

# AN ALGEBRA OF OBSERVABLES FOR CROSS RATIOS UNE ALGÈBRE D'OBSERVABLES POUR LES BIRAPPORTS

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**ABSTRACT.** Nous introduisons une algèbre de Poisson, l'*algèbre d'échange*, définie à l'aide de l'intersection des courbes dans le disque. Nous interprétons l'*algèbre des multifractions* – une sous-algèbre de l'algèbre des fractions de l'algèbre d'échange – comme une algèbre de fonctions sur l'espace des birapports et donc en particulier comme une algèbre de fonctions sur la composante de Hitchin ainsi que sur l'espace des  $\mathrm{SL}(n, \mathbb{R})$ -opers d'holonomie triviale. Nous relierons alors notre structure de Poisson à la structure de Poisson de Drinfel'd-Sokolov ainsi qu'à la structure symplectique d'Atiyah-Bott-Goldman.

**ABSTRACT.** We define a Poisson Algebra called the *swapping algebra* using the intersection of curves in the disk. We interpret a subalgebra of the fraction swapping algebra – called the *algebra of multifractions* – as an algebra of functions on the space of cross ratios and thus as an algebra of functions on the Hitchin component as well as on the space of  $\mathrm{SL}(n, \mathbb{R})$ -opers with trivial holonomy. We finally relate our Poisson structure to the Drinfel'd-Sokolov structure and to the Atiyah-Bott-Goldman symplectic structure.

## 1. VERSION FRANÇAISE ABRÉGÉE

Si  $(X, x, Y, y)$  est un quadruplet de points distincts du cercle, l'*intersection*  $\mathfrak{I}(X, x, Y, y)$  des couples  $(X, x)$  et  $(Y, y)$  est l'intersection dans le disque des deux courbes orientées joignant respectivement  $X$  à  $x$  et  $Y$  à  $y$ . Cette intersection s'étend à tous les couples de points (voir formule 2) en prenant sa valeur dans  $\{-1, -1/2, 0, 1/2, 1\}$ . Nous noterons désormais  $Xx$  le couple de points  $(X, x)$ .

Soit  $\mathcal{P}$  un sous-ensemble de points du cercle. L'*algèbre d'échange*  $\mathcal{Z}(\mathcal{P})$  est l'algèbre associative commutative – c'est-à-dire l'algèbre polynomiale – engendrée sur  $\mathbb{Q}$  par les couples  $Xx$  avec les relations  $Xx = 0$  si  $X = x$ , où  $X$  et  $x$  appartiennent à  $\mathcal{P}$ . On définit le *crochet d'échange de couples* sur les générateurs par

$$\{Xx, Yy\} = \mathfrak{I}(X, x, Y, y) Xy.Yx. \quad (1)$$

On étend ce crochet à toute l'algèbre  $\mathcal{Z}(\mathcal{P})$  de façon à ce que  $u \rightarrow \{u, v\}$  et  $u \rightarrow \{v, u\}$  soient des dérivations pour tout  $v$ .

Notre premier résultat – théorème 1 – est que cette algèbre est une algèbre de Poisson. Notre but dans cette note est d'annoncer deux résultats reliant cette algèbre de Poisson à deux structures symplectiques connues

- la structure symplectique de Drinfel'd-Sokolov sur les  $\mathrm{SL}(n, \mathbb{R})$ -opers [9, 2].

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- la structure symplectique d'Atiyah-Bott-Goldman sur la variété des caractères des représentations d'un groupe de surface orientée dans  $\mathrm{SL}(n, \mathbb{R})$  [1, 4].

Une telle relation avait été prévue par Witten dans [10]. Nous allons relier ces différentes structures grâce à la notion de birapport utilisée dans [6, 7].

Rappelons qu'un *birapport faible*  $\mathbf{b}$  sur  $\mathcal{P}$  est une fonction à valeurs réelles définie sur  $\mathcal{P}^{4*} := \{(x, y, z, t) \in \mathcal{P}^4 \mid x \neq t \text{ et } y \neq z\}$  et vérifiant les relations

$$\begin{aligned} x = y \text{ ou } z = t &\Rightarrow \mathbf{b}(x, y, z, t) = 0 , & x = z \text{ ou } y = t &\Rightarrow \mathbf{b}(x, y, z, t) = 1 , \\ \mathbf{b}(x, y, z, t) &= \mathbf{b}(x, y, w, t)\mathbf{b}(w, y, z, t), & \mathbf{b}(x, y, z, t) &= \mathbf{b}(x, y, z, w)\mathbf{b}(x, w, z, t). \end{aligned}$$

Une *bifraction* est un élément de l'algèbre des fractions de  $\mathcal{Z}(\mathcal{P})$  de la forme

$$[X; x; Y; y] := \frac{Xy.Yx}{Xx.Yy},$$

L'*algèbre des multifractions* est la sous-algèbre commutative  $\mathcal{B}(\mathcal{P})$  de l'algèbre des fractions de  $\mathcal{Z}(\mathcal{P})$  engendrée par les bifractions, c'est aussi le sous-espace vectoriel engendré par les expressions de la forme (4). L'algèbre des multifractions est stable par l'extension du crochet d'échange de couples. Une bifraction donne naissance à une fonction sur l'espace  $\mathbb{B}(\mathcal{P})$  des birapports sur  $\mathcal{P}$  : la valeur de la bifraction  $[X; x; Y; y]$  pour le birapport  $\mathbf{b}$  est donnée par

$$[X; x; Y; y](\mathbf{b}) := \mathbf{b}(X, x, Y, y).$$

Cette application s'étend en un morphisme de l'algèbre des multifractions dans l'algèbre des fonctions sur  $\mathbb{B}(\mathcal{P})$ .

Rappelons maintenant que l'espace des  $\mathrm{SL}(n, \mathbb{R})$ -opers d'holonomie triviale et la composante de Hitchin (voir [5]) de  $\mathrm{Rep}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$  s'interprètent tous les deux comme des sous-espaces de  $\mathbb{B}(\mathcal{P})$  :

- il est classique que les  $\mathrm{SL}(n, \mathbb{R})$ -opers d'holonomie triviale s'interprètent comme les *courbes de Frenet* de classe  $C^\infty$  à valeurs dans  $\mathbb{RP}^{n-1}$  (voir [9, 2, 3]). Par ailleurs, une courbe de Frenet donne naissance à un birapport par la formule (5).
- De même d'après [6, 7], les représentations de Hitchin s'interprètent comme des birapports sur le bord à l'infini  $\partial_\infty \pi_1(S)$  du groupe fondamental de  $S$ .

Autrement dit, nous pouvons interpréter une multifraction à la fois comme une fonction sur l'espace des  $\mathrm{SL}(n, \mathbb{R})$ -opers et comme une fonction sur la composante de Hitchin. Les résultats annoncés dans cette note sont les suivants

- (1) Nous montrons que le crochet de Poisson de Drinfel'd-Sokolov coïncide avec le crochet d'échange de couples pour les multifractions (voir Théorème 2).
- (2) Nous montrons le crochet d'Atiyah-Bott-Goldman coïncide asymptotiquement avec le crochet d'échange de couples pour certaines multifractions et pour des suites particulières de sous-groupes d'indice finis de  $\pi_1(S)$  (voir Théorème 3).

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## 2. THE SWAPPING ALGEBRA

**2.1. Intersection of ordered pairs of points on the circle.** We recall that, if  $(X, x, Y, y)$  is a quadruple of points of the interval  $]0, 1[$ , the *intersection*  $\mathfrak{I}(X, x, Y, y)$  of  $(X, x)$  and  $(Y, y)$  is

$$\frac{1}{2} (\text{Sign}(X-x) \text{Sign}(X-y) \text{Sign}(y-x) - \text{Sign}(X-x) \text{Sign}(X-Y) \text{Sign}(Y-x)), \quad (2)$$

where  $\text{Sign}(u) = -1, 0, 1$  whenever  $u < 0$ ,  $u = 0$  and  $u > 0$  respectively. Now, if  $(X, x, Y, y)$  is a quadruple of points of the oriented circle  $\mathbb{T}$ , we check that the intersection of  $(X, x, Y, y)$  in the interval  $\mathbb{T} \setminus \{z\}$  does not depend on  $z$  if  $z \notin \{X, x, Y, y\}$  and is thus declared to be the intersection of  $(X, x, Y, y)$  in  $\mathbb{T}$ .

When the four points  $(X, x, Y, y)$  are pairwise distinct,  $\mathfrak{I}(X, x, Y, y)$  is the intersection of the oriented curves joining  $X$  to  $x$  and joining  $Y$  to  $y$  in the disk. In this case, the intersection belongs  $\{-1, 0, 1\}$ , in general the intersection belongs to  $\{-1, -1/2, 0, 1/2, 1\}$ .

**2.2. The Poisson swapping algebra.** Let  $\mathcal{P}$  be a subset of the circle. We represent an ordered pair  $(X, x)$  of points of  $\mathcal{P}$  by the expression  $Xx$ . We consider the associative commutative algebra  $\mathcal{Z}(\mathcal{P})$  generated over  $\mathbb{Q}$  by ordered pairs of points on  $\mathcal{P}$ , together with the relations  $Xx = 0$  when  $X = x$ .

Let  $\alpha$  be any real number. The *swapping bracket* is defined on generators by

$$\{Xx, Yy\}_\alpha = \mathfrak{I}(X, x, Y, y)(\alpha \cdot Xx \cdot Yy + Xy \cdot Yx). \quad (3)$$

and extended to  $\mathcal{Z}(\mathcal{P})$  so that  $u \rightarrow \{u, v\}_\alpha$  and  $u \rightarrow \{v, u\}_\alpha$  are derivations. The *swapping algebra*  $\mathcal{Z}(\mathcal{P})_\alpha$  is the algebra  $\mathcal{Z}(\mathcal{P})$  equipped with the swapping bracket.

**Theorem 1.** *The bracket  $\{\cdot, \cdot\}_\alpha$  satisfies the Jacobi identity. Hence, the algebra  $\mathcal{Z}(\mathcal{P})_\alpha$  is a Poisson Algebra.*

This theorem only uses formal properties of the intersection and can be generalised in a more abstract setting.

## 3. THE ALGEBRA OF MULTIFRACTIONS

Let again  $\mathcal{P}$  be a subset of the circle. A *cross fraction* is an element of the algebra of fractions  $\mathcal{Q}(\mathcal{P})$  of  $\mathcal{Z}(\mathcal{P})$  of the form

$$[X; x; Y; y] := \frac{Xy \cdot Yx}{Xx \cdot Yy},$$

where  $X \neq x$  and  $Y \neq y$ . More generally, a *multiplication* is an element of  $\mathcal{Q}(\mathcal{P})$  of the form

$$\frac{X_1 x_{\sigma(1)} \dots X_n x_{\sigma(n)}}{X_1 x_1 \dots X_n x_n}. \quad (4)$$

where  $\sigma$  is a permutation of  $\{1 \dots n\}$  and for all  $i$ ,  $X_i \neq x_i$ .

Let  $\mathcal{B}(\mathcal{P})$  be the vector space generated by multifractions. Observe that  $\mathcal{B}(\mathcal{P})$  is the associative commutative algebra generated by cross fractions and moreover is stable by the Poisson bracket and is thus a Poisson algebra. Finally the restriction  $\{\cdot, \cdot\}_W$  of the bracket  $\{\cdot, \cdot\}_\alpha$  is independent of  $\alpha$ .

Then, the *algebra of multifractions* is the vector space  $\mathcal{B}(\mathcal{P})$  equipped with the commutative associative product and the Poisson bracket  $\{\cdot, \cdot\}_W$ .

**3.1. Multifractions and cross ratios.** We want to see the algebra of multifractions as an algebra of observables, that is an algebra of functions on a space. We first see the algebra of multifractions as a subalgebra of functions on the set of all cross ratios. Recall from [7] that a *weak cross ratio* on a set  $\mathcal{P}$  is a real valued function  $\mathbf{b}$  on  $\mathcal{P}^{4*} := \{(x, y, z, t) \in \mathcal{P}^4 \mid x \neq t, \text{ and } y \neq z\}$  which satisfies the following rules

$$\begin{aligned} x = y \text{ or } z = t &\Rightarrow \mathbf{b}(x, y, z, t) = 0, & x = z \text{ or } y = t &\Rightarrow \mathbf{b}(x, y, z, t) = 1, \\ \mathbf{b}(x, y, z, t) &= \mathbf{b}(x, y, w, t)\mathbf{b}(w, y, z, t), & \mathbf{b}(x, y, z, t) &= \mathbf{b}(x, y, z, w)\mathbf{b}(x, w, z, t). \end{aligned}$$

If  $\Gamma$  is a group acting on  $\mathcal{P}$ , we say that a weak cross ratio is *invariant* under  $\Gamma$  if it is invariant under the diagonal action. Every cross fraction on  $\mathcal{P}$  defines a natural function on the set  $\mathbb{B}(\mathcal{P})$  of weak cross ratios on  $\mathcal{P}$  by

$$[X; x; Y; y](\mathbf{b}) := \mathbf{b}(X, x, Y, y).$$

More generally, this definition gives rise to an homomorphism of the associative algebra  $\mathcal{B}(\mathcal{P})$  into the algebra of functions on  $\mathbb{B}(\mathcal{P})$ . Therefore, in some sense our Theorem 1 gives a Poisson structure on the set  $\mathbb{B}(\mathcal{P})$ .

**3.2. Frenet curves and cross ratios.** A curve  $\xi$  defined from the circle  $\mathbb{T}$  to  $\mathbf{P}(\mathbb{R}^n)$  is a *Frenet curve* if there exists a curve  $(\xi^1, \xi^2, \dots, \xi^{n-1})$  defined on  $\mathbb{T}$ , called the *osculating flag curve*, with values in the flag variety such that for every  $x$  in  $\mathbb{T}$ ,  $\xi(x) = \xi^1(x)$ , and moreover

- For every pairwise distinct points  $(x_1, \dots, x_l)$  in  $\mathbb{T}$  and positive integers  $(n_1, \dots, n_l)$  such that  $\sum_{i=1}^{i=l} n_i \leq n$ , then the sum  $\xi^{n_1}(x_1) + \dots + \xi^{n_l}(x_l)$  is direct.
- For every  $x$  in  $\mathbb{T}$  and positive integers  $(n_1, \dots, n_l)$  such that  $p = \sum_{i=1}^{i=l} n_i \leq n$ , then  $\lim_{\substack{(y_1, \dots, y_l) \rightarrow x, \\ y_i \text{ all distinct}}} \left( \bigoplus_{i=1}^{i=l} \xi^{n_i}(y_i) \right) = \xi^p(x)$ .

We call  $\xi^{n-1}$  the *osculating hyperplane*.

Let  $\xi$  be a Frenet curve and  $\xi^*$  be its associated osculating hyperplane curve. The *weak cross ratio* associated to this pair of curves is the function on  $\mathbb{T}^{4*}$  defined by

$$\mathbf{b}_{\xi, \xi^*}(x, y, z, t) = \frac{\langle \widehat{\xi}(x) | \widehat{\xi}^*(y) \rangle \langle \widehat{\xi}(z) | \widehat{\xi}^*(t) \rangle}{\langle \widehat{\xi}(z) | \widehat{\xi}^*(y) \rangle \langle \widehat{\xi}(x) | \widehat{\xi}^*(t) \rangle}, \quad (5)$$

where for every  $u$ , we choose an arbitrary nonzero vector  $\widehat{\xi}(u)$  and  $\widehat{\xi}^*(u)$  respectively in  $\xi(u)$  and  $\xi^*(u)$ .

#### 4. TWO INCARNATIONS OF THE ALGEBRA OF MULTIFRACTIONS

Our aim now is to relate the Poisson structure on  $\mathbb{B}(\mathcal{P})$  to two classical Poisson structures namely

- the Drinfel'd-Sokolov structure on the space of  $\mathrm{SL}(n, \mathbb{R})$ -opers,
- the Atiyah-Bott-Goldman symplectic structure on the character variety of a surface group in  $\mathrm{SL}(n, \mathbb{R})$ .

Witten in [10] has foreshadowed a relation between these two spaces which were proved to be related in [6, 7]. Our purpose here is to relate their symplectic structures.

**4.1. Multifractions and opers.** The space of smooth Frenet curves carries a Poisson structure from the Drinfel'd-Sokolov reduction – and is identified to the space of  $\mathrm{SL}(n, \mathbb{R})$ -opers with trivial holonomy – whose Poisson bracket is denoted by  $\{\cdot, \cdot\}_{\mathrm{DS}}$  (see [9, 2, 3]). Thus, a multifraction being a function on  $\mathbb{B}(\mathcal{P})$  is also function on the space of Frenet curves.

Our second theorem identifies the two Poisson brackets.

**Theorem 2.** *The swapping Poisson bracket coincides with the Drinfel'd-Sokolov bracket for multifractions. That is, for every multifractions  $b_0$  and  $b_1$ ,*

$$\{b_0, b_1\}_{\mathrm{DS}} = \{b_0, b_1\}_{\mathrm{W}}.$$

#### 4.2. Multifractions and the Goldman algebra.

**4.2.1. The Atiyah-Bott-Goldman structure and the Goldman algebra.** Let  $S$  be a closed surface. For any semi-simple Lie group  $G$ , the character variety  $\mathrm{Rep}(\pi_1(S), G)$  of conjugacy classes of homomorphisms of  $\pi_1(S)$  in  $G$  admits a Poisson structure (see [1, 4]). When  $G = \mathrm{SL}(n, \mathbb{R})$ , a preferred component called the *Hitchin component* has been identified with a space of  $\pi_1(S)$ -invariant cross ratios on  $\partial_\infty \pi_1(S)$  in [6, 7]. We denote by  $\mathcal{A}(S)$  the Poisson algebra of smooth functions on the Hitchin component and  $\{\cdot, \cdot\}_S$  its Poisson bracket. Since representations in the Hitchin component are cross ratios, we have a homomorphism  $F_S$  of associative algebras from  $\mathcal{B}(\partial_\infty \pi_1(S))$  to  $\mathcal{A}(S)$ . If  $S_m$  is a finite covering of  $S$ , we also denote by  $R_S$  the restriction map from  $\mathcal{A}(S_m)$  to  $\mathcal{A}(S)$ .

**4.2.2. Coverings.** Let  $\mathcal{P}$  be the subset of  $\partial_\infty \pi_1(S)$  which consists of fixed points of non trivial elements of  $\pi_1(S)$ . Let  $\mathcal{G}$  be the set of ordered pairs of points  $\gamma = (\gamma^-, \gamma^+)$  in  $\mathcal{P}^2$  which corresponds to fixed points by a non trivial element of the group  $\pi_1(S)$ . Observe that given any finite index subgroup  $\Gamma$  of  $\pi_1(S)$ , the set  $\mathcal{G}$  is in bijection with the set of primitive elements of  $\Gamma$ , where by definition a primitive element of  $\Gamma$  is an element  $g$  that is not of the form  $h^p$  with  $p > 1$  and  $h \in \Gamma$ . In the sequel, we shall freely identify elements of  $\mathcal{G}$  with primitive elements in any finite index subgroup of  $\Gamma$ .

We say a nested sequence  $\{\Gamma_m\}_{m \in \mathbb{N}}$  of finite index subgroups of  $\pi_1(S)$  is *vanishing* if the following holds: let  $\gamma$  and  $\eta$  be elements in  $\mathcal{G}$ , let  $\gamma_m$  and  $\eta_m$  be the corresponding primitive elements in  $\Gamma_m$ , then there exists  $m_0$  so that for every  $m \geq m_0$  the geodesics corresponding to  $\gamma_m$  and  $\eta_m$  have at most one intersection point and moreover the geometric intersection is  $\mathfrak{I}(\gamma^-, \gamma^+, \eta^-, \eta^+)$ . It follows from the double coset separability proved by G. Niblo in [8] that vanishing sequences exist.

Observe finally that associated with a sequence  $\sigma = \{\Gamma_m\}_{m \in \mathbb{N}}$  of nested finite index subgroups of  $\pi_1(S)$  is the inverse limit  $S_\sigma$  of  $S_m := \tilde{S}/\Gamma_m$ . Similarly we consider the inverse limit  $\mathcal{A}(S_\sigma)$  of  $\mathcal{A}(S_m)$ . Then the homomorphism  $F_\sigma$  from  $\mathcal{B}(\mathcal{P})$  to  $\mathcal{A}(S_\sigma)$  is injective.

Let  $\{g_m\}_{m \in \mathbb{N}}$  be a sequence of functions, so that  $g_m \in \mathcal{A}(S_m)$ , we say that  $\{g_m\}_{m \in \mathbb{N}}$  converges to the function  $h$  in  $\mathcal{A}(S_\sigma)$  and write

$$\lim_{m \rightarrow \infty} g_m = h,$$

if for any  $p$ , we have  $\lim_{n \rightarrow \infty} R_{S_p}(g_n) = R_{S_p}(h)$ .

4.2.3. *Atiyah-Bott-Goldman Poisson bracket and the swapping bracket.* Our third theorem relates the Atiyah-Bott-Goldman Poisson bracket  $\{\cdot, \cdot\}_S$  on the Hitchin component of  $S$  with the swapping bracket.

**Theorem 3.** *Let  $\{\Gamma_m\}_{m \in \mathbb{N}}$  be a vanishing sequence of subgroups of  $\pi_1(S)$ , let  $b_0$  and  $b_1$  be two elements of  $\mathcal{B}(\mathcal{P})$ , then*

$$\lim_{n \rightarrow \infty} \{\mathsf{F}_{S_n}(b_0), \mathsf{F}_{S_n}(b_1)\}_{S_n} = \mathsf{F}_\sigma(\{b_0, b_1\}_W).$$

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