \zeta_0\rceil\lceil g_0, h\rceil &= \frac{(g_1^+, h^-)(g_0^+, g_1^-)(g_0^+, h^-)(h^+, g_0^-)}{(g_1^+, g_1^-)(g_0^+, h^-)(g_0^+, g_0^-)(h^+, h^-)} \\ &= \frac{(g_1^+, h^-)(h^+, g_0^-)(g_0^+, g_1^-)}{(g_1^+, g_1^-)(h^+, h^-)(g_0^+, g_0^-)} = \lceil g_1, h, g_0\rceil .\end{aligned} \tag{90}$$

Similarly

$$\frac{\lceil g_1, h, g_0\rceil}{\lceil g_1, h\rceil\lceil g_0, h\rceil} = \frac{(g_0^+, g_1^-)(h^+, h^-)}{(g_0^+, h^-)(h^+, g_1^-)} = \lceil (g_0^+, h^-), (h^+, g_1^-)\rceil . \tag{91}$$

The result then follows from equations (90) and the fact that $\gamma_0 = (g_0^+, h^-)$ and $\gamma_1 = (h^+, g_1^-)$. □

---

[8]Observe that γ_0 and γ_1 are not phantom geodesics by hypothesis.

9.2. **Proof of the Sine Formula Theorem 9.0.1.** We are now in position to prove the sine formula.

Proof. By the Representation Theorem and its corollary 8.3.2

$$\{\mathsf{L}_\mu, \{\mathsf{L}_\mu, \mathsf{L}_\nu\}\}(\rho) = \int_{C^3/\Gamma} \mathsf{T}_{[g_1,[g_0,h]]}(\rho)\, \mathrm{d}\mu(g_0)\mathrm{d}\mu(g_1)\mathrm{d}\nu(h)\ .$$

Since the support of μ consists of non intersecting geodesics, we have for g_0 and g_1 in the support of μ, $[g_0, g_1] = 0$. Hence the Jacobi identity for the swapping bracket gives

$$[g_0, [g_1, h]] = [g_1, [g_0, h]]\ ,$$

for g_0 and g_1 in the support of μ. It follows that

$$\int_{C^3/\Gamma} \mathsf{T}_{[g_1,[g_0,h]]}(\rho)\, \mathrm{d}\mu(g_0)\mathrm{d}\mu(g_1)\mathrm{d}\nu(h) = 2\int_{C^{3,+}/\Gamma} \mathsf{T}_{[g_1,[g_0,h]]}(\rho)\, \mathrm{d}\mu(g_0)\mathrm{d}\mu(g_1)\mathrm{d}\nu(h)\ .$$

Then we use lemma 9.1.1 to conclude. ◻

9.3. **Positivity.** Recall that a projective representation ρ has a positive cross ratio (according to Martone–Zhang[22]) if for all g, h intersecting geodesics $0 < \mathsf{T}_{\lceil g,h\rceil}(\rho) < 1$. Our goal is the following.

Proposition 9.3.1 (Sign proposition). *Assume ρ is a projective representation with a positive cross ratio. Let g_1, g_0 be such that $\varepsilon(g_0, g_1) = 0$. Then we have the inequality*

$$\mathsf{T}_{[g_1,[g_0,h]]}(\rho) \geqslant 0\ .$$

Furthermore the inequality is strict if and only if h intersects both g_0, g_1 in their interiors.

We first give an equivalent definition of positivity.

Lemma 9.3.2. *A projective representation ρ has a positive cross ratio if and only if for all (X, Y, y, x) cyclically oriented*

$$\mathsf{T}_{\lceil (X,x),(Y,y)\rceil}(\rho) > 1\ .$$

Proof. Let X, x, Y, y be 4 points. We observe that (X, Y, y, x) is cyclically oriented if and only if geodesics $(X, y), (Y, x)$ intersect. The result then follows from

$$\lceil (X, x), (Y, y)\rceil = \frac{(X, y)\ (Y, x)}{(X, x)\ (Y, y)} = \left(\frac{(X, x)\ (Y, y)}{(X, y)\ (Y, x)}\right)^{-1} = \lceil (X, y), (Y, x)\rceil^{-1}\ . \quad ◻$$

We now prove proposition 9.3.1

Proof of proposition 9.3.1. The Jacobi identity for the swapping bracket 5.3.2 gives that $[g_0, [g_1, h]] = [g_1, [g_0, h]]$ since $[g_0, g_1] = 0$. Thus in the statement of the proposition, we can always assume that not only h intersects both g_1 and g_0, but furthermore that h intersects g_1 before g_0. By lemma 9.1.1, to prove the proposition it is enough to prove that

$$\varepsilon_1\varepsilon_0 \mathsf{T}_{\lceil g_1,h,g_0\rceil - \lceil g_1,h\rceil\lceil g_0,h\rceil}(\rho) \geqslant 0\ , \tag{92}$$

and furthermore the inequality is strict if and only if h intersects both g_0, g_1 in their interiors (i.e. if and only if $|\varepsilon_0\varepsilon_1| = 1$). By Lemma 9.1.1 we have, since g_1 meets h before g_0.

$$\varepsilon_0\varepsilon_1 \left(\lceil g_1, h, g_0\rceil - \lceil g_1, h\rceil\lceil g_0, h\rceil\right) = \varepsilon_1\varepsilon_0 \lceil g_1, h\rceil\ \lceil g_0, h\rceil\ \left(\lceil \gamma_0, \gamma_1\rceil - 1\right)\ .$$

where $\gamma_0 := (g_0^+, h^-)$ and $\gamma_1 := (h^+, g_1^-)$. We will also freely use that if $x^+ = y^-$ or $x^- = y^+$, then $\mathsf{T}_{\lceil x,y\rceil} = 0$, while if $x^+ = y^+$ or $x^- = y^-$ then $\mathsf{T}_{\lceil x,y\rceil} = 1$.

First case: $\varepsilon_0\varepsilon_1 = 0$. In that case, we have equality.

Second case: $0 < |\varepsilon_0\varepsilon_1| < 1$. In that situation one of the end point of h is an end point of g_0 or g_1.

(1) Firstly, the cases $g_0^\pm = h^-$ or $g_1^\pm = h^+$ are impossible since h meets g_1 before g_0.
(2) Secondly if $g_1^+ = h^-$ or $g_0^- = h^+$, then $\mathsf{T}_{\lceil g_1,h\rceil}\mathsf{T}_{\lceil g_0,h\rceil}(\rho) = 0$.
(3) Finally, if $g_1^- = h^-$ or $g_0^+ = h^+$, then either $\gamma_0^+ = \gamma_1^+$ or $\gamma_0^- = \gamma_1^-$. In both cases, $\mathsf{T}_{\lceil \gamma_0,\gamma_1\rceil}(\rho) = 1$ and hence it follows that $\mathsf{T}_{\lceil g_1,h,g_0\rceil - \lceil g_1,h\rceil\lceil g_0,h\rceil}(\rho) = 0$.

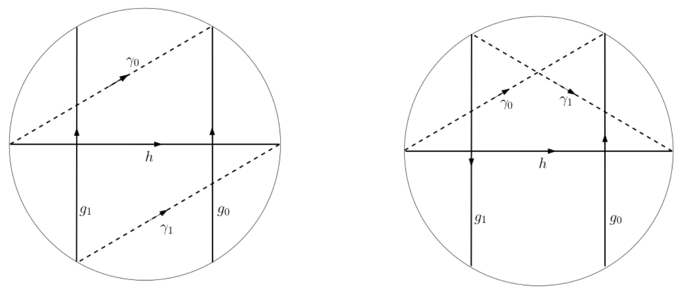
FINAL CASE: $|\varepsilon_0\varepsilon_1| = 1$.

As both g_0 and g_1 intersect h and ρ has a positive cross ratio, then by lemma 9.3.2,

$$\mathsf{T}_{\lceil g_1,h\rceil\,\lceil g_0,h\rceil}(\rho) = \mathsf{T}_{\lceil g_1,h\rceil}(\rho)\mathsf{T}_{\lceil g_0,h\rceil}(\rho) > 0\ . \tag{93}$$

We can then split into two cases as in figure (5):

(1) If $\varepsilon_0\varepsilon_1 > 0$, then γ_0 and γ_1 do not intersect, and (h^-, g_0^+, h^+, g_1^-) is a cyclically oriented quadruple. Hence, by definition $\mathsf{T}_{\lceil\gamma_0,\gamma_1\rceil}(\rho) > 1$. See figure (5i))
(2) If now $\varepsilon_0\varepsilon_1 < 0$, then γ_0 and γ_1 intersect, and by lemma 9.3.2 $\mathsf{T}_{\lceil\gamma_0,\gamma_1\rceil}(\rho) < 1$.(see figure (5ii))



(I) $\varepsilon_0\varepsilon_1 > 0$ (II) $\varepsilon_0\varepsilon_1 < 0$

FIGURE 5. Curves γ_0 and γ_1

Combining both cases, we get that

$$\varepsilon_0\varepsilon_1\left(\mathsf{T}_{\lceil\gamma_0,\gamma_1\rceil}(\rho) - 1\right) > 0\ . \tag{94}$$

The result follows from equations (94) and (93). □

9.4. **Proof of the Convexity Theorem 9.0.2.**

Proof. By the representation theorem and its corollary 8.3.2

$$\{\mathsf{L}_\mu, \{\mathsf{L}_\mu, \mathsf{L}_\nu\}\}(\rho) = \int_{C^3/\Gamma} \mathsf{T}_{[g_1,[g_0,h]]}(\rho)\ \mathrm{d}\mu(g_0)\mathrm{d}\mu(g_1)\mathrm{d}\nu(h)\ .$$

Since by the Sign Proposition 9.3.1, the integrand is non-negative, the integral is non-negative.

Let us finally treat the equality case. If $i(\mu,\nu) = 0$ then for all g in the support of μ and h in the support of ν, $|\varepsilon(g,h)| < 1$. Thus by the equality case of proposition 9.3.1 for g_0, g_1 in the support of μ and h in the support of ν then

$$\mathsf{T}_{[g_1,[g_0,h]]}(\rho) = 0\ .$$

Thus the integral is zero for $i(\mu,\nu) = 0$.

If $i(\mu,\nu) \neq 0$ then there exists g_0, h in the supports of μ,ν respectively such that $|\varepsilon(g_0,h)| = 1$. If h is descends to a closed geodesic then it is invariant under an element γ of Γ then we let $g_1 = \gamma g_0$. Then the triple (g_1, g_0, h) is in the support of $\mu\otimes\mu\otimes\nu$. Thus $\mathsf{T}_{[g_1,[g_0,h]]}(\rho) > 0$ and the integral is positive. If h does not descend to a closed geodesic, then as any geodesic current is a limit of a discrete geodesic currents, it follows that h intersects $g_1 = \gamma g_0$ for some γ in Γ. Again the triple (g_1, g_0, h) are in the support of $\mu\otimes\mu\otimes\nu$ with $\mathsf{T}_{[g_1,[g_0,h]]}(\rho) > 0$. Thus the integral is positive. This completes the proof of Theorem 9.0.2. □

10. Commuting subalgebras

Our second application allows us to construct commuting subalgebras in the Poisson algebra of correlation functions for projective Ansov representations. Let $\mathcal{L}$ be a geodesic lamination. Associated to this lamination we get several functions that we called *associated to the lamination*

(1) The length functions associated to geodesic currents supported on the lamination,
(2) functions associated to any complementary region of the lamination.

Let $F_{\mathcal{L}}$ be the vector space generated by these functions. Our result is then,

Theorem 10.0.1. *Let $\mathcal{L}$ be a geodesic lamination, then the vector space $F_{\mathcal{L}}$ consists of pairwise Poisson commuting functions.*

An interesting example is the case of the maximal geodesic lamination coming from a decomposition into pair of pants. An easy check gives that there are $6g-6$ length functions, and $4g-4$ triangle functions. Thus we have $10g-10$ commuting functions. However in the $\mathsf{PSL}_3(\mathbb{R})$ case the dimension of the space is $16g-16$ and it follows that there are relations between these functions. It is interesting to notice that these relations may not be algebraic ones: In that specific case some relations are given by the higher identities [20] generalizing Mirzakhani–McShane identities.

As we said before the fact that these subalgebras are commuting is related, in the special case of Hitchin representations, to the coordinate systems discussed in [6, 26, 25].

10.1. Triangle functions and double brackets.

Let $\delta_0 = (a_1, a_2, a_3)$ be an oriented ideal triangle, we associate to such a triangle the configuration

$$t_0 := \lceil a_1, a_3, a_2 \rceil \,. \tag{95}$$

The reader should notice the change of order.

One can make the following observation. First $t\,\bar{t} = 1$. Thus for a self-dual representation ρ, we have $\mathsf{T}_t(\rho)^2 = 1$ and in particular T_t is constant along self dual representations.

Lemma 10.1.1 (Brackets of triangle functions). *Let $t_0 = \lceil a_1, a_3, a_2 \rceil$ be a triangle, then*

$$[t_0, g] \quad = \quad \sum_{j\in\{1,2,3\}} \varepsilon(a_j, g)\, t_0 \left(\lceil g, a_j\rceil + \lceil g, \bar{a}_j\rceil\right) \,.$$

Let $t_1 = \lceil b_1, b_3, b_2 \rceil$. Then

$$[t_1, t_0] = t_1 \cdot t_0 \sum_{i,j\in\{1,2,3\}} \varepsilon(a_i, b_j)(\lceil a_i, b_j\rceil + \lceil a_i, \overline{b}_j\rceil + \lceil \overline{a}_i, b_j\rceil + \lceil \overline{a}_i, \overline{b}_j\rceil) = t_0 \sum_{i\in\{1,2,3\}} [t_1, \overline{a}_i - a_i] \,.$$

Proof. Observe first that the hypothesis imply that $[t_0, t_1] = 0$. Thus, by the Jacobi identity,

$$[t_0, [t_1, g]] = [t_1, [t_0, g]] \,.$$

The ghost polygon associated to t is $(a_1, \overline{a}_2, a_3, \overline{a}_1, a_2, \overline{a}_3)$. Thus

$$[t_0, g] = t_0 \sum_{j\in\{1,2,3\}} \varepsilon(a_j, g)\lceil g, a_j\rceil - \varepsilon(\overline{a}_j, g)\lceil g, \bar{a}_j\rceil = t_0 \sum_{j\in\{1,2,3\}} \varepsilon(a_j, g)\left(\lceil g, a_j\rceil + \lceil g, \bar{a}_j\rceil\right) \,.$$

For t_0, t_1 we have

$$\begin{aligned}
[t_1, t_0] \quad &= \quad t_1 \cdot t_0 \sum_{i,j\in\{1,2,3\}} \varepsilon(a_i, b_j)\lceil a_i, b_j\rceil - \varepsilon(a_i, \overline{b}_j)\lceil a_i, \overline{b}_j\rceil - \varepsilon(\overline{a}_i, b_j)\lceil \overline{a}_i, b_j\rceil + \varepsilon(\overline{a}_i, \overline{b}_j)\lceil \overline{a}_i, \overline{b}_j\rceil \\
&= \quad t_0 \cdot t_1 \sum_{i,j\in\{1,2,3\}} \varepsilon(a_i, b_j)(\lceil a_i, b_j\rceil + \lceil a_i, \overline{b}_j\rceil + \lceil \overline{a}_i, b_j\rceil + \lceil \overline{a}_i, \overline{b}_j\rceil) \\
&= \quad t_0 \sum_{i\in\{1,2,3\}} [t_1, \overline{a}_i - a_i]
\end{aligned}$$

∎

10.2. **Proof of Commuting Subalgebra Theorem 10.0.1.** We now consider $\mathcal{L}$ a maximal lamination and $F_{\mathcal{L}}$ the vector space of functions associated to $\mathcal{L}$ generated by triangle functions and length functions supported on $\mathcal{L}$. The proof of Theorem C that $F_{\mathcal{L}}$ is a commuting subalgebra now follows from the below proposition.

Proposition 10.2.1. *Let g be disjoint from the interior of ideal triangle δ. Then g and the triangle function t commute. Similarly let δ_0, δ_1 be ideal triangles with disjoint interiors. Then the associated triangle functions t_0, t_1 commute.*

Proof. We first make an observation. If $\varepsilon(g,h) = \pm 1/2$ then

$$\lceil g, h \rceil + \lceil g, \overline{h} \rceil = 1 \ .$$

To see this, assume $g^+ = h^-$. Then $\lceil g, h\rceil = 0$ and $\lceil g, \overline{h}\rceil$ has ghost polygon $(g, \overline{h}, \overline{h}, g)$ giving

$$\lceil g, \overline{h} \rceil = \frac{\overline{h} \cdot g}{g \cdot \overline{h}} = 1 \ .$$

By symmetry, this holds for all g, h with $\varepsilon(g,h) = \pm 1/2$.

Let g be disjoint from the interior of ideal triangle $\delta = (a_1, a_2, a_3)$. Then from above

$$[g, t] = t \sum_{i \in \{1,2,3\}} \varepsilon(g, a_i)(\lceil g, a_i \rceil + \lceil g, \overline{a}_i \rceil) = t \sum_{i \in \{1,2,3\}} \varepsilon(g, a_i) \ .$$

If $\varepsilon(g, a_i) = 0$ for all i then trivially $[g,t] = 0$. Thus we can assume $\varepsilon(g, a_1) = 0$ and $\varepsilon(g, a_2), \varepsilon(g, a_3) \neq 0$. If $g = a_1$ then as $\varepsilon(a_1, a_2) = -\varepsilon(a_1, a_3)$ then $[g, t] = 0$. Similarly for $g = \overline{a_1}$.

Otherwise g, a_2, a_3 share a common endpoint and a_2, a_3 have opposite orientation at the common endpoint. Therefore as g is not between a_2 and a_3 in the cyclic ordering about their common endpoint, then $\varepsilon(g, a_2) = -\varepsilon(g, a_3)$ giving $[g, t] = 0$.

Let t_0, t_1 be the triangle function associated to ideal polygons δ_0, δ_1 with $t_0 = [a_1, a_3, a_2]$. Then from above

$$[t_1, t_0] = t_0 \sum_i [t_1, \overline{a}_i - a_i] \ .$$

Thus if t_0, t_1 have ideal triangles with disjoint interiors then by the above, $[a_i, t_1] = [\overline{a}_i, t_1] = 0$ giving $[t_0, t_1] = 0$. ◻

Part 4
Addendum

Appendix A. Fundamental domain and L^1-functions

If Γ is a countable group acting on X preserving a measure μ, a *μ-fundamental domain* for this action is a measurable set Δ so that $\sum_{\gamma \in \Gamma} \mathbf{1}_{\gamma(\Delta)} = 1$, μ-almost everywhere. A function F on X is Γ-invariant if for every γ in Γ, $F = F \circ \gamma$, μ–almost everywhere. Then

Lemma A.0.1. *For any Γ-invariant positive function, if Δ_0 and Δ_1 are μ-fundamental domain then*

$$\int_{\Delta_0} F \, \mathrm{d}\mu = \int_{\Delta_1} F \, \mathrm{d}\mu \, .$$

Proof. Using the γ-invariance of F

$$\int_{\Delta_0} F = \sum_{\gamma\in\Gamma} \int_X F \cdot \mathbf{1}_{\Delta_0\cap\gamma(\Delta_1)} \mathrm{d}\mu = \sum_{\eta\in\Gamma} \int_X F \cdot \mathbf{1}_{\eta(\Delta_0)\cap\Delta_1} \mathrm{d}\mu = \int_{\Delta_1} F \, .$$ □

Let Γ be a group acting properly on X_0 and X_1 preserving μ_0 and μ_1 respectively. Assume that Δ_0 – respectively Δ_1 – is a fundamental domain for the action of Γ on X_0 and X_1, then.

Lemma A.0.2. *Let F be a positive function on $X_0 \times X_1$ which is Γ invariant, where Γ acts diagonally and the action on each factor preserves measures called μ_0 and μ_1 and admits a fundamental domain called Δ_0 and Δ_1, then*

$$\iint_{\Delta_0\times X_1} F \, \mathrm{d}\mu_0 \otimes \mathrm{d}\mu_1 = \iint_{X_0\times\Delta_1} F \, \mathrm{d}\mu_0 \otimes \mathrm{d}\mu_1 \, .$$

Proof. Indeed $\Delta_0 \times X_1$ and $X_0 \times \Delta_1$ are both fundamental domains for the diagonal action of Γ on $X_0 \times X_1$. The lemma then follows from the previous one and Fubini's theorem. □

Let f be a continuous function defined on a topological space X. Let μ be a Radon measure on X. Then the following lemma holds as a consequence of Lebesgue dominated convergence.

Lemma A.0.3. *Assume that there exists a real constant k so that for every exhausting sequence $\{K_m\}_{m\in\mathbb{N}}$ of compacts of X, $\lim_{m\to\infty} \int_{K_m} f \, \mathrm{d}\mu = k$. Then f belongs to $L^1(X, \mu)$ and $\int_X f \, \mathrm{d}\mu = k$.*

Appendix B. A lemma in hyperbolic geometry

Lemma B.0.1. *For any geodesic g and g_0, where g_0 is parametrized by arclength, the following holds. If $R > 1$ and $d(g_0(R), g) < 2$, while $d(g_0(R-1), g) \geqslant 2$, then*

$$d(g_0(0), g) \geqslant R \, .$$

Proof. We let h be a geodesic with $d(g_0(R), h) = d(g_0(R-1), h) = 2$. Then we observe that $d(g_0(0), g) \geqslant d(g_0(0), h)$. We drop perpendiculars from $g_0(R-1)$, $g_0(R-\frac{1}{2})$ and $g_0(0)$ to h. The perpendicular from $g_0(R-1)$ to h is length 2 and let a be the length of the perpendicular from $g_0(R-\frac{1}{2})$. For the Lambert quadrilateral with opposite sides of length a and 2 we have

$$\sinh(a)\cosh\left(\frac{1}{2}\right) = \sinh(2) \quad , \quad \sinh(a)\cosh\left(R - \frac{1}{2}\right) = \sinh D \, ,$$

where $D = d(g_0(0), h)$. It follows easily that

$$\frac{e^D}{2} \geqslant \sinh(D) = \sinh(a)\cosh\left(R - \frac{1}{2}\right) \geqslant \frac{\sinh(a)}{2} e^{R-1/2} \, .$$

Thus

$$d(g_0(0), g) \geqslant D \geqslant R - \frac{1}{2} + \log(\sinh(a)) \geqslant R \, . \quad \square$$

Appendix C. The Jacobi identity for the Θ-ghost bracket

We now explain the Jacobi identity for polygons with disjoint set of vertices.

C.1. **Linking number on a set.** Let us recall some constructions from [19]. Let P be a set, $\mathcal{G}_1$ be the set of pair of points of P. We denote temporarily the pair (X, x) with the symbol Xx. We also define a *linking number* on P to be a map from P^4 to a commutative ring $\mathbb{A}$ $(X, x, Y, y) \to \varepsilon(Xx, Yy)$, so that for all points X, x, Y, y, Z, z the following conditions are satisfied

$$\begin{aligned}
\varepsilon(Xx, Yy) + \varepsilon(Xx, yY) = \varepsilon(Xx, Yy) + \varepsilon(Yy, Xx) &= 0\,, \\
\varepsilon(zy, XY) + \varepsilon(zy, YZ) + \varepsilon(zy, ZX) &= 0\,, \\
\varepsilon(Xx, Yy).\varepsilon(Xy, Yx) &= 0\,.
\end{aligned}$$

The second author proved in [19, Proposition 2.1.3] the following.

Proposition C.1.1 (The hexagonal relation)**.** *Let (X, x, Z, z, Y, y) be 6 points on the set* P *equipped with an linking number, then*

$$\varepsilon(Xy, Zz) + \varepsilon(Yx, Zz) = \varepsilon(Xx, Zz) + \varepsilon(Yy, Zz). \tag{96}$$

Moreover, if $\{X, x\} \cap \{Y, y\} \cap \{Z, z\} = \emptyset$, then

$$\varepsilon(Xx, Yy)\varepsilon(Xy, Zz) + \varepsilon(Zz, Xx)\varepsilon(Zx, Yy) + \varepsilon(Yy, Zz)\varepsilon(Yz, Xx) = 0\,, \tag{97}$$

$$\varepsilon(Xx, Yy)\varepsilon(Yx, Zz) + \varepsilon(Zz, Xx)\varepsilon(Xz, Yy) + \varepsilon(Yy, Zz)\varepsilon(Zy, Xx) = 0\,. \tag{98}$$

C.2. **The ghost algebra of a set with a linking number.**

C.2.1. *Ghost polygons and edges.* We say a *geodesic* is a pair of points in P. We write $g = (g_-, g_+)$. A *configuration* $G := \lceil g_1, \ldots g_n \rceil$ is a tuple of geodesics $(g_1, \ldots g_n)$ up to cyclic ordering, with $n \geqslant 1$. The positive integer n is the rank of the configuration.

To a configuration of rank greater than 1, we associate a *ghost polygon*, also denoted G which is a tuple $G = (\theta_i, \ldots, \theta_{2n})$ where $g_i = \theta_{2i}$ are the *visible edges* and $\varphi_i = \theta_{2i+1} := ((g_{i+1})_-, (g_i)_+)$ are the ghost edges.

The *ghost index* i_e of an edge e is an element of $\mathbb{Z}/2\mathbb{Z}$ which is zero for a visible edge and one for a ghost edge. In other words $i_{\theta_k} := k$ [2].

We will then denote by $G_\circ$ the set of edges (ghost or visible) of the configuration G.

Geodesics, or rank 1 configurations, play a special role. In that case $G = \lceil g \rceil$, by convention $G_\circ$ consists of of single element g which is a visible edge.

C.2.2. *Opposite edges.* We now define the *opposite* of an edge in a configuration. Recall that a configuration is a tuple up to cyclic permutation. In this section we will denote a tuple by $\lfloor g_1, \ldots g_n \rfloor$. We denote by $\bullet$ the concatenation of tuples:

$$\lfloor g_1, \ldots g_n \rfloor \bullet \lfloor h_1, \ldots h_p \rfloor := \lfloor g_1, \ldots g_n, h_1, \ldots h_p \rfloor\,.$$

We introduce the following notation. If θ is a visible edge of G, we define $\theta_+ = \theta_- = \theta$ and if θ is a ghost edge of G then we define θ_+ to be the visible edge after θ and θ_- the visible edge before. The opposite of an edge is $\theta^* := \lfloor \theta_+ \ldots \theta_- \rfloor$ where the ordering is an increasing ordering of visible edges from θ_+ to θ_-. More specifically

(1) For a visible edge g_i, the opposite is the tuple $g_i^* = \lfloor g_i, g_{i+1}, \ldots g_{i-1}, g_i \rfloor$,
(2) while for a ghost edge φ_i the opposite is $\varphi_i^* = \lfloor g_{i+1}, g_{i+2}, \ldots, g_{i-1}, g_i \rfloor$.
(3) if $\lceil h \rceil$ is a rank 1 configuration. The opposite of its unique edge h is h itself.

C.3. **Ghost bracket and our main result.** We now define the *ghost algebra* of P to be the polynomial algebra $\mathcal{A}_0$ freely generated by ghost polygons and geodesics. The ghost algebra is equipped with the antisymmetric *ghost bracket*, given on the generators $\mathcal{A}$ by, for two ghosts polygons B and C and geodesics g and h,

$$[B, C] = \sum_{(b,c) \in B_\circ \times C_\circ} \varepsilon(c, b)(-1)^{i_b + i_c} \lceil c^*, b^* \rceil\,. \tag{99}$$

It is worth writing down the brackets of two geodesics g and h, as well as the bracket of a geodesic g and a configuration B,

$$-[g,B]=[B,g] = \sum_{b\in B_\circ} \varepsilon(g,b)(-1)^{i_b+1}\lceil g,b^*\rceil\,, \tag{100}$$

$$-[g,h]=[h,g] = \varepsilon(g,h)\lceil g,h\rceil\,. \tag{101}$$

Our goal in this section is to prove

Theorem C.3.1 (Jacobi identity). *Let A, B, C be three ghost polygons with no common vertices:*

$$V_A\cap V_B\cap V_C=\emptyset, \tag{102}$$

where V_G is the set of vertices of the ghost polygon G. Then the ghost bracket satisfies the Jacobi identity for A, B, C:

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0\,.$$

As the formula for the bracket differs based on whether ghost polygons are rank 1 or higher, we will need to consider the different cases based on the rank of the three elements. We will denote rank 1 elements by a,b,c and higher rank by A,B,C. For a, b and c edges in A, B, C ghost or otherwise, we label their ghost indexes by i_a ,i_b, i_c and their opposites by a^*,b^*,c^*.

C.4. **Preliminary: more about opposite edges.** We also use the following notation: if θ_k and θ_l are two edges, ghost or visible of a ghost polygon, then

$$G(\theta_k,\theta_l)=\lfloor\theta_{k+}\ldots\theta_{l-}\rfloor\,,$$

where again this is an increasing ordering of visible edges. The tuple $G(\theta_k,\theta_l)$ is an "interval" defined by θ_k and θ_l. In order to continue our description of the triple brackets, we need to understand, in the above formula, what are the opposite of φ^* in $[b^*,c^*]$. Our preliminary result is the following.

Lemma C.4.1 (Opposite edges in a bracket). *Let B and C be two ghost polygons, b and c edges in B and C respectively. Let φ be an edge in $\lceil b^*,c^*\rceil$, then we have the following eight possibilities*

1: *Either φ is an edge of B, different from b or a ghost edge, then*

$$\varphi^*=G(\varphi,b)\bullet c^*\bullet G(b,\varphi)\,,$$

2: *b is a visible edge, φ is the initial edge b in b^* and then*

$$\varphi^*=b^*\bullet c^*\bullet b\,.$$

3: *b is a visible edge, φ is the final edge b in b^* and then*

$$\varphi^*=b\bullet c^*\bullet b^*\,.$$

4, 5, 6: *Or φ is an an edge of C, and the three items above apply with some obvious symmetry, giving three more possibilities.*

7: *or φ is the edge $u_{b,c}:=(c_-^-,b_+^+)$ of $\lceil b^*,c^*\rceil$ which is neither an edge of b nor an edge of c, a ghost edge, and*

$$\varphi^*=\lfloor c^*,b^*\rfloor\,.$$

8: *φ is the edge $u_{c,b}:=(b_-^-,c_+^+)$ of $\lceil b^*,c^*\rceil$ which is neither an edge of b nor an edge of c, a ghost edge, and*

$$\varphi^*=\lfloor b^*,c^*\rfloor\,.$$

Proof. This follows from a careful book-keeping and the previous definitions. □

C.5. **Cancellations.** Let us introduce the following quantities for any triple of polygons A, B, C whatever their rank. They will correspond to the cases obtained corresponding to the cases observed in lemma C.4.1:

$$\begin{aligned}
\text{Case 1: } P_1(A,B,C) &:= \sum_{\substack{(a,c,b,\varphi)\in A_\circ\times C_\circ\times B_\circ^2\\ \varphi\neq b}} \varepsilon(a,\varphi)\varepsilon(c,b)(-1)^{i_a+i_\varphi+i_b+i_c}\lceil a^*\bullet G(\varphi,b)\bullet c^*\bullet G(b,\varphi)\rceil ,\\
\text{Case 2: } P_2(A,B,C) &:= \sum_{\substack{(a,b,c,\varphi)\in A_\circ\times B_\circ\times C_\circ^2\\ \varphi\neq c}} \varepsilon(a,\varphi)\varepsilon(c,b)(-1)^{i_a+i_\varphi+i_b+i_c}\lceil a^*\bullet G(\varphi,c)\bullet b^*\bullet G(c,\varphi)\rceil ,\\
\text{Case 3: } Q_1(A,B,C) &:= \sum_{(a,b,c)\in A_\circ\times B_\circ\times C_\circ} \varepsilon(a,b)\varepsilon(c,b)(-1)^{i_a+i_c}\lceil a^*\bullet b\bullet c^*\bullet b^*\rceil ,\\
\text{Case 4: } Q_2(A,B,C) &:= \sum_{(a,b,c)\in A_\circ\times B_\circ\times C_\circ} \varepsilon(a,b)\varepsilon(c,b)(-1)^{i_a+i_c}\lceil a^*\bullet b^*\bullet c^*\bullet b\rceil ,\\
\text{Case 5: } R_1(A,B,C) &:= \sum_{(a,b,c)\in A_\circ\times B_\circ\times C_\circ} \varepsilon(a,c)\varepsilon(c,b)(-1)^{i_a+i_b}\lceil a^*\bullet c\bullet b^*\bullet c^*\rceil ,\\
\text{Case 6: } R_2(A,B,C) &:= \sum_{(a,b,c)\in A_\circ\times B_\circ\times C_\circ} \varepsilon(a,c)\varepsilon(c,b)(-1)^{i_a+i_b}\lceil a^*\bullet c^*\bullet b^*\bullet c\rceil ,\\
\text{Case 7: } S_1(A,B,C) &:= \sum_{(a,b,c)\in A_\circ\times B_\circ\times C_\circ} \varepsilon(a,u_{b,c})\varepsilon(c,b)(-1)^{i_a+i_c+i_b}\lceil a^*\bullet c^*\bullet b^*\rceil ,\\
\text{Case 8: } S_2(A,B,C) &:= \sum_{(a,b,c)\in A_\circ\times B_\circ\times C_\circ} \varepsilon(a,u_{c,b})\varepsilon(c,b)(-1)^{i_a+i_c+i_b}\lceil a^*\bullet b^*\bullet c^*\rceil .
\end{aligned}$$

We then have,

Lemma C.5.1 (Cancellations). *We have the following cancellations, where the two last ones use the hypothesis* (102)

$$\begin{aligned}
P_1(A,B,C)+P_2(C,A,B) &= 0\,, \text{ first cancellation },\\
R_1(A,B,C)+Q_2(B,C,A) &= 0\,, \text{ second cancellation-1 },\\
R_2(A,B,C)+Q_1(B,C,A) &= 0\,, \text{ second cancellation-2 },\\
S_1(A,B,C)+S_1(B,C,A)+S_1(C,A,B) &= 0\,, \text{ hexagonal cancellation-1 },\\
S_2(A,B,C)+S_2(B,C,A)+S_2(C,A,B) &= 0\,, \text{ hexagonal cancellation-2 }.
\end{aligned} \tag{103}$$

Proof. For the first cancellation, we have

$$\begin{aligned}
&P_1(A,B,C)+P_2(C,A,B)\\
= &\sum_{\substack{(a,c,b,\varphi)\in A_\circ\times C_\circ\times B_\circ^2\\ \varphi\neq b}} \varepsilon(a,\varphi)\varepsilon(c,b)(-1)^{i_a+i_\varphi+i_b+i_c}\lceil a^*\bullet G(\varphi,b)\bullet c^*\bullet G(b,\varphi)\rceil\\
+ &\sum_{\substack{(c,a,b,\varphi)\in C_\circ\times A_\circ\times B_\circ^2\\ \varphi\neq b}} \varepsilon(c,\varphi)\varepsilon(b,a)(-1)^{i_a+i_\varphi+i_b+i_c}\lceil c^*\bullet G(\varphi,b)\bullet a^*\bullet G(b,\varphi)\rceil\\
= &\sum_{\substack{(a,c)\in A_\circ\times C_\circ\\ (b_0,b_1)\in B_\circ^2\\ b_0\neq b_1}} \left(\varepsilon(a,b_1)\varepsilon(c,b_0)+\varepsilon(c,b_0)\varepsilon(b_1,a)\right)(-1)^{i_a+i_{b_0}+i_{b_1}+i_c}\lceil a^*\bullet G(\varphi,b)\bullet c^*\bullet G(b,\varphi)\rceil = 0\,,
\end{aligned}$$

where we used the change of variables $(b_0,b_1)=(b,\varphi)$ in the second line and $(b_0,b_1)=(\varphi,b)$ in the third and used the cyclic invariance.

The second cancellation-1 follows by a similar argument

$$\begin{aligned} R_1(A,B,C) + Q_2(B,C,A) &= \sum_{(a,b,c)\in A_\circ\times B_\circ\times C_\circ} \varepsilon(a,c)\varepsilon(c,b)(-1)^{i_a+i_b}\lceil b^*\bullet c^*\bullet a^*\bullet c)\rceil \\ &+ \sum_{(a,b,c)\in A_\circ\times B_\circ\times C_\circ} \varepsilon(b,c)\varepsilon(a,c)(-1)^{i_a+i_b}\lceil b^*\bullet c^*\bullet a^*\bullet c)\rceil \ = 0\ . \end{aligned}$$

Similarly for the second cancellation-2. Finally the hexagonal cancellation-1 follows from the hexagonal relation

$$\varepsilon(a,u_{b,c})\varepsilon(c,b) + \varepsilon(b,u_{c,a})\varepsilon(a,c) + \varepsilon(c,u_{a,b})\varepsilon(b,a) = 0\ ,$$

which is itself a consequence of lemma C.1.1 and the assumption (102). A similar argument works the second hexagonal relation. □

C.6. **The various possibilities for the triple bracket.** We have to consider 3 different possibilities for the triple brackets $[A,[B,C]]$ taking in account whether B and C have rank 1.

The following lemma will be a consequence of lemma C.4.1. We will also use the following conventions:

$$\begin{aligned} &\text{if } Q_1(U,V,W) = Q_2(U,V,W)\ , \text{ then we write } Q(U,V,W) := Q_1(U,V,W) = Q_2(U,V,W)\ ,\\ &\text{if } R_1(U,V,W) = R_2(U,V,W)\ , \text{ then we write } R(U,V,W) := R_1(U,V,W) = R_2(U,V,W)\ . \end{aligned}$$

Lemma C.6.1 (Triple bracket). *We have the following four possibilities (independent of the rank of U) for the triple brackets*

(1) *The polygons V and W have both rank greater than 1, then*

$$\begin{aligned} [U,[V,W]] &= P_1(U,V,W) + P_2(U,V,W) + Q_1(U,V,W) + Q_2(U,V,W) \\ &+ R_1(U,V,W) + R_2(U,V,W) + S_1(U,V,W) + S_2(U,V,W)\ . \end{aligned} \tag{104}$$

(2) *Both $v := V$ and $w := W$ have rank 1, then*

$$[U,[v,w]] = Q(U,v,w) + R(U,v,w) + S_1(U,v,w) + S_2(U,v,w)\ . \tag{105}$$

(3) *The polygon W has rank greater than 1, while $v := V$ has rank 1, then*

$$[U,[v,W]] = P_2(U,v,W) + Q(U,v,W) + R_1(U,v,W) + R_2(U,v,W) + S_1(U,v,W) + S_2(U,v,W)\ .$$

(4) *The polygon W has rank greater than 1, while $v := V$ has rank 1, then*

$$\begin{aligned} [U,[V,w]] &= P_2(U,W,v) + R(U,W,v) + Q_1(U,W,v) + R_2(U,W,v) \\ &+ S_1(U,W,v) + S_2(U,W,v)\ . \end{aligned}$$

Proof. This is deduced from lemma C.4.1. Indeed we deduce from that lemma that we have

(1) if V is a geodesic, then case 1 does not happen, and case 3 and case 4 coincide, thus

$$P_1(U,V,W) = 0\ ,\ Q_1(U,V,W) = Q_2(U,V,W) =: Q(U,V,W)\ .$$

(2) Symmetrically, if W is a geodesic, then case 2 does not happen, and case 5 and case 6 coincide, thus

$$P_2(U,V,W) = 0\ ,\ R_1(U,V,W) = R_2(U,V,W) =: R(U,V,W)\ . \quad □$$

C.7. **Proof of the Jacobi identity.** We will use freely in that paragraph lemma C.6.1

Proof 1: all three ghost polygons have rank greater than 1. The previous discussion gives

$$\begin{aligned}
[A,[B,C]] &= P_1(A,B,C)+P_2(A,B,C)+Q_1(A,B,C)+Q_2(A,B,C)\\
&+ R_1(A,B,C)+R_2(A,B,C)+S_1(A,B,C)+S_2(A,B,C)\ ,\\
[B,[C,A]] &= P_1(B,C,A)+P_2(B,C,A)+Q_1(B,C,A)+Q_2(B,C,A)\\
&+ R_1(B,C,A)+R_2(B,C,A)+S_1(B,C,A)+S_2(B,C,A)\ ,\\
[C,[A,B]] &= P_1(C,A,B)+P_2(C,A,B)+Q_1(C,A,B)+Q_2(C,A,B)\\
&+ R_1(C,A,B)+R_2(C,A,B)+S_1(C,A,B)+S_2(C,A,B)\ .
\end{aligned}$$

The proof of the Jacobi identity then follows from the cancellations (103). □

Proof 2: all three ghost polygons have rank 1. , then writing $a := A$, $b := B$ and $c := C$, we have

$$\begin{aligned}
[a,[b,c]] &= Q(a,b,c)+R(a,b,c)+S_1(a,b,c)+S_2(a,b,c)\ ,\\
[b,[c,a]] &= Q(b,c,a)+R(b,c,a)+S_1(b,c,a)+S_2(b,c,a)\\
[c,[b,a]] &= Q(c,a,b)+R(c,a,b)+S_1(c,a,b)+S_2(c,a,b)\ .
\end{aligned}$$

The Jacobi identity follows from the cancellations (103). □

Proof 3: exactly one of the three polygons has rank 1. Assume $a := A$ is a geodesic, B and C has rank greater than 1. Then

$$\begin{aligned}
[a,[B,C]] &= P_1(a,B,C)+P_2(a,B,C)+Q_1(a,B,C)+Q_2(a,B,C)\\
&+ R_1(a,B,C)+R_2(a,B,C)+S_1(a,B,C)+S_2(a,B,C)\ ,\\
[C,[a,B]] &= P_2(C,a,B)+Q_1(C,a,B)+Q_2(C,a,B)+R(C,a,B)+S_1(C,a,B)+S_2(C,a,B)\ ,\\
[B,[C,a]] &= P_1(B,C,a)+R_1(B,C,a)+R_2(B,C,a)+Q(B,C,a)+S_1(B,C,a)+S_2(B,C,a)\ .
\end{aligned}$$

Then again the cancellations (103), yields the Jacobi identity in that case. □

Proof of the final possibility : exactly two of the three polygons have rank 1. We have here that A has rank greater than 1, while $b := B$ and $c := C$ are geodesics, then

$$\begin{aligned}
[A,[b,c]] &= Q(A,b,c)+R(A,b,c)+S_1(A,b,c)+S_2(A,b,c)\ ,\\
[b,[c,A]] &= P_1(b,c,A)+Q_1(b,c,A)+Q_2(b,c,A)+R(b,c,A)+S_1(b,c,A)+S_2(b,c,A)\ ,\\
[c,[A,b]] &= P_2(c,A,b)+R_1(c,A,b)+R_2(c,A,b)+Q(c,A,b)+S_1(c,A,b)+S_2(c,A,b)\ .
\end{aligned}$$

For the last time, the cancellations (103), yields the Jacobi identity in that case. □

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