LORENTZ–EPSTEIN SURFACES AND A LIOUVILLE ACTION FOR POSITIVE CURVES

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Abstract. We investigate and define in this paper, in the context of the correspondence between anti-de Sitter 3-space and $(1,1)$-conformal metrics, the analogs of $\mathcal{W}$-volume, Epstein surfaces, and Liouville action. These notions were well-studied in the correspondence between $3d$-hyperbolic manifolds and $2d$ conformal metrics. We apply our construction to positive curves in flag manifolds equipped with a positive structure to obtain invariants of these curves that are finite in the case of *piecewise circles*.

1. Introduction

Renormalized volume is motivated by the holographic principle of AdS/CFT correspondence in String Theory [8, 13], allowing one to renormalize the volume of an infinite volume Einstein manifold using a truncation procedure determined by a metric on the conformal boundary.

In the mathematics literature, the most studied case is the correspondence between convex co-compact hyperbolic 3-manifolds M, with its conformal boundary $\partial_\infty M$ consisting of Riemann surfaces. The renormalized volume is defined as the $\mathcal{W}$-volume of a submanifold N_g obtained by "truncating" M using Epstein surface [6] determined by a choice of the conformal metric g on $\partial_\infty M$ [3, 14]:

$$\mathcal{W}(M, g) := \mathrm{vol}(N_g) - \frac{1}{2}\int_{\partial N_g} H \,\mathrm{d}a\,, \tag{1.1}$$

where vol is the volume form of M, H is the mean curvature, and da the area form on ∂N_g induced from M.

The case of $2+1$ dimensions was particularly interesting, as it was shown in [13, 14] that the $\mathcal{W}$-volume holographically expresses the Liouville action of Takhtajan and Zograf [25] on the conformal boundary. In particular, its Weyl

Date: March 10, 2026.

F. L. and J. T. acknowledge funding by the European Research Council under ERC-Advanced grant 101095722. J. T. acknowledges the support of the Institut Universitaire de France. Y. W. is supported by the Swiss State Secretariat for Education, Research and Innovation (SERI): MB25.00004. F. L. is supported by the Swiss State Secretariat for Education, Research and Innovation (SERI): MB25.00031.

anomaly follows Polyakov's formula,

$$\mathcal{W}(M, g) - \mathcal{W}(M, e^{2\sigma} g) \propto \int_{\partial_\infty M} \frac{1}{2}|\nabla\sigma|^2 + K_g \sigma \, \mathrm{d}\,\mathrm{vol}_g ,$$

where K_g is the Gauss curvature of g. The right-hand side is proportional to the Liouville action $\mathcal{S}_g(\sigma)$ with zero cosmological constant in the physics literature.

One of the first motivations for studying the Liouville action is the uniformization theorem. In fact, within the same conformal class of fixed area metrics, the critical point of the Liouville action is that associated with the constant curvature metric in $\partial_\infty M$ (i.e., the hyperbolic metric if the area is well-chosen). Moreover, as a function on the Teichmüller space (i.e., when we vary the hyperbolic structure on $\partial_\infty M$), the $\mathcal{W}$-volume turns out to be a Kähler potential for the Weil–Petersson metric [14,20] on Teichmüller space of the boundary. See [19] for a recent survey.

The $\mathcal{W}$-volume can also be used to describe the geometry of Jordan curves on the Riemann sphere $\mathbf{CP}^1 = \partial_\infty \mathbf{H}^3$. In fact, each Jordan curve γ determines two Epstein surfaces, meeting at γ, determined by the hyperbolic metrics on the connected components $\mathbf{CP}^1 \setminus \gamma$. The $\mathcal{W}$-volume of the 3-manifold between these two surfaces is proportional to the universal Liouville action (introduced by [21]) of the curve [4], which also has a deep link to the theory of random curves SLE [23,24]. We also mention that the definition of Epstein (hyper)-surface is not limited to $2+1$ dimension. In fact, even in the simpler $1+1$ dimension, the "renormalized area" of the hyperbolic disk truncated by the Epstein curve coincides with the Schwarzian action [16].

Nevertheless, the construction of Epstein hypersurfaces and renormalized volume is only studied in the Riemannian setup (i.e., in hyperbolic spaces, which are called Euclidean anti-de Sitter spaces by physicists).

The goal of the present paper is to explore the definition of Epstein surfaces associated with a conformal metric of type $(1, 1)$ and the corresponding $\mathcal{W}$-volume and Liouville action for the Lorentzian anti-de Sitter space $\mathbf{H}^{2,1}$: while the hyperbolic 3-space replaced by the $(2, 1)$-Anti De Sitter space $\mathbf{H}^{2,1}$, we consequently replace $\mathbf{CP}^1$ with its conformal structure – geometrically the boundary at infinity of the hyperbolic 3-space – by the *Einstein Universe* $\mathbf{Ein}^{1,1}$ which is topologically a 2-dimensional torus and which has a conformal structure of type $(1, 1)$. We will therefore proceed by analogy, as in the Riemannian case, to define Epstein surfaces and the $\mathcal{W}$-volume in that context. Our construction gives rise to an invariant for positive curves, curves which are objects of interest in [1,12], in particular for smooth hyperconvex curves in real projective spaces [7,15].

We first construct the analogs of Epstein surfaces in our context. More precisely, we show in Theorem 3.1:

Theorem A. *Let (S, g) a Lorentzian surface, and ϕ a conformal immersion from (S, g) to $\mathbf{Ein}^{1,1}$ satisfying a topological hypothesis. Then there exists a holonomic surface Σ_g in the space of tangent vectors of norm 1, $\mathsf{U}_+\mathbf{H}^{2,1}$, whose first fundamental form at infinity is g.*

The terminology of this theorem requires some explanation and definitions that are given in the main part of the paper. For the moment, we just remark the following:

- The topological hypothesis is to say that the pullback of a geometrically natural line bundle is trivial, this is made explicit in Theorem 3.1.
- We explain what a holonomic surface is in section 4.2. For the sake of this introduction, we just say that a typical example of a holonomic surface is the set of normal vectors $n(S_g)$ to a surface S_g of type $(1, 1)$ in $\mathbf{H}^{2,1}$.
- In this example of typical holonomic surface, the first fundamental form at infinity is $\frac{1}{2}(\mathrm{I} + 2\mathrm{II} + \mathrm{III})$ where I, II and III are respectively the first, second, and third fundamental forms of S_g as we shall see in Proposition 4.3.

The precise statement of the theorem provides a uniqueness result. We explain what this theorem means in terms of the envelope of AdS horospheres in paragraph 4.3.2, recovering a classical feature of Epstein surfaces.

Once Epstein surfaces are defined, we can proceed to the definition of the $\mathcal{W}$-volume. This $\mathcal{W}$-volume is an invariant of a 3-manifold immersed in $\mathsf{U}\mathbf{H}^{2,1}$ whose boundary is the union of two surfaces S_1 and S_2 "equal outside a compact set". The definition of the $\mathcal{W}$-volume uses natural differential forms on $\mathsf{U}\mathbf{H}^{2,1}$. In the case where the two boundary surfaces of N project to immersed surfaces S_1 and S_2 *equal outside of a compact set*, hence bounding a 3-manifold M in $\mathbf{H}^{2,1}$ — see paragraph 5.1.2 — we show in Proposition 5.2

$$\mathcal{W}(N) = \mathbf{Vol}(M) - \frac{1}{2}\int_{\partial M} H \,\mathrm{d}a \,,$$

where H is the mean curvature of ∂M and a its volume form. Observe the perfect parallel with equation (1.1).

We then prove the variational formula — Theorem 5.3 — for this $\mathcal{W}$-volume from which we draw two conclusions. For the simplicity of statement and our applications, we will restrict ourselves to the case of the *split annulus* $\mathbf{A}$ — conformal to $\mathbf{dS}^{1,1}$ — see paragraph 2.3.

(1) The $\mathcal{W}$-volume only depends on the first fundamental form at infinity on S_1 and S_2. We can therefore define the *Liouville action* $\mathcal{S}(h_1, h_2)$ of two metrics conformal to the de Sitter surface $\mathbf{dS}^{1,1}$ as the $\mathcal{W}$-volume bounded by any two Epstein surfaces associated with h_1 and h_2 that are equal outside a compact set: Corollary 6.1.
(2) Surfaces whose first fundamental form at infinity has constant curvature are critical points of this Liouville action with respect to compactly supported deformations: Corollary 6.3.

Our definition of Liouville action is proportional to the Lorentzian Liouville action in the physics literature with zero cosmological constant (or among metrics of the same area). See Remark 6.6 and, e.g., [22].

So far, this discussion has been concerned with surfaces that are equal at infinity and associated metrics that coincide outside a compact set. We now extend it to a larger set of metric pairs. More precisely, we define an equivalence relation (Lemma 6.8) between metrics. The equivalence classes of this relation are called $\mathcal{S}$*-classes*, see Definition 6.7. We are then able to define the Liouville action in Definition 6.9 for two $(1,1)$- metrics g and h in the same $\mathcal{S}$-class, related by the conformal factor u such that $h = e^{2u}g$ as

$$\mathcal{S}(g,h) = -\frac{1}{2}\int_{\mathbf{A}} uF_g + \frac{1}{4}\int_{\mathbf{A}} u\, \mathrm{d}(\mathrm{d}u \circ \mathrm{I})\ ,$$

where F_g is the curvature form of g, I is the *split involution* associated with the split annulus, see Lemma 2.2 for details. Again, we see the analogy with the Polyakov formula, where we replace the complex structure by the split involution. We now summarize the properties of the Liouville action, which is, in particular, consistent with the previous definition using the $\mathcal{W}$-volume.

Theorem B. *Let g, h and k be three metrics in the same $\mathcal{S}$-class, and u defined by $h = e^{2u}g$ then*

$$\mathcal{S}(g,h) = \mathcal{S}(g,k) + \mathcal{S}(k,h)\ , \quad \text{(Chasles Formula)} \tag{1.2}$$

$$\mathcal{S}(g,h) = -\frac{1}{4}\int_{\mathbf{A}} u(F_g + F_h)\ , \quad \text{(Monotonicity Formula)} \tag{1.3}$$

$$\mathcal{S}(g,h) = 0\ , \quad \textit{when } g \textit{ and } h \textit{ both have constant curvature } c\ . \tag{1.4}$$

Finally, when g and h are equal outside a compact set, and if S and Σ are Epstein surfaces associated with, respectively, g and h and an immersion ϕ of $\mathbf{A}$ in $\mathbf{Ein}^{1,1}$,

$$\mathcal{S}(g,h) = \mathcal{W}(S,\Sigma)\ .$$

We also prove that constant curvature surfaces are exactly the critical point of the Liouville action for area preserving deformations. More precisely, let us say that an *area preserving deformation* of g_0 is a 1-parameter family of conformal metric $(g_t)_{t\in\mathbb{R}}$ on $\mathbf{A}$, equal outside a compact set K satisfying

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{vol}_{g_t}(K) = 0\ ,$$

Theorem C. *If $(g_t)_{t\in\mathbb{R}}$ is a one-parameter family of area preserving deformation such that g_0 has constant curvature, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{S}(g_t, g_0) = 0\ .$$

Conversely, if for any area preserving variation of metrics,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{S}(g_t, g_0) = 0 \ ,$$

then, g_0 has constant curvature.

Observe that equation (1.4) of Theorem B indeed makes sense: given a split annulus, there are several metrics of constant curvature in the same conformal class and in the same $\mathcal{S}$-class (see Proposition 6.13 for details).

This theorem allows us to define the *Liouville action* for a metric h in the $\mathcal{S}$-class of a constant curvature metric h_0, as

$$\mathcal{S}(h) \coloneqq \mathcal{S}(h, h_0) \ .$$

This definition is unambiguous in the choice of constant curvature metric h_0 in the conformal class of h, thanks to Chasles relation (1.2) and the assertion (1.4).

We now apply these results for (locally) positive curves in a flag manifold $\mathcal{F}$ associated with a group G equipped with a positive structure. Positive structures were introduced in [11] and positive curves in [12]. Among classical examples are spacelike curves in the Einstein universe of arbitrary dimension; they may also arise as convex curves in $\mathbf{P}(\mathbb{R}^3)$ or more generally hyperconvex curves in $\mathbf{P}(\mathbb{R}^n)$. They are discussed in section 7.

We sketch now how positive curves give rise to $(1,1)$-metrics on $\mathbf{A}$: if we consider a C^1 positive curve c from $\mathbf{P}(\mathbb{R}^1)$ to $\mathcal{F}$, we obtain a C^1-immersion from $\mathbf{A} = \mathbf{P}(\mathbb{R}^1) \times \mathbf{P}(\mathbb{R}^1) \setminus \Delta$ to $\mathcal{G} = \mathcal{F} \times \mathcal{F} \setminus \Delta$. The latter is equipped with a (p, p) metric, and the induced metric h_c on $\mathbf{A}$ is $(1, 1)$.

We now restrict our discussion to *piecewise circles*: a special case of positive curves arising from *circles* which are orbits of *positive* $\mathsf{PSL}_2(\mathbb{R})$ – see Section 7.2. The corresponding metric on $\mathbf{A}$ has constant curvature k (since $\mathsf{PSL}_2(\mathbb{R})$ is a transitive group of isometries), and the value of the constant curvature will depend on the choice of the conjugacy class of $\mathsf{PSL}_2(\mathbb{R})$.

We then define – see definition 7.3 – a *piecewise circle* as a C^1-curve which is piecewise a circle for a given conjugacy class of $\mathsf{PSL}_2(\mathbb{R})$. This generalizes the notion of piecewise Möbius curves discussed in [2, 17]. We can then compare the metric h_c with the metric h_0 of constant curvature coming from a circle. We show the following result.

Theorem D. *The metric h_c is in the $\mathcal{S}$-class of h_0. Consequently its Liouville action $\mathcal{S}(h_c, h_0)$ is finite.*

Thus given G, a positive structure on a flag manifold $\mathcal{F}$ for G, a positive $\mathsf{PSL}_2(\mathbb{R})$, we obtain an invariant (under the action of G) of a piecewise circle (with respect to our choice of $\mathsf{PSL}_2(\mathbb{R})$) map c in a $\mathcal{F}$ as

$$\mathcal{S}(c) \coloneqq \mathcal{S}(h_c, h_0) \ .$$

Our main result is then that $\mathcal{S}(c)$ is finite and circles are critical points of this action $\mathcal{S}(c)$ by Corollary 6.3. However, we generally do not expect that circles are a local minimum of $\mathcal{S}$.

Contents

2. Conformal Lorentz surfaces

2.1. **Split structure.** Let V be a real vector of dimension 2. We fix an orientation on V.

2.1.1. *Split vector spaces.* A *split structure* on V is a pair (V_1, V_2) of distinct lines in V. A *split basis* is then an oriented basis (v_1, v_2) of V with v_i spanning V_i. The *canonical involution* I is the involution of V preserving each V_i and such that $\mathrm{I}_{|V_i} = (-1)^i$.

A *Lorentz product* is a quadratic form of signature $(1,1)$) on V. Split structures are linked with Lorentz metrics as follows. For a Lorentz metric $\mathbf{q}$, there is a unique split structure (V_1, V_2) on V such that each V_i is $\mathbf{q}$-isotropic (namely, $\mathbf{q}_{|V_i} = 0$) and $\mathbf{q}(v_1, v_2)$ is positive for any split basis (v_1, v_2). The quadratic form $\mathbf{q}$ is then said to be *compatible* with the split structure (V_1, V_2).

Recall that an endomorphism φ of $(V, \mathbf{q})$ is *conformal* if $\varphi^*\mathbf{q} = e^{\lambda}\mathbf{q}$ for some real number λ. The group $\mathrm{Conf}(V, \mathbf{q})$ of conformal endomorphisms of V has 4 connected components, and we denote by $\mathrm{Conf}_+(V, \mathbf{q})$ the index 2 subgroup consisting of orientation-preserving conformal endomorphisms. Observe that $\mathrm{Conf}_+(V, \mathbf{q})$ is isomorphic to the nonconnected Lie group $\mathrm{Conf}_+(1,1) = \mathbb{R}_{>0} \times \mathsf{SO}(1,1)$.

Lemma 2.1. *An orientation-preserving endomorphism of* $(V, \mathbf{q})$ *is conformal if and only if it preserves the associated split structure.*

Proof. Let φ be an orientation-preserving endomorphism of V and let (v_1, v_2) be a split basis.

If φ is conformal, then it maps v_i to an isotropic vector, so it globally preserves $V_1 \cup V_2$. To prove that φ cannot exchange V_1 and V_2, just observe that if $\varphi(v_i) = \mu_i v_{i+1}$ then the conformality of φ implies $\mu_1\mu_2 > 0$ and so φ reverses the orientation.

Conversely, if φ preserves each V_i, since it preserves orientation, there exists μ_i such that $\varphi(v_i) = \mu_i v_i$ with $\mu_1\mu_2 > 0$. So φ is conformal. □

In particular, a split structure on V is equivalent to a conformal class of Lorentz structures.

2.1.2. *Split surfaces and Lorentz metric.* Let S be a smooth, oriented surface. A *split structure* $\boldsymbol{\sigma}$ on S is a pair $(\mathcal{L}_1, \mathcal{L}_2)$ of transverse 1-dimensional foliations. A *split surface* is then a pair $(S, \boldsymbol{\sigma})$ where S is an oriented surface and $\boldsymbol{\sigma}$ is a split structure on S. A *split framing* is a frame (u_1, u_2) of $\mathsf{T}S$ such that at any point x, the pair $(u_1(x), u_2(x))$ is a split basis of T_xS with split structure $(\mathsf{T}_x\mathcal{L}_1, \mathsf{T}_x\mathcal{L}_2)$.

A (C^k) *Lorentz metric* on S, is a (C^k) field of Lorentz product on $\mathsf{T}S$. Two Lorentz metric g and h are *conformally equivalent* or *conformal* is there is a function f on S such that $g = e^{2f}h$.

Lemma 2.1 has the following consequence:

Lemma 2.2. *Given a smooth, oriented surface* S*, the following structures are equivalent:*

1. *a split structure,*
2. *a conformal class of Lorentz metric,*
3. *a field of involution* I *in* $\Gamma(S, \mathrm{End}(\mathsf{T}S))$ *with* 1*-dimensional eigenspaces,*
4. *a reduction of the structure group of the bundle of oriented frames to* $\mathrm{Conf}_+(1,1)$.

Remark 2.3. In the equivalence described above, the leaves of the foliation $\mathcal{L}_i$ are the integral curves of the distribution $\mathrm{Ker}(\mathrm{I} - (-1)^i\,\mathrm{Id})$ on S.

Observe that the existence of a split structure on a surface S implies that the tangent bundle of S is trivial. In particular, if S is closed, then it is diffeomorphic to the torus.

A *split map* between two split surfaces $(S_1, \boldsymbol{\sigma}_1)$ and $(S_2, \boldsymbol{\sigma}_2)$ is a homeomorphism f from S_1 to S_2 that sends lightlike geodesics to lightlike geodesics. We will only be interested in split diffeormorphisms, which can be characterised as maps which are conformal with respect to the underlying Lorentz conformal structures. The horizontal and vertical lines define a standard split structure $\boldsymbol{\sigma}_0$ on $\mathbf{R}^2$ and an *isothermal coordinate* on $(S, \boldsymbol{\sigma})$ is a local chart with values in $(\mathbf{R}^2, \boldsymbol{\sigma}_0)$ which is a split map.

Lemma 2.4 (Existence of isothermal coordinates)**.** *Let $(S, \boldsymbol{\sigma})$ be a split surface. Then locally $(S, \boldsymbol{\sigma})$ admits isothermal coordinates.*

Proof. Let (X_1, X_2) be a split framing around a point p. One can find positive functions f_1 and f_2 such that the new split framing (Y_1, Y_2) with $Y_i = f_i X_i$ satisfies $[Y_1, Y_2] = 0$. In particular, the flows commute, and the inverse of the map

$$F(s,t) = \Phi^s_{Y_1} \circ \Phi^t_{Y_2}(p)$$

defines isothermal coordinates around p. □

2.1.3. *Compatible Lorentz structure.* Let $(S, \boldsymbol{\sigma})$ be a split surface with canonical involution I, and let V_i be the distribution tangent to $\mathcal{L}_i$ for $i = 1, 2$.

Definition 2.5. A Lorentz metric g on S is *compatible with* $\boldsymbol{\sigma}$ if its conformal class coincides with $\boldsymbol{\sigma}$ (see Lemma 2.2). For $k \geq 1$, we denote by $\mathcal{M}^k(S, \boldsymbol{\sigma})$ the space of C^k-Lorentz metrics on S compatible with $\boldsymbol{\sigma}$.

Remark 2.6. Observe that $C^k(S)$ acts $\mathcal{M}^k(S, \boldsymbol{\sigma})$ where the action is given for u a function and g a metric by $(u, g) \mapsto e^{2u} g$. This action is simply transitive, since any two metrics in $\mathcal{M}^k(S, \boldsymbol{\sigma})$ are conformal. In fancy terms, the space $\mathcal{M}^k(S, \boldsymbol{\sigma})$ is a $C^k(S)$-torsor.

Remark 2.7. For any Lorentz metric g in $\mathcal{M}^k(S, \boldsymbol{\sigma})$, its *volume form* ω_g compatible with the orientation satisfies

$$\omega_g(u, v) = g(u, \mathrm{I}v) \ .$$

Indeed, the standard flat Lorentz metric on $\mathbb{R}^2$ is given by

$$g_{flat} = \mathrm{d}x \, \mathrm{d}y$$

where ∂_x spans V_1 and ∂_y spans V_2, and the corresponding volume form is $\omega_{flat} = \mathrm{d}x \wedge \mathrm{d}y$. In particular, if (v_1, v_2) is a split basis for g satisfying $g(v_1, v_2) = 1$, then

$$\omega_g(v_1, v_2) = 1 \ .$$

It follows that the map

$$\varphi : \left\{ \begin{array}{rcl} \mathcal{M}(S, \boldsymbol{\sigma}) & \to & \Omega^2_+(S) \\ g & \mapsto & \omega_g \end{array} \right.$$

defines a one-to-one correspondence between $\mathcal{M}^k(S,\boldsymbol{\sigma})$ and the set $\mathbf{Vol}^k_+(S)$ of C^k-volume forms compatible with the orientation.

Remark 2.8. Observe finally that a C^{k+1}-split diffeomorphism φ from $(S_1,\boldsymbol{\sigma}_1)$ to $(S_2,\boldsymbol{\sigma}_2)$, then the pull-back φ^* induces a one-to-one correspondence from $\mathcal{M}^k(S_2,\boldsymbol{\sigma}_2)$ to $\mathcal{M}^k(S_1,\boldsymbol{\sigma}_1)$.

2.1.4. *Curvature of a Lorentz surface.* Let g be a compatible metric on a split surface $(S,\boldsymbol{\sigma})$, and denote by ∇ the associated Levi-Civita connection. The *d'Alembertian* $\square_g$ is the differential operator defined on a C^2 function f by

$$\square_g f = \mathrm{tr}_g(\nabla \mathrm{d} f)\ .$$

We recall that for a symmetric bilinear form Q associated with the linear operator A by $Q(u,v) = g(Au,v)$ then

$$\mathrm{tr}_g(Q) := 2Q(v_1,v_2) = \mathrm{tr}(A)$$

where (v_1,v_2) is a split basis for g satisfying $g(v_1,v_2) = 1$. Observe in particular that $\mathrm{tr}_g(g) = 2$.

Let R_g be the curvature tensor of g, and let I define the split structure. The *sectional curvature* of g is denoted by $K(g)$ and defined by the relation

$$R_g = -K(g)\omega_g \otimes \mathrm{I}\ .$$

The *curvature 2-form* of g is $F_g := K(g)\omega_g$. Thus $R_g = -F_g \otimes I$.

Proposition 2.9 (Conformal change). *Let g and h be metrics in $\mathcal{M}(S,\boldsymbol{\sigma})$ with $h = e^{2u}g$. Then*

$$\square_h = e^{-2u}\,\square_g\ , \tag{2.1}$$

$$(\square_g\, u)\omega_g = \mathrm{d}(\mathrm{d}u \circ \mathrm{I}) = F_g - F_h\ . \tag{2.2}$$

Equivalently, we have

$$\square_g\, u = K(g) - e^{2u}K(h)$$

which is the Lorentzian analog of the conformal change formula of the sectional curvature.

To see this, we first compute the relevant terms in isothermal coordinates.

Lemma 2.10. *Let g be a C^2-Lorentz metric on S given by $g = e^{2u}\mathrm{d}x\mathrm{d}y$ in some isothermal coordinates (x,y). Let f be a function on S.*

(1) *The Levi-Civita connection ∇ of g is given by*

$$\nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 0\ \ ,\ \ \nabla_{\partial_x}\partial_x = 2(\partial_x u)\,\partial_x\ \ ,\ \ \nabla_{\partial_y}\partial_y = 2(\partial_y u)\,\partial_y\ .$$

(2) *The d'Alembertian with respect to g is given by*

$$\square_g f = 2e^{-2u}\partial^2_{xy} f\ .$$

(3) *The sectional curvature of g satisfies*

$$K(g) = -\,\square_g\, u\ .$$

Proof. Differentiating the equations $g(\partial_x,\partial_x) = g(\partial_y,\partial_y) = 0$, one obtains that both V_1 and V_2 are parallel with respect to ∇. Thus, ∇ splits into $\nabla = \nabla^1 \oplus \nabla^2$ with ∇^i being a connection of the bundle V_i. Using $[\partial_x,\partial_y] = 0$ we get

$$\nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x \in V_1 \cap V_2 = \{0\} .$$

Differentiating $g(\partial_x,\partial_y) = e^{2u}$ then gives the expression of ∇. This concludes the proof of item (1).

For item (2), given a function f, using the definition of the d'Alembertian, we have

$$\nabla \mathrm{d}f = \tfrac{1}{2}(\Box_g f)g + H_0 ,$$

where H_0 is traceless and thus $H_0(\partial_x,\partial_y) = 0$. This gives

$$\begin{aligned}\Box_g f &= \frac{2\nabla \mathrm{d}f(\partial_x,\partial_y)}{g(\partial_x,\partial_y)} \\ &= \frac{2\left(\partial_x(\mathrm{d}f(\partial_y)) - \mathrm{d}f(\nabla_{\partial_x}\partial_y)\right)}{e^{2u}} \\ &= 2e^{-2u}\partial^2_{xy} f .\end{aligned}$$

Finally, for item (3), using the previous items, we have

$$\begin{aligned}R_g(\partial_x,\partial_y)\partial_y &= \nabla_{\partial_x}\nabla_{\partial_y}\partial_y \\ &= 2(\partial^2_{xy}u)\,\partial_y \\ &= (\Box_g u)\omega_g(\partial_x,\partial_y)\mathrm{I}(\partial_y) .\end{aligned}$$

The expression of $K(g)$ follows. □

Proof of Proposition 2.9. Observe first that item (2) of Lemma 2.10 implies that

$$\Box_h = e^{-2u}\,\Box_g .$$

Write $g = e^{2v}\mathrm{d}x\mathrm{d}y$ in isothermal coordinates. Thus, the previous lemma gives

$$K(h) = -\Box_h(u+v) = -e^{-2u}\,\Box_g(u+v) .$$

Thus

$$K(g) - e^{2u}K(h) = -\Box_g v + \Box_g u + \Box_g v = \Box_g u .$$

Now we have

$$\mathrm{d}(\mathrm{d}u \circ \mathrm{I}) = 2\partial^2_{xy}u\,\mathrm{d}x \wedge \mathrm{d}y = (\Box_g u)\omega_g ,$$

hence

$$F_g - F_h = \left(K(g) - e^{2u}K(h)\right)\omega_g = (\Box_g u)\omega_g = \mathrm{d}(\mathrm{d}u \circ \mathrm{I}) .$$

This concludes the proof. □

2.2. **De Sitter surface.** Given a 3-dimensional real vector space E equipped with a quadratic form Q of signature $(2,1)$, the *de Sitter surface* is

$$\mathbf{dS}^{1,1} = \{x \in E\ ,\ Q(x) = 1\}\ .$$

The quadratic form Q restricts to a Lorentz product on each tangent space $\mathsf{T}_x\mathbf{dS}^{1,1}$ since this tangent space is identified with $x^\perp$. The resulting metric has constant curvature +1 and the group $\mathsf{SO}_0(Q)$ acts by isometries.

Given a point x in $\mathbf{dS}^{1,1}$ and a split basis (v_1, v_2) at $\mathsf{T}_x\mathbf{dS}^{1,1}$, the leaf $\mathcal{L}_\alpha$ through x is given by $x + \mathbb{R}v_\alpha$ and so its projection to $\mathbf{P}(E)$ intersects the quadric $\mathbf{RP}^1 \cong \{x \in \mathbf{P}(E)\ ,\ \mathbf{q}_{|x} = 0\}$ in $[v_\alpha]$. This defines a map

$$\Phi : \begin{cases} \mathbf{dS}^{1,1} & \rightarrow \quad (\mathbf{RP}^1 \times \mathbf{RP}^1) \setminus \Delta, \\ \qquad p & \mapsto \quad ([v_1], [v_2])\ , \end{cases}$$

where Δ is the diagonal. Observe that given two distinct points (x_1, x_2) in $(\mathbf{RP}^1 \times \mathbf{RP}^1) \setminus \Delta$, the planes $x_1^\perp$ and $x_2^\perp$ intersect along a positive line and one can find a unique point p in $\mathbf{dS}^{1,1}$ on this line such that V_α is contained in x_α. Thus, Φ is bijective and is indeed a diffeomorphism. In this model, the split structure is given by the fiber of the projection on each factor.

Lemma 2.11. *Any affine chart on* $\mathbf{RP}^1$ *defines an open set in* $(\mathbf{RP}^1 \times \mathbf{RP}^1) \setminus \Delta$ *in which the de Sitter metric* g_0 *satisfies*

$$g_0 = \frac{2\,\mathrm{d}x\,\mathrm{d}y}{(x-y)^2}\ .$$

Proof. Any split metric on $(\mathbb{R} \times \mathbb{R}) \setminus \Delta$ invariant under the affine group acting diagonally on $\mathbb{R} \times \mathbb{R}$ is of the form $\frac{\lambda \mathrm{d}x\mathrm{d}y}{(x-y)^2}$ for some positive λ. The value of λ is determined by the condition that the curvature is equal to 1. □

2.3. **Split annulus.**

Definition 2.12. The *split annulus* is the split surface $\mathbf{A}$ underlying $\mathbf{dS}^{1,1}$, that is, $\mathbf{A} = (\mathbf{RP}^1 \times \mathbf{RP}^1) \setminus \Delta$ where the split structure $(\mathcal{L}_1, \mathcal{L}_2)$ is given by the fibers of the projection on the i^{th}-factor.

Proposition 2.13. *Every* C^k*-split map of* $\mathbf{A}$ *is of the form*

$$\Phi : (x, y) \mapsto (\varphi(x), \varphi(y))\ .$$

where φ *is a* C^k*-diffeomorphism (or homeomorphism for* $k = 0$*) of* $\mathbf{RP}^1$*. The map* Φ *is an isometry of* $(\mathbf{A}, g_0)$ *if and only if* φ *is projective.*

Proof. Let Φ be a split homeomorphism of $\mathbf{A}$ to itself, namely, Φ sends each oriented foliation to itself. Let us write $\Phi(x, y) = (f_y(x), g_x(y))$. For every y, f_y is a homeomorphism from $\mathbf{RP}^1 \setminus \{y\}$ to $\mathbf{RP}^1 \setminus \{z\}$, for some $z \in \mathbf{RP}^1$ which we denote $z := \varphi(y)$. We can thus extend f_y to $\mathbf{RP}^1$ as a homeomorphism by setting $f_y(y) = \varphi(y)$. Similarly for g_x. As Φ is a split homeomorphism, we observe that $g_x(y)$ does not depend on x and $f_y(x)$ does not depend on y. It follows that $\Phi(x,y) = (f(x), g(y))$. Since Φ sends the diagonal to itself, we have $f(z) = g(z) = \varphi(z)$ for all $z \in \mathbf{RP}^1$. We also obtain that φ is a homeomorphism of $\mathbf{RP}^1$ as $\Phi(x,y) = (\varphi(x), \varphi(y))$ is a homeomorphism of $\mathbf{A}$. This completes the proof. □

Observe that, unlike the hyperbolic disk, the group of conformal diffeomorphisms of $\mathbf{A}$ is infinite dimensional and thus much greater than the finite dimensional group of isometries.

3. Isotropic surfaces

In this section, we consider a four-dimensional vector space W equipped with a quadratic form $\mathbf{q}$ of signature $(2,2)$ and denote by $\langle \cdot , \cdot \rangle$ the corresponding polar form.

3.1. The split structure of the Einstein torus.

The *2-dimensional Einstein Universe* or *Einstein Torus* is the quadric

$$\mathbf{Ein}^{1,1} = \{x \in \mathbf{P}(W) \ , \ \mathbf{q}(x) = 0\} \ .$$

Let τ be the *tautological line bundle* over the Einstein Torus $\mathbf{Ein}^{1,1}$, that is, the line bundle whose fiber τ_x over a point x is the isotropic line defined by x in W.

The Einstein Torus is naturally equipped with a canonical split structure $(\mathcal{L}_1, \mathcal{L}_2)$ which we now describe. Let (V, ω) be a 2-dimensional real vector space equipped with a volume form ω, and consider $(V \otimes V, \mathbf{q})$, where $\mathbf{q} := -\omega \otimes \omega$. Explicitly, we have

$$-\omega \otimes \omega(u_1 \otimes u_2, v_1 \otimes v_2) = -\omega(u_1, v_1)\omega(u_2, v_2) \ ,$$

so $-\omega \otimes \omega$ is a signature $(2,2)$ quadratic form on the 4-dimensional vector space $V \otimes V$. Once we choose an isomorphism between $(W, \mathbf{q})$ and $(V \otimes V, -\omega \otimes \omega)$, the image of the Segre embedding

$$\mathcal{S} : \left\{ \begin{array}{rcl} \mathbf{P}(V) \times \mathbf{P}(V) & \to & \mathbf{P}(V \otimes V) \ , \\ ([v_1], [v_2]) & \mapsto & [v_1 \otimes v_2] \ , \end{array} \right. \tag{3.1}$$

is exactly $\mathbf{Ein}^{1,1}$. This defines an isomorphism between $\mathbf{Ein}^{1,1}$ and $\mathbf{RP}^1 \times \mathbf{RP}^1$. Define $\mathcal{L}_i$ as the foliation of $\mathbf{Ein}^{1,1}$ whose leaves are the fibers of the projection on the i^{th} factor.

3.2. Isotropic surfaces.

An *isotropic surface* is an immersion σ of a surface S in W whose image of every point is an isotropic vector and such that $\sigma(s)$ is transverse to the image I_s of $\mathsf{T}_s\sigma$. Equivalently, we have for any s in S, for any u in T_sS,

$$\langle \sigma, \sigma \rangle = 0 \ , \dim(\operatorname{span}\{\mathrm{D}_u\sigma, \sigma(s)\}) = 2 \ .$$

Observe that projecting σ to $\mathbf{P}(W)$ defines an immersion $[\sigma]$ from S to $\mathbf{Ein}^{1,1}$. Two isotropic surfaces σ_0 and σ_1 are equivalent if $\sigma_0 = \pm\sigma_1$.

Our first result is the following.

Theorem 3.1. *Let ϕ be an immersion of S in $\mathbf{Ein}^{1,1}$, such that $L := \phi^*\tau$ is trivializable over S. Let g be any C^k-Lorentz metric on S compatible with the induced split structure. Then there exists a unique equivalence class of C^k-isotropic surface σ from S to W, such that $[\sigma] = \phi$ and*

$$g(u, v) = \langle \mathrm{D}_u\sigma, \mathrm{D}_v\sigma\rangle \ .$$

We remark that although the formula defining g seems at first sight to depend on the first derivative of σ, due to the "lightlike" nature of σ, g only depends on σ pointwise as we shall see in the proof.

Let ϕ be an immersion of S in $\mathbf{Ein}^{1,1}$. Observe that the natural inclusion of L in W defines an immersion i from the complement L^* of the zero section in L to W. We start with a remark: in the sequel, we shall freely identify (equivalence classes of) isotropic surfaces with sections of L.

Now, the proof of the theorem follows from the following lemma:

Lemma 3.2. *Assume that L is trivial and let L^+ be a connected component of the complement of the zero section in L.*

Then, the map which associates with a section σ of L^+, the metric g_σ defined by $g_\sigma(u, v) = \langle \mathrm{D}_u\sigma, \mathrm{D}_v\sigma\rangle$ is a bijection from the space $\Gamma^k(L^+)$ of C^k-sections with the $\mathcal{M}^k(S)$ of C^k-metrics on S, in the conformal class determined by the split structure on S.

Proof. Let σ_0 and σ_1 be two sections of L^+. Let us write $\sigma_0 = e^f\sigma_1$. Then

$$g_{\sigma_0}(u, v) = \langle \mathrm{D}_u(e^f\sigma_1), \mathrm{D}_v(e^f\sigma_1)\rangle = e^{2f}\,\langle \mathrm{D}_u\sigma_1, \mathrm{D}_v\sigma_1\rangle = e^{2f} g_{\sigma_1}(u, v)\ . \qquad (3.2)$$

In the second equality, we used the fact that for any split surface σ, then $\langle\sigma, \sigma\rangle = 0$. Hence, after differentiation, for any vector u, $\langle \mathrm{D}_u\sigma, \sigma\rangle = 0$. Finally, the lemma follows immediately from equation (3.2). ◻

3.3. **Dual isotropic surfaces and forms at infinity.** Given an isotropic surface σ, we call a *dual isotropic surface* to σ a map η such that

$$\langle\eta, \sigma\rangle = 1\ ,\ \langle\eta, \eta\rangle = 0\ , \langle\eta, \mathrm{D}_u\sigma\rangle = 0\ .$$

Proposition 3.3. *Let k be a positive integer Let σ be a C^k-isotropic surface, then there exists a unique C^{k-1}-dual isotropic surface.*

Proof. Let us consider the two dimensional space

$$V(s) := \operatorname{Im} \mathrm{D}\sigma(s)\ .$$

Then V has signature $(1, 1)$ and is transverse to its $\mathbf{q}$-orthogonal $V^\perp$ which also has signature $(1, 1)$. The split structure on V defines a split structure on $V^\perp$. We then define uniquely η such that (σ, η) is a split basis of $V^\perp$ and $\langle\sigma, \eta\rangle = 1$. By construction, if σ is C^k then V is C^{k-1}, hence, η is C^{k-1}. ◻

We observe that if σ is C^2, then σ is the dual isotropic surface to its dual isotropic surface.

Following the terminology of Krasnov and Schlenker, we define

Definition 3.4. Given vector fields X, Y on S, we define

(1) The *first fundamental form at infinity* as
$$\mathrm{I}^*(X,Y) := \langle \mathrm{D}_X\sigma, \mathrm{D}_Y\sigma\rangle \ .$$
(2) The *second fundamental form at infinity* as
$$\mathrm{II}^*(X,Y) := -\langle \mathrm{D}_X\sigma, \mathrm{D}_Y\eta\rangle \ .$$
(3) The *third fundamental form at infinity* as
$$\mathrm{III}^*(X,Y) := \langle \mathrm{D}_X\eta, \mathrm{D}_Y\eta\rangle \ .$$
(4) The *shape operator at infinity B^** is defined by
$$\mathrm{II}^*(X,Y) := \mathrm{I}^*(\mathrm{B}^*(X),Y)$$

We then observe that
$$\mathrm{III}^*(X,Y) := \mathrm{I}^*(\mathrm{B}^*(X),\mathrm{B}^*(Y)) \ .$$

4. Isotropic, holonomic and Epstein surfaces

Let $(W, \mathbf{q})$ be a real vector space of four dimensions equipped with a signature $(2,2)$ quadratic form as before.

4.1. Anti-de Sitter geometry.

The *anti-de Sitter* 3-space is
$$\mathbf{H}^{2,1} = \{x \in \mathbf{P}(W) \mid \mathbf{q}(x) < 0\} \ ,$$
its double cover is
$$\mathbf{H}^{2,1}_+ = \{x \in W \mid \mathbf{q}(x) = -1\} \ ,$$
and the covering involution ι is given by the action of $(-\operatorname{Id})$ in $\mathsf{SO}(\mathbf{q})$.

Given a point x in $\mathbf{H}^{2,1}_+$, the tangent space $\mathsf{T}_x\mathbf{H}^{2,1}_+$ is identified with the $\mathbf{q}$-orthogonal of x, so the restriction of $\mathbf{q}$ defines an $\mathsf{SO}(\mathbf{q})$-invariant metric of signature $(2,1)$ and curvature -1. Moreover, this metric is ι-invariant and therefore descends to a Lorentz metric on $\mathbf{H}^{2,1}$.

Finally, note that the Einstein torus $\mathbf{Ein}^{1,1}$ is the boundary of $\mathbf{H}^{2,1}$ in $\mathbf{P}(W)$.

4.1.1. *Frame bundle.* An *orthonormal frame* of a 3-dimensional oriented vector space equipped with a signature $(2,1)$ quadratic form Q, is an oriented basis $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of pairwise Q-orthogonal vectors with $Q(\varepsilon_1) = Q(\varepsilon_2) = -Q(\varepsilon_3) = 1$. Such an orthogonal frame is fully defined by the first two vectors $(\varepsilon_1, \varepsilon_2)$. We define in this way the *(orthonormal) frame bundle* $\mathcal{F}(M)$ of an oriented Lorentz 3-manifold as the set of pairs (x, ε) where x is a point of M and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is a frame of T_xM. By projecting any frame ε to ε_1, we obtain an $\mathsf{SO}(1,1)$-principal bundle
$$\mathcal{F}(M) \mapsto \mathsf{U}M \ ,$$
where $\mathsf{U}M := \{(x,n) \in \mathsf{T}M \mid \mathbf{q}(n) = 1\}$ is the *(spacelike) unit tangent bundle* of M. The Levi-Civita connection of M induces a natural connection on this bundle.

Moreover, for $M = \mathbf{H}^{2,1}_+$, we observe that $\mathsf{SO}(\mathbf{q})$ acts simply transitively on $\mathcal{F}(W)$ turning the latter into an $\mathsf{SO}(\mathbf{q})$ torsor.

4.2. **Unit tangent bundle, holonomic and isotropic surfaces.**

4.2.1. *Holonomic surfaces.* We briefly recall basic results on jet bundles and contact geometry. We refer to [18] for details.

Let M be any manifold of dimension n, equipped with an non degenerate inner product g of type (p, q). Recall that the (spacelike) unit tangent bundle is

$$\mathsf{U}M = \{u \in \mathsf{T}M \mid g(u, u) = 1\} .$$

Later on, we will only consider the case of $M = \mathbf{H}^{2,1}_+$, but the discussion is better understood in full generality.

Our goal is to define holonomic surfaces in the spirit of how holonomic surfaces are defined in Riemannian geometry.

We first recall the construction of the *contact form* or *the Liouville form* of $\mathsf{U}M$. The construction runs as follows: $\mathsf{T}M$ inherits a symplectic form ω by duality (defined by the quadratic form) with T^*M. Then the contact form (or *Liouville form*) on $\mathsf{U}M$ is $i_X\omega$ where X is the vector field generating the (spacelike) geodesic flow on $\mathsf{U}\mathbf{H}^{2,1}_+$. The kernel of this Liouville form is called the *contact distribution*: it is a field of hyperplanes in the tangent space of $\mathsf{U}M$.

A submanifold Σ of dimension $n-1$ in $\mathsf{U}M$ is *holonomic* if it is always tangent to the contact distribution. The following classical lemma, whose proof is left to the reader and can be found in [18, Theorem 4.3.15], helps to interpret what a holonomic submanifold is.

Lemma 4.1. *Let S be an immersed submanifold in M of type $(p-1, q)$ and n its normal vector field. Then $n(S)$ is a holonomic submanifold in $\mathsf{U}M$.*

Conversely, if Σ is a holonomic submanifold in $\mathsf{U}M$ transverse to the fiber of the projection to M, then $\Sigma = n(S)$ where S is an immersed surface in M of type $(p-1, q)$.

We call a holonomic surface of the form $n(S)$ where S is a $(1, 1)$-surface in M a *typical holonomic surface associated with S*

We will sometimes keep track of the type of the underlying submanifold and speak about $(p-1, q)$-holonomic submanifold.

4.2.2. *Back to anti-de-Sitter geometry.* We now specify this discussion for the (spacelike) *unit tangent bundle* of $\mathbf{H}^{2,1}_+$.

We have the following identification:

$$\mathsf{U}\mathbf{H}^{2,1}_+ = \{(x, n) \in W \times W \mid \langle x, n\rangle = 0 \ , \ \mathbf{q}(x) = -\mathbf{q}(n) = -1\} . \tag{4.1}$$

Under this identification, the tangent space splits as

$$\mathsf{T}_{(x,n)}\mathsf{U}\mathbf{H}^{2,1}_+ = \{(u_1, u_2) \in W \times W \mid \langle u_1, x\rangle = \langle u_2, n\rangle = \langle u_1, n\rangle + \langle x, u_2\rangle = 0\} . \tag{4.2}$$

The involution ι lifts to $\mathsf{U}\mathbf{H}^{2,1}_+$, and the quotient is $\mathsf{U}\mathbf{H}^{2,1}$, the unit tangent bundle of $\mathbf{H}^{2,1}$. In this identification again, the contact distribution is given by

$$\ker(\lambda_{(x,n)}) = \{(u_1, u_2) \mid \langle n, u_1\rangle = \langle x, u_2\rangle = 0\}.$$

4.2.3. *Holonomic and isotropic surfaces.* Finally, in this situation, holonomic surfaces are identified with isotropic surfaces. A direct computation gives the following proposition.

Proposition 4.2. *If σ defines an isotropic surface, let η be the dual isotropic surface. Then*

$$(x, n) : \begin{cases} S & \rightarrow & \mathsf{U}\mathbf{H}^{2,1}_+ \\ s & \mapsto & \left(\frac{\sqrt{2}}{2}(\sigma(s) - \eta(s)), \frac{\sqrt{2}}{2}(\sigma(s) + \eta(s)\right)) \end{cases}$$

is a holonomic surface.

Conversely if (x, n) is a holonomic surface then

$$(\sigma, \eta) : \begin{cases} S & \rightarrow & W \times W \\ s & \mapsto & \left(\frac{\sqrt{2}}{2}(x(s) + n(s)), \frac{\sqrt{2}}{2}(n(s) - x(s))\right) \end{cases}$$

is such that σ is an isotropic surface and η the dual isotropic surface. It follows that any isotropic surface gives a holonomic surface.

The following proposition identifies the first fundamental form at infinity.

Proposition 4.3. *The first fundamental form at infinity of a typical holonomic surface associated with a surface S in $\mathbf{H}^{2,1}$ is $\frac{1}{2}(\mathrm{I} + 2\mathrm{II} + \mathrm{III})$ where I, II and III are respectively the first, second, and third fundamental forms of S.*

Proof. We have by the previous proposition that σ is given by $\frac{\sqrt{2}}{2}(x + n)$. Thus

$$\langle \mathrm{D}_u\sigma, \mathrm{D}_u\sigma\rangle = \frac{1}{2}\left(\langle \mathrm{D}_u x, \mathrm{D}_u x\rangle + 2\langle \mathrm{D}_u x, \mathrm{D}_u n\rangle + \langle \mathrm{D}_u n, \mathrm{D}_u n\rangle\right) \tag{4.3}$$

$$= \frac{1}{2}(\mathrm{I}(u, u) + 2\mathrm{II}(u, u) + \mathrm{III}(u, u)) . \tag{4.4}$$

This completes the proof. □

Proposition 4.4. *Let S be a holonomic surface immersed in $\mathsf{U}\mathbf{H}^{2,1}$ that we consider as a subset of $W \times W$. The first fundamental form at infinity on S is the induced metric from the metric on $W \times W$ given by the quadratic form*

$$Q((u, v)) = \frac{1}{2}\langle u + v, u + v\rangle \ .$$

Proof. Indeed, the first fundamental form on S is given by

$$\frac{1}{2}\left(\langle \mathrm{D}_u x, \mathrm{D}_u x\rangle + 2\langle \mathrm{D}_u x, \mathrm{D}_u n\rangle + \langle \mathrm{D}_u n, \mathrm{D}_u n\rangle\right) = \frac{1}{2}\langle \mathrm{D}_u x + \mathrm{D}_u n, \mathrm{D}_u x + \mathrm{D}_u n\rangle \ .$$

The result follows. □

4.3. **Epstein surface.**

Definition 4.5. Let ϕ be an immersion of S in $\mathbf{Ein}^{1,1}$ such that $\phi^*\tau$ is trivial. Let g be a C^2 metric on S. Then the *g-Epstein surface* ε_g is the holonomic surface in $\mathbf{H}^{2,1}_+$ associated (by Proposition 4.2) to the isotropic surface σ defined by g (by Theorem 3.1).

We now relate this definition for the sake of completeness to the notion of envelope of horospheres, thus showing that our Epstein surfaces are the analogs in $\mathbf{H}^{2,1}$ of the classical Epstein surfaces constructed in the hyperbolic 3-space.

4.3.1. *Horospheres.* Any isotropic vector x_0 in W defines a *horosphere* in $\mathbf{H}^{2,1}_+$ via the formula

$$H(x_0) := \left\{p \in \mathbf{H}^{2,1}_+ \mid \langle x_0, p\rangle = -\frac{\sqrt{2}}{2}\right\} .$$

An horosphere in $\mathbf{H}^{2,1}$ is then the projection of an horosphere in $\mathbf{H}^{2,1}_+$. Observe that the vectors $\pm x_0$ define the same horospheres in $\mathbf{H}^{2,1}$.

The tangent space to $H(x_0)$ at a point p is identified with $\operatorname{span}\{x_0, p\}^\perp$ and so the induced metric on $H(x_0)$ is Lorentzian. Moreover, $H(x_0)$ is intrinsically flat.

4.3.2. *Envelope.* Let S, ϕ, g and σ be as in the definition. We then have a family of horospheres

$$\mathcal{H}(g) = \{H(\sigma(s)) \mid s \in S\}.$$

An *envelope* of $\mathcal{H}(g)$ is then a smooth map ε_g from S to $\mathbf{H}^{2,1}$ such that for every x

$$\varepsilon_g(x) \in H(\sigma(x)) \text{ and } \mathrm{d}_x\varepsilon_g(\mathsf{T}_xS) \subset \mathsf{T}_{\varepsilon_h(x)}H(\sigma(x)) .$$

Proposition 4.6. *Given a g-Esptein surface ψ, the map $\pi \circ \psi$ is an envelope for $\mathcal{H}(g)$, where π the projection to $\mathbf{H}^{2,1}$*

Proof. Let g, σ and S as in the definition, and let η be the dual isotropic surface to σ as in paragraph 3.3. In particular, for any vector fields X, Y on S we have

$$\langle\sigma, \eta\rangle - 1 = \langle\sigma, \sigma\rangle = \langle\eta, \eta\rangle = \langle \mathrm{D}_X\sigma, \eta\rangle = 0 \quad \text{and} \quad \langle \mathrm{D}_X\sigma, \mathrm{D}_Y\sigma\rangle = g(X, Y) .$$

We have

$$\pi \circ \psi : \begin{cases} S & \to \quad \mathbf{H}^{2,1} , \\ x & \mapsto \quad \frac{\sqrt{2}}{2}(\sigma(x) - \eta(x)) , \end{cases}$$

so

$$\langle\sigma(x), \pi \circ \psi(x)\rangle = -\frac{\sqrt{2}}{2} ,$$

and for any tangent vector u in T_xS

$$\langle \mathrm{d}_x(\pi \circ \psi)(u), \sigma(x)\rangle = \langle \mathrm{d}_x(\pi \circ \psi)(u), \eta(x)\rangle = 0 .$$

So $\pi \circ \psi$ is an envelope for $\mathcal{H}(g)$. □

One can easily see from the above computation that the induced metric on $\pi \circ \psi$ is equal to $\frac{1}{2}\mathrm{I}^* + \mathrm{II}^* + \frac{1}{2}\mathrm{III}^*$. In particular, it is not always immersed. We denote by Σ_g its image.

5. The $\mathcal{W}$-volume

We define in this section the $\mathcal{W}$-volume for 3-manifolds with boundary in $\mathbf{H}^{2,1}$. As in [4], we work in a noncompact setting and this requires some technical adjustments. Also the $\mathcal{W}$-volume is not actually defined for 3-manifolds with boundary in $\mathbf{H}^{2,1}$, but rather for 3-manifolds with boundary immersed in $\mathsf{U}\mathbf{H}^{2,1}$ with some holonomy condition on the boundary.

5.1. Preliminary: the geometry of $\mathsf{U}\mathbf{H}^{2,1}$ and cobordism.

5.1.1. *Differential forms on* $\mathsf{U}\mathbf{H}^{2,1}$. The unit tangent bundle $\mathsf{U}\mathbf{H}^{2,1}$ is equipped with a set of differential forms that we now describe. We use the decomposition described in equation (4.2) and write $u = (u_1, u_2)$ the two components of a vector u in $\mathsf{T}_{(x,n)}\left(\mathsf{U}\mathbf{H}^{2,1}_+\right)$. We introduce and consider the following forms.

(1) The pull-back ω of the volume form on $\mathbf{H}^{2,1}_+$ via the projection is a closed 3-form on $\mathsf{U}\mathbf{H}^{2,1}_+$ whose value at (x, n) is given by

$$\omega(u, v, w) = \pi^* \operatorname{vol}_{\mathbf{H}^{2,1}}(u, v, w) = \det(x, u_1, v_1, w_1) .$$

(2) The 2-form α whose value at (x, n) is given by

$$\alpha(u, v) = \frac{1}{4}\left(\det(x, n, u_2, v_1) + \det(x, n, u_1, v_2)\right) .$$

We remark that all those differential forms are invariant under the involution ι and so descend to forms on $\mathsf{U}\mathbf{H}^{2,1}$ that we denote the same way.

5.1.2. *Corbordism constant at infinity.* As a first example of cobordism constant at infinity, we have a C^k map ϕ from M into $\mathsf{U}\mathbf{H}^{2,1}_+$, where

- $M = S \times [0, 1]$ and S is a possibly noncompact surface,
- $\phi(S \times \{0\})$ and $\phi(S \times \{1\})$ are holonomic surfaces,
- there exists K a compact subset of S, such that $\phi(x, t)$ is constant in t for all x not in K.

In that case, we call (M, ϕ) a *lens cobordism*.

More generally, we want the compact K to have some more complicated topology. Let N_0 and N_1 be two oriented surfaces which will be non compact as well as ϕ_0 and ϕ_1 from N_0 and N_1 respectively to $\mathsf{U}\mathbf{H}^{2,1}_+$ such that $\phi_i(N_i)$ are holonomic surfaces.

A *cobordism constant at infinity* between (N_0, ϕ_0) and (N_1, ϕ_1) is a pair (M, ϕ), where

(1) M is a possibly non compact 3-manifold with boundary $\partial M = N_0 \sqcup \overline{N}_1$.
(2) ϕ is a C^∞ map from M to $\mathsf{U}\mathbf{H}^{2,1}_+$, such $\phi|_{N_i} = \phi_i$.
(3) Moreover ϕ is *constant at infinity*: there is a compact K in M, such that K^c is homeomorphic to $U \times [0, 1]$, with $U \times \{0\}$, respectively $U \times \{1\}$, is a subset of N_0, respectively N_1, and ϕ is constant in the second factor of $U \times [0, 1]$.

Note that ϕ_1 and ϕ_0 agree outside a compact set. Observe that given a cobordism as above $\phi^*\tau$ is a trivial bundle.

5.2. **Differential forms and the $\mathcal{W}$-volume.**

Definition 5.1. We define the *$\mathcal{W}$-volume* of the cobordism (M, ϕ),

$$\mathcal{W}(M, \phi) = \int_M \phi^*\omega - \int_{\partial M} \phi^*\alpha \ .$$

In this equation, we first observe that $\phi^*\omega$ vanishes outside a compact set. We also used an abuse of language for the last term, since $\phi|_{U\times\{0\}}$ and $\phi|_{U\times\{1\}}$ agree outside of a compact set.

Because ω is closed,

$$\mathcal{W}(M, \phi) = \mathcal{W}(M, \psi) \ ,$$

whenever ϕ and ψ agree outside a compact in $\mathrm{Int}(M)$. In particular, for a lens cobordism, where we write $\partial M = S_0 \sqcup S_1$, $\mathcal{W}(M, \phi)$ depends only on the restriction of ϕ_0 on S_0, and ϕ_1 on S_1. In that situation, we write

$$\mathcal{W}(S_0, S_1) := \mathcal{W}(M, \phi) \ . \tag{5.1}$$

We will most of the time write $\mathcal{W}(M, \phi) =: \mathcal{W}(M)$. Then note that since one can glue cobordism, the $\mathcal{W}$-volume satisfies *Chasles relation*

$$\mathcal{W}(M \sharp N) = \mathcal{W}(M) + \mathcal{W}(N) \ .$$

5.2.1. *Classical formula.*

Proposition 5.2. *Let (M, ϕ) be a cobordism. Assume that the holonomic surface $\phi(\partial M)$ is the lift of a surface S in $\mathbf{H}^{2,1}_+$. Then*

$$\mathcal{W}(M, \phi) = \mathrm{vol}(\pi \circ \phi(M)) - \frac{1}{2}\int_S H \, \mathrm{d}a \ ,$$

where H and $\mathrm{d}a$ are, respectively, the mean curvature and the area form of S.

Proof. If f is an immersion of type $(1, 1)$ from a surface S into $\mathbf{H}^{2,1}_+$, the tangent vectors to the holonomic lift F of S have the form $(u, B(u))$ where u is tangent to $f(S)$ and B is the shape operator. Given a pair of vectors (u, v) of S, we get

$$F^*\alpha(u, v) = \frac{1}{4}(\mathrm{d}a(B(u), v) + \mathrm{d}a(u, B(v))) = \frac{1}{4}\,\mathrm{tr}(B)\,\mathrm{d}a(u, v) \ .$$

The result follows. ∎

5.3. **Variational formula.** We now prove the variational formula for the $\mathcal{W}$-volume:

Theorem 5.3 (Variational formula). *Let $(M, \phi_t)_{t\in\mathbb{R}}$ be a smooth family of cobordism, pairwise equal at infinity. Let g_t be the induced first fundamental form at infinity on ∂M and u the compactly supported function such that $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_t = 2ug_0$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{W}(M, \phi_t) = -\frac{1}{2}\int_{\partial M} uF_g \ .$$

5.3.1. *More differential forms.* We need to introduce more differential forms and objects to understand the variations of ω and α.

(1) The 1-forms x^* and n^* given by

$$x^*(u) = \langle x, u_2\rangle \ ,$$
$$n^*(u) = \langle u_1, n\rangle \ .$$

Recall that $x^* + n^* = 0$ and that the kernel of any of these forms is the contact distribution $\mathcal{P}$ in $\mathsf{UH}^{2,1}$. As a vector subspace of $W \times W$,

$$\mathcal{P}_{(x,n)} = \{(u, v) \mid \langle x, u\rangle = \langle x, v\rangle = \langle n, u\rangle = \langle n, v\rangle = 0\}$$

(2) The 2-forms θ_1, θ_2 and α whose value at (x, n) is given by

$$\theta_1(u, v) = \det(x, n, u_1, v_1) \ ,$$
$$\theta_2(u, v) = \det(x, n, u_2, v_2) \ .$$

(3) Let $\mathcal{F}(Q)$ be the space of pairwise orthogonal triples (x, u, n) satisfying

$$-\langle x, x\rangle = \langle n, n\rangle = \langle u, u\rangle = 1 \ .$$

Let β be the 1-form on $\mathcal{F}(Q)$ defined by

$$\beta_{(x,n,u)}(w) = -\det(x, n, u, w_3) \ , \tag{5.2}$$

Remarks

(1) All these differential forms are again invariant under the involution ι and therefore descend to forms on $\mathsf{UH}^{2,1}$ that we denote the same way.
(2) We have a projection π from $\mathcal{F}(Q)$ to $\mathsf{UH}^{2,1}$ defined by $\pi(x, n, u) = \pi(x, n)$.

We first need to identify precisely β. Let Q be the subdistribution or rank 2 of $\mathcal{P}$ defined by

$$Q \coloneqq \{(u, v) \in \mathcal{P} \mid u = v\} \ .$$

We observe that we have an identification of $=Q_{x,n}$ with the orthogonal of (x, n) in W. Then $\mathcal{F}(Q)$ is the positive unit tangent bundle of Q.

Proposition 5.4. *The form $\mathrm{d}\beta$ is the curvature form of the bundle Q, equipped with the $(1,1)$-metric and the connection from its embedding as a subbundle of the trivial bundle $W \oplus W$ over $\mathsf{UH}^{2,1}$.*

Proof. By definition, if (x^t, n^t, u_1^t) is a curve in $\mathcal{F}(Q)$ seen as a subset of W^3

$$\beta_{(x^0,n^0,u^0)}(\dot{x}, \dot{n}, \dot{u}) = \langle \dot{u}, e \rangle \ ,$$

where e is the vector of norm that satisfies $\langle e, e \rangle = -1$, orthogonal to (x, n, u) and such that $\det(x, n, e, u) = 1$. Thus

$$\beta_{(x^0,n^0,u_1^0)}(\dot{x}, \dot{n}, \dot{u}) = -\det(x, n, u, \dot{u}).$$

This is what we wanted to prove □

Proposition 5.5 (Fundamental equations)**.** *We have given*

$$\mathrm{d}\beta = \theta_1 - \theta_2 \ , \tag{5.3}$$

$$\mathrm{d}\alpha = \frac{1}{2}(n^* \wedge \theta_2 - x^* \wedge \theta_1) \ , \tag{5.4}$$

Proof. We first compute $\mathrm{d}\beta$. Recall that

$$\mathrm{d}\beta(w, v) = \mathrm{D}_w\beta(v) - \mathrm{D}_w\beta(u) \ .$$

From equation (5.2), we have

$$\mathrm{D}_w\beta(v) := -\det(w_1, n, u, v_3) - \det(x, w_2, u, v_3) - \det(x, n, w_3, v_3) \ .$$

Recall that $\mathcal{F}(Q)$ is the subset of W^3 consisting of triples (x, n, u) which are pairwise orthogonal and satisfy

$$-\langle x, x \rangle = \langle n, n \rangle = \langle u, u \rangle = 1 \ .$$

Differentiating these equations, and introducing the vector e orthogonal to x, u and n, such that $\det(x, n, u, e) = 1$, recall that $\langle n, n \rangle = \langle u, u \rangle = 1$ while $\langle x, x \rangle = \langle e, e \rangle = -1$

- $\langle u, w_3 \rangle = 0 = \langle v_3, u \rangle$ and thus
$$\det(x, n, w_3, v_3) = 0 \ . \tag{5.5}$$
- $\langle w_2, n \rangle = 0$ and thus
$$\begin{aligned} \det(x, w_2, u, v_3) &= \langle v_3, n \rangle \det(x, w_2, u, n) \\ &= -\langle w_2, e \rangle \langle v_3, n \rangle \det(x, e, u, n) \\ &= \langle w_2, e \rangle \langle v_3, n \rangle \ . \end{aligned} \tag{5.6}$$
- $\langle w_1, x \rangle = 0$ and thus
$$\begin{aligned} \det(w_1, n, u, v_3) &= -\langle v_3, x \rangle \det(w_1, n, u, x) \\ &= \langle w_1, e \rangle \langle v_3, x \rangle \det(e, n, u, x) \\ &= -\langle w_1, e \rangle \langle v_3, x \rangle \ . \end{aligned} \tag{5.7}$$

It follows that

$$\mathrm{D}_w\beta(v) = \langle w_2, e \rangle \langle v_3, n \rangle - \langle w_1, e \rangle \langle v_3, x \rangle \ .$$

Finally deriving $\langle u, x \rangle = \langle u, n \rangle = 0$, we get

$$\langle v_3, x \rangle + \langle u, v_1 \rangle = 0 \ , \ \langle v_3, n \rangle + \langle u, v_2 \rangle = 0 \ ,$$

hence

$$\mathrm{D}_w\beta(v) = \langle w_1, e\rangle \langle v_1, u\rangle - \langle w_2, e\rangle \langle v_2, u\rangle \ . \tag{5.8}$$

Since

$$\det(x, n, w_2, v_2) = \langle w_2, e\rangle\langle v_2, u\rangle - \langle w_2, u\rangle\langle v_2, e\rangle \ , \tag{5.9}$$

$$\det(x, n, w_1, v_1) = \langle w_1, e\rangle\langle v_1, u\rangle - \langle w_1, u\rangle\langle v_1, e\rangle \ , \tag{5.10}$$

it follows that

$$\begin{aligned}\mathrm{d}\beta(w, v) &= \det(x, n, w_1, v_1) - \det(x, n, w_2, v_2)\\ &= \theta_1(w, v) - \theta_2(w, v) \ .\end{aligned}$$

Let us now compute the second differential. Recall that

$$\alpha(u, v) = \frac{1}{4}\left(\det(x, n, u_2, v_1) + \det(x, n, u_1, v_2)\right) \ .$$

We have

$$\mathrm{d}\alpha(u, v, w) = (\mathrm{D}_u\alpha)(v, w) + (\mathrm{D}_v\alpha)(w, u) + (\mathrm{D}_w\alpha)(u, v) \ .$$

Then

$$\begin{aligned}4(\mathrm{D}_u\alpha)(v, w) = &\det(u_1, n, v_2, w_1) + \det(x, u_2, v_2, w_1)\\ &+ \det(u_1, n, w_2, v_1) + \det(x, u_2, w_2, v_1) \ .\end{aligned}$$

Now we use the fact that u_1, v_1 and w_1 are all normal to x, while u_2, v_2 and w_2 are normal to n. Thus

$$\begin{aligned}4(\mathrm{D}_u\alpha)(v, w) &= -\langle v_2, x\rangle \det(u_1, n, x, w_1) + \langle w_1, n\rangle \det(x, u_2, v_2, n)\\ &\quad - \langle w_2, x\rangle \det(u_1, n, x, v_1) + \langle v_1, n\rangle \det(x, u_2, w_2, n)\\ &= -\langle v_2, x\rangle\, \theta_1(w, u) + \langle w_1, n\rangle\, \theta_2(u, v) - \langle w_2, x\rangle\, \theta_1(u, v) + \langle v_1, n\rangle\, \theta_2(u, w) \ .\end{aligned}$$

Thus $4\mathrm{d}\alpha = 2n^* \wedge \theta_2 - 2x^* \wedge \theta_1$. □

The following proposition identifies β as the connection form of a holonomic surface. Let S be an holonomic surface, and then we have a linear map from $\mathsf{T}S_{(x,n)}S$ to $\mathcal{Q}_{(x,n)}$, both seen as subspaces of $W \oplus W$, given by

$$\Lambda : (u, v) \mapsto \tfrac{1}{2}(u + v, u + v) \ ,$$

Proposition 5.6. *The linear map* Λ *is an isometry from* $\mathsf{T}_{x,n}S$ *equipped with its first fundamental form at infinity with* $\mathcal{Q}_{(x,n)}$ *equipped with the induced metric from* $W \oplus W$.

Proof. Let $U = (\mathrm{D}_u x, \mathrm{D}_u n)$ be a tangent vector to S. Then

$$\langle \Lambda(U), \Lambda(U)\rangle = \tfrac{1}{2}\langle \mathrm{D}_u x + \mathrm{D}_u n, \mathrm{D}_u x + \mathrm{D}_u n\rangle = \mathrm{I}^*(U, U) \ .$$

This completes the proof . □

5.3.2. *Variation of surfaces.* To show Theorem 5.3, we will use the following lemma. Let σ_t be the isotropic surface associated with $g_t =: e^{2u_t} g$ and η_t its dual surface. Write $(\sigma, \eta) = (\sigma_0, \eta_0)$ and denote by $\dot{\sigma}$ and $\dot{\eta}$ the derivative at $t = 0$.

Lemma 5.7. *Using the above notations, we have*

$$\dot{\sigma} = u\sigma \quad \text{and} \quad \dot{\eta} = -u\eta - \nabla^g u \ ,$$

where $\nabla^g u$ is the g-gradient of u, that is the vector field on S satisfying $g(\nabla^g u, \cdot\,) = \mathrm{d}u$.

Proof. By construction, the sections (σ_t, η_t) satisfy

$$\sigma_t = e^{u_t}\sigma \ \ , \ \ \langle \eta_t, \eta_t \rangle = 0 \ \ , \ \ \text{and} \ \ \langle \sigma_t, \eta_t \rangle = 1 \ .$$

Derivating the first equation gives $\dot{\sigma} = u\sigma$. The second gives $\langle \dot{\eta}, \eta \rangle = 0$ and the third gives $\langle \dot{\eta}, \sigma \rangle = -u$. In particular, there is a vector field X on S such that

$$\dot{\eta} = -u\eta + \mathrm{D}_X \sigma \ .$$

We now obtain the expression of X. Let Y be a tangent vector on S. Then the last equation to be used is

$$\langle \mathrm{D}_Y \sigma_t, \eta_t \rangle = 0 \ .$$

We differentiate in time the last equation to get

$$\langle \mathrm{D}_Y \dot{\sigma}, \eta \rangle + \langle \mathrm{D}_Y \sigma, \dot{\eta} \rangle = 0 \ .$$

Using the previous identifications, we obtain

$$0 = \mathrm{d}u(Y) + u\,\langle \mathrm{D}_Y \sigma, \eta \rangle - u\,\langle \mathrm{D}_Y \sigma, \sigma \rangle + \langle \mathrm{D}_Y \sigma, \mathrm{D}_X \sigma \rangle = \mathrm{d}u(Y) + \langle \mathrm{D}_Y \sigma, \mathrm{D}_X \sigma \rangle \ .$$

Using the equality $\langle \mathrm{D}_X \sigma, \mathrm{D}_Y \sigma \rangle = g(X, Y)$, we finally obtain

$$\mathrm{d}u(Y) + g(X, Y) = 0 \ ,$$

which yields $X = -\nabla^g u$ and the result. □

Proof of Theorem 5.3. Let $(\Sigma_t)_{t \in \mathbb{R}}$ be the family of holonomic surfaces associated with $e^{2u_t} g$. From Proposition 4.2 and Lemma 5.7, we see that the variation ζ of the family of holonomic surfaces along Σ_0 is given by

$$\begin{aligned} \zeta &= (\zeta_1, \zeta_2) \\ &= \frac{\sqrt{2}}{2} \left(u\sigma + u\eta + \nabla^g u, u\sigma - u\eta - \nabla^g u \right) \\ &= \left(nu + \frac{\sqrt{2}}{2} \nabla^g u, xu - \frac{\sqrt{2}}{2} \nabla^g u \right) . \end{aligned}$$

Given a family of compact cobordisms ϕ_t to define the Liouville action, we obtain

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \mathcal{S}_\phi(g, e^{2tu} g) = - \int_S \phi^* \iota_\zeta (\omega - \mathrm{d}\alpha) \ .$$

A first computation gives for U and V tangent to Σ_0

$$\begin{aligned}\iota_\zeta\omega(U,V) &= \det(x,\zeta_1,U_1,V_1)\\ &= u\det(x,n,U_1,V_1)\\ &= u\theta_1(U,V)\ ,\end{aligned}$$

where we used the fact that $\nabla^g u$ is tangent to Σ_0. Similarly for U and V tangent to Σ_0

$$\begin{aligned}2\iota_\zeta \mathrm{d}\alpha(U,V) &= (n^*\wedge\theta_2 - x^*\wedge\theta_1)(\zeta,U,V)\\ &= \langle n,\zeta_1\rangle\ \theta_2(U,V) - \langle x,\zeta_2\rangle\ \theta_1(U,V)\\ &= u\,(\theta_2(U,V)+\theta_1(U,V))\ .\end{aligned}$$

It then follows that

$$\iota_\zeta(\omega - \mathrm{d}\alpha) = \frac{1}{2}u(\theta_1-\theta_2) = \frac{1}{2}u\ \mathrm{d}\beta\ ,$$

where the last equation comes from Proposition 5.5. Now $\mathrm{d}\beta$ is the curvature of the bundle Q over S, thus from Proposition 5.6, $\mathrm{d}\beta$ is also the curvature of the bundle $\mathsf{T}S$ equipped with the first fundamental form at infinity. ◻

6. Liouville action on the split annulus

6.1. **Liouville action for metrics equal outside of a compact.** We now concentrate on the special case of a lens cobordism (M,ϕ) between split annuli. Recall that we denote by $\mathbf{A}$ the split annulus defined in Section 2.2. We assume that $M = \mathbf{A}\times[0,1]$, ϕ is a map from M to $\mathsf{U}\mathbf{H}^{2,1}$, such that

(1) ϕ restricted to ∂M is a holonomic surface.
(2) Let $\phi_t(x) := \phi(x,t)$, then ϕ_t is constant in t outside a compact set in $\mathbf{A}$.

we observe that $\mathcal{W}(M,\phi)$ does not depend on ϕ but just ϕ_0 and ϕ_1, and we write

$$\mathcal{W}(M,\phi) = \mathcal{W}(\phi_0,\phi_1)\ ,$$

Observe also that, thanks to Theorem 3.1, given an immersion ψ of $\mathbf{A}$ in $\mathbf{Ein}^{1,1}$ and two metrics g_0 and g_1 on $\mathbf{A}$ outside of a compact set, we obtain two holonomic surfaces ϕ_0 and ϕ_1, and we define

$$\mathcal{W}_\psi(g_0,g_1) := \mathcal{W}(\phi_0,\phi_1)\ .$$

We have two immediate and important corollaries

Corollary 6.1. *The quantity $\mathcal{W}_\psi(g_0,g_1)$ does not depend on ψ.*

Definition 6.2 (Liouville action)**.** We will therefore define unambiguously the *Liouville action* between two metrics h_0 and h_1 on $\mathbf{A}$ equal outside a compact set as

$$\mathcal{S}(h_0,h_1) := \mathcal{W}_\psi(h_0,h_1)\ ,$$

where ψ is any immersion of $\mathbf{A}$ in $\mathbf{Ein}^{1,1}$.

The following corollary of the variational formula justifies the name of Liouville action.

Corollary 6.3 (Critical point). *With the same notation as in Theorem 5.3, if g in $\mathcal{M}(\mathbf{A})$ is a critical point for all compact Weyl scaling preserving the area, then g has constant curvature. More precisely, if for all u compactly supported functions on $\mathbf{A}$ such that $\int_{\mathbf{A}} u\mathrm{d}a_g = 0$, we have*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{S}(g, e^{2tu}g) = 0,$$

then, g has constant curvature.

Remark 6.4.
(1) The two terminologies *Liouville action* for pairs of metrics, or $\mathcal{W}$*-volume* for pairs of Epstein surfaces are here for historical reasons in the Riemannian context [14, 20] and the choice depends on whether one wants to insist on metrics or surfaces.
(2) Since we can glue isotopies, as a corollary of the same relation for composing cobordisms

$$\mathcal{S}(h_1, h_2) + \mathcal{S}(h_2, h_3) = \mathcal{S}(h_1, h_3) .$$

6.1.1. *First properties.* We now summarize some of the consequences of the variational formula for the Liouville action. Let S be a surface and ϕ be an immersion of S in $\mathbf{Ein}^{1,1}$ with $\phi^*\tau$ trivial.

Proposition 6.5. *Let g and h be two metrics in $\mathcal{M}(\mathbf{A})$ with $h = e^{2u}g$ for a compactly supported smooth function u, then*

(1) *The Liouville action can be expressed in terms of the conformal factor u by*

$$\mathcal{S}(g, h) = -\frac{1}{2}\int_{\mathbf{A}} uF_g + \frac{1}{4}\int_{\mathbf{A}} u\ \mathrm{d}(\mathrm{d}u \circ \mathrm{I}) .$$

(2) *The* monotonicity formula *holds*

$$\mathcal{S}(g, h) = -\frac{1}{4}\int_{\mathbf{A}} u(F_g + F_h) .$$

(3) *For any split diffeomorphism φ of S, we have the* split invariance

$$\mathcal{S}(\varphi^* g, \varphi^* h) = \mathcal{S}(g, h) .$$

Proof. Write $g_t = e^{2tu}g$ so $g = g_0$ and $h = g_1$. By Theorem 5.3 and the Chasles relation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(g, e^{2tu}g) = -\frac{1}{2}\int_{\mathbf{A}} uF_{g_t} .$$

Using Proposition 2.9, we have

$$F_{g_0} = F_{g_t} + t\mathrm{d}(\mathrm{d}u \circ \mathrm{I}) .$$

This gives

$$
\begin{aligned}
\mathcal{W}(g,h) &= \int_{t=0}^{1} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(g, e^{2tu}g)\,\mathrm{d}t \\
&= \int_{t=0}^{1} \left(-\frac{1}{2}\int_{\mathbf{A}} uF_{g_0} + \frac{t}{2}\int_{S} u\,\mathrm{d}(\mathrm{d}u\circ \mathrm{I}) \right)\mathrm{d}t \\
&= -\frac{1}{2}\int_{\mathbf{A}} uF_g + \frac{1}{4}\int_{\mathbf{A}} u\,\mathrm{d}(\mathrm{d}u\circ \mathrm{I}) \\
&= -\frac{1}{4}\int_{\mathbf{A}} u(F_g + F_h)
\end{aligned}
$$

where for the last equality, we used $\mathrm{d}(\mathrm{d}u\circ \mathrm{I}) = F_g - F_h$. Thus, items (1) and (2) follow. The split invariance is a direct consequence of the change of coordinates formula. □

Remark 6.6. If $g = \mathrm{d}x\,\mathrm{d}y$ and u are compactly supported, then

$$
\mathcal{S}(e^{2u}g, g) = -\frac{1}{2}\int_{\mathbf{A}} u\,\partial^2_{xy}u\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_{\mathbf{A}} \partial_x u\,\partial_y u\,\mathrm{d}x\,\mathrm{d}y.
$$

More generally, if u is compactly supported, the expression in Proposition 6.5 above can be rewritten as

$$
\mathcal{S}(e^{2u}g, g) = \frac{1}{4}\int_{\mathbf{A}} (2uK_g - u\,\Box_g\,u)\,\mathrm{d}a_g,
$$

where $\mathrm{d}a_g$ is the area form of g. This expression coincides with the Liouville action with zero cosmological constant in the physics literature, see, e.g., [22].

6.2. **Liouville action for general metrics.** We are moving beyond metrics that are equal at infinity. In this section, $\mathbf{A}$ is the split annulus.

6.2.1. *The $\mathcal{S}$-class.* We denote by $\mathcal{M}(\mathbf{A})$ the space of *weak metrics*, that is, the space of volume forms on $\mathbf{A}$ defining measures absolutely continuous to the de Sitter volume form.

We also denote by $\mathcal{M}_b(\mathbf{A})$ the set of C^2 metrics in $\mathcal{M}(\mathbf{A})$ whose sectional curvature is bounded (namely, in $\mathsf{L}^\infty(\mathbf{A})$).

Definition 6.7. Let g be a metric in $\mathcal{M}_b(\mathbf{A})$. The *$\mathcal{S}$-class of g* is the set $\mathcal{S}_g$ of weak metrics h in $\mathcal{M}(\mathbf{A})$ of the form $e^{2u}g$ where u satisfies

(1) the function u is to $\mathsf{L}^\infty(\mathbf{A}, \mathrm{d}a_g)$ where $\mathrm{d}a_g$ is the area form of g associated with the volume form ω_g,
(2) the function u tends to 0 uniformly on $\partial\mathbf{A}$ (that is, for any $\varepsilon > 0$, we have $|u| < \varepsilon$ away from a compact set),
(3) the d'Alembertian $\Box_g\,u$ belongs to $\mathsf{L}^\infty(\mathbf{A}, \mathrm{d}a_g)$ and $\mathsf{L}^1(\mathbf{A}, \mathrm{d}a_g)$
(4) there is a polygonal curve P such that $\mathrm{VB}(u, P)$ is finite.

Observe that if u is C^0, then item (1) follows from item (2).
Moreover, condition (4) is easily satisfied, in particular when u is in C^1.
We denote by $\mathcal{S}_0$ the $\mathcal{S}$-class of the de Sitter metric g_0.

A *polygonal curve* in $\mathbf{A}$ is an oriented loop P made of finitely many lightlike segments and whose vertices project on each factor $\mathbf{RP}^1$ to a cyclically oriented tuple (see Figure 1). We say that a tuple $(\alpha_1, \ldots, \alpha_{2k})$ *represents* P if

- the projection of $(\alpha_1, \ldots, \alpha_{2k})$ on each factor is cyclically oriented,
- for any i the points α_{2i} and α_{2i+1} are the extremities of a vertical segment $[\alpha_{2i}, \alpha_{2i+1}]$ contained in P,
- for any i, the points α_{2i+1} and α_{2i+1-2} are the extremities of a horizontal segment $[\alpha_{2i+1}, \alpha_{2i+2}]$ contained in P,
- $P = \bigcup_{i=1}^{2k} [\alpha_i, \alpha_{i+1}]$ with $\alpha_{2k+1} = \alpha_1$.

Observe that we do not assume that $\alpha_i \neq \alpha_{i+1}$, so we have many sequences that represent the same curve. For any function f on $\mathbf{A}$ and polygonal curve P, we define

$$\mathrm{VB}(f, P) = \sup \left\{ \sum_{i=1}^{n} |u(\alpha_{2i-1}) - u(\alpha_{2i})| \ , \ (\alpha_1, \ldots \alpha_{2n}) \text{ representing } P \right\} .$$

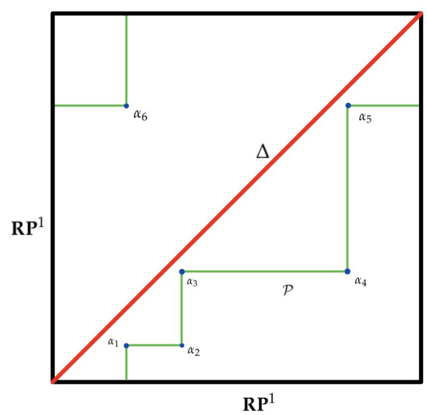

FIGURE 1. A polygonal curve $\mathcal{P}$ represented by $(\alpha_1, ..., \alpha_6)$

Lemma 6.8. *Let g be a metric in $\mathcal{M}_b(\mathbf{A})$. If h is a C^2 metric in $\mathcal{S}_g$, then h is in $\mathcal{M}_b(\mathbf{A})$ and $\mathcal{S}_g = \mathcal{S}_h$ — in particular, being C^2 and in the same $\mathcal{S}$-class defines an equivalence relation on $\mathcal{M}_b(\mathbf{A})$.*

Proof. Write $h = e^{2u} g$. Using Proposition 2.9 we have

$$\square_g \, u = e^{2u} K(h) - K(g) \, ,$$

so $K(h)$ belongs to $\mathsf{L}^\infty(\mathbf{A})$ since $K(g)$ and u do. It follows that h is in $\mathcal{M}_b(\mathbf{A})$.

Since u belongs to $\mathsf{L}^\infty(\mathbf{A})$, there exists a positive A such that $A^{-1} < e^{2u} < A$. In particular, since $\mathrm{d}a_h = e^{2u}\mathrm{d}a_g$, we have $\mathsf{L}^1(\mathbf{A}, \mathrm{d}a_g) = \mathsf{L}^1(\mathbf{A}, \mathrm{d}a_h)$.

Let f be a metric in $\mathcal{S}_g$, of the form $f = e^{2v}g$. Then $f = e^{2(v-u)}h$. It follows that $w = v - u$ satisfies items (1) and (2) of the definition above. Moreover, by equation (2.1)

$$\square_h w = e^{-2u} \square_g w ,$$

it follows that item (3) of the definition is satisfied. Hence f belongs to $\mathcal{S}_h$. Similarly for item (4), we observe that the condition is satisfied by w which is C^2. The result follows. ◻

For a weak metric h in the S-class of g with $h = e^{2u}g$, we write

$$F_h := F_g - \mathrm{d}(\mathrm{d}u \circ I) .$$

Observe that this notation is coherent for C^2-metrics by Proposition 2.9. and moreover, F_h does not depend on the choice of the C^2-metric g such that h belongs to the $\mathcal{S}$-class of g. Finally, one observes that F_h belongs to $L^\infty(\mathbf{A})$.

6.2.2. *The Liouville action.*

Definition 6.9. Let g and h be two metrics in $\mathcal{M}(\mathbf{A})$ of the same $\mathcal{S}$ class. Write as before $h = e^{2u}g$. The Liouville action is then

$$\mathcal{S}(g,h) = -\frac{1}{2}\int_{\mathbf{A}} uF_g + \frac{1}{4}\int_{\mathbf{A}} u\ \mathrm{d}(\mathrm{d}u \circ \mathrm{I}) .$$

Observe that since $F_g = K(g)\mathrm{d}a_g$ and $\mathrm{d}(\mathrm{d}u \circ \mathrm{I}) = \square_g u\ \mathrm{d}a_g$, our assumptions on the $\mathcal{S}$-class imply that both integrals are defined.

Proposition 6.10. *Let g, h and k be metrics in $\mathcal{M}_b(\mathbf{A})$ in the same $\mathcal{S}$-class, write $h = e^{2u}g$ and let φ be a C^2 split diffeomorphism of $\mathbf{A}$. Then*

(1) *The monotonicity formula holds*

$$\mathcal{S}(g,h) = -\frac{1}{4}\int_{\mathbf{A}} u(F_g + F_h) .$$

(2) *The metrics φ^*g and φ^*h are in the same $\mathcal{S}$-class and*

$$\mathcal{S}(\varphi^*g, \varphi^*h) = \mathcal{S}(g,h) .$$

(3) *Chasles relation holds*

$$\mathcal{S}(g,h) + \mathcal{S}(h,k) = \mathcal{S}(g,k) .$$

Lemma 6.11. *Given any two embedded curves P_0 and P_1 then*

$$\left|\mathrm{VB}(f, P_0) - \mathrm{VB}(f, P_1)\right| \leqslant \tfrac{1}{2}\int_{\mathbf{A}} |\square_g(f)|\ \mathrm{d}a_g .$$

Proof. Let (α_i) and (β_j) be sequences representing P_0 and P_1 respectively, such that α_i and β_i are on the same vertical segment. In other words, $(\alpha_{2i-1}, \alpha_{2i}, \beta_{2i-1}, \beta_{2i})$ are the vertices of a diamond Δ. Observe now that

$$\int_\Delta |\square_g(f)|\, \mathrm{d}a_g \geqslant \left|\int_\Delta \mathrm{d}(\mathrm{d}f \circ I)\right| = 2\left|f(\beta_{2i-1}) - f(\beta_{2i}) + f(\alpha_{2i}) - f(\alpha_{2i-1})\right| .$$

Thus

$$|f(\beta_{2i-1}) - f(\beta_{2i})| \leqslant |f(\alpha_{2i-1}) - f(\alpha_{2i})| + \tfrac{1}{2}\int_\Delta |\square_g(f)|\, \mathrm{d}a_g .$$

The result follows. □

This lemma has a useful corollary

Corollary 6.12. *Let u be a C^2 function in the $\mathcal{S}$-class of g. Then there is a constant K_u such that for any polynomial curve P.*

$$\mathrm{VB}(u, P) \leqslant K_u .$$

Proof. Since u is C^2 then for any polygonal curve P_0 such that $\mathrm{VB}(u, P_0)$ is finite. Moreover, by Lemma 6.11 we have that for any polygonal curve P

$$\mathrm{VB}(u, P_0) \leqslant \mathrm{VB}(u, P) + \tfrac{1}{2}\|\square_g(u)\|_1 .$$

The result follows. □

Proof of Proposition 6.10. Item (1) follows from Proposition 2.9 since $\mathrm{d}(\mathrm{d}u \circ I) = F_h - F_g$. Item (2) follows from the change of variable formula.

For item (3), write $h = e^{2u}g$ and $k = e^{2v}h$. Then by item (1) we have

$$\begin{aligned}
\mathcal{S}(g,h) + \mathcal{S}(h,k) - \mathcal{S}(g,k) &= -\frac{1}{4}\int_{\mathbf{A}} \Big(u(F_g + F_h) + v(F_h + F_k) - (u+v)(F_g + F_k)\Big) \\
&= -\frac{1}{4}\int_{\mathbf{A}} \Big(u(F_h - F_k) - v(F_g - F_h)\Big) \\
&= -\frac{1}{4}\int_{\mathbf{A}} (u\, \mathrm{d}(\mathrm{d}v \circ \mathrm{I}) - v\, \mathrm{d}(\mathrm{d}u \circ \mathrm{I})) .
\end{aligned}$$

Recall that $\mathbf{A} = \mathbf{P}(V) \times \mathbf{P}(V) \setminus \Delta$ and parametrize $\mathbf{P}(V)$ by the circle $\mathbb{R}/\mathbb{Z}$.

Let $\{C_N\}_{N\in\mathbb{N}}$ be a sequence of polygonal curves in $\mathbf{A}$ that converges uniformly to Δ, and denote by $V_N = \{\alpha_1^N, ..., \alpha_{2k_N}^N\}$ the (cyclically oriented) vertices of C_N. By construction

$$\lim_{N\to\infty} \sup_{x \in V_N}(|u(x)|, |v(x|) = 0 .$$

Let $\mathbf{A}_N$ be the bounded connected component of $\mathbf{A}\setminus C_N$. Since $\mathrm{d}u \wedge \mathrm{d}v \circ \mathrm{I} = \mathrm{d}v \wedge \mathrm{d}u \circ \mathrm{I}$, applying Stokes to $\mathbf{A}_N$ gives

$$\mathcal{S}(g,h) + \mathcal{S}(h,k) - \mathcal{S}(g,k) = -\frac{1}{4}\lim_{N\to\infty}\int_{C_N} (u\mathrm{d}v \circ \mathrm{I} - v\mathrm{d}u \circ \mathrm{I}) .$$

Then denoting c_i^N the lightlike arc joining α_{2i}^N to α_{2i+1}^N and d_i^N the lightlike arc joining α_{2i+1}^N to α_{2i+2}^N, we have

$$\begin{aligned}\left|\int_{C_N}(u\mathrm{d}v\circ \mathrm{I}-v\mathrm{d}u\circ \mathrm{I})\right| &\leqslant \left|\sum_{i=1}^{k_N}\left(\int_{c_i^N}(u\mathrm{d}v+v\mathrm{d}u)-\int_{d_i^N}(u\mathrm{d}v+v\mathrm{d}u)\right)\right|\\ &= 2\left|\sum_{i=1}^{N}u(\alpha_{2i-1}^N)v(\alpha_{2i-1}^N)-u(\alpha_{2i}^N)v(\alpha_{2i}^N)\right|\\ &\leqslant \left(\sum_{i=1}^{k_N}|u(\alpha_{2i-1}^N)-u(\alpha_{2i}^N)|\right)\sup_{i\in\{1,N\}}(|v(\alpha_i^N)|)\\ &+\left(\sum_{i=1}^{N}|v(\alpha_{2i-1}^N)-v(\alpha_{2k_N}^N)|\right)\sup_{i\in\{1,2k_N\}}(|u(\alpha_i^N)|)\\ &\leqslant (\mathrm{VB}(u,c_N)+\mathrm{VB}(v,c_N))\sup_{x\in V_N}(|u(x)|,|v(x|).\end{aligned}$$

The result now follows from Corollary 6.12. ☐

6.3. **The Liouville action between two uniformizing metrics.** A *uniformization* of the conformal annulus $\mathbf{A}$ is a C^3 split diffeomorphism between $\mathbf{A}$ and $\mathbf{dS}_+^{1,1}$. The *uniformizing metric* is then the pullback of the de Sitter metric by uniformization. In particular, any uniformizing metric is of the form $\Phi^* g_0$ where g_0 is the "standard" de Sitter metric and Φ is a C^2 conformal diffeomorphism of $\mathbf{A}$. Here we prove the following.

Proposition 6.13. *Any uniformizing metric h on $\mathbf{A}$ is in $\mathcal{S}_0$ and $\mathcal{S}(h,g_0)=0$.*

Proof. Let us first prove that h is in the $\mathcal{S}$-class of g_0. We have $h=\Phi^* g_0$ for a C^2 conformal diffeomorphism of $\mathbf{A}$. By Proposition 2.13, Φ is given in the splitting $\mathbf{A}=\mathbf{RP}^1\times\mathbf{RP}^1\setminus\Delta$ by $\Phi(x,y)=(\varphi(x),\varphi(y))$ where φ is an orientation preserving C^2-diffeomorphism of $\mathbf{RP}^1$. In particular, identifying $\mathbf{RP}^1$ with $\mathbb{R}\cup\{\infty\}$ we have

$$h=\Phi^*\left(2\frac{\mathrm{d}x\,\mathrm{d}y}{(x-y)^2}\right)=2\frac{\varphi'(x)\varphi'(y)}{(\varphi(x)-\varphi(y))^2}\mathrm{d}x\,\mathrm{d}y\,.$$

Thus, we can write $h=e^{2u}g_0$ for

$$u=\frac{1}{2}\log\left(\frac{(x-y)^2\varphi'(x)\varphi'(y)}{(\varphi(x)-\varphi(y))^2}\right).$$

Lemma 6.14. *Using the notation above, we have the following estimates*

$$u(x,y)=\frac{1}{3}(x-y)^2S_\varphi(x)+o(x-y)^2,$$

where S_φ is the Schwarzian derivative of φ.

Proof. Let us prove the second item first. For $y = x + \varepsilon$, the Taylor expansion of φ and φ' at x gives

$$(x-y)^2\varphi'(x)\varphi'(y) = \varepsilon^2\varphi'(x)^2\left(1 + \varepsilon\frac{\varphi''(x)}{\varphi'(x)} + \varepsilon^2\frac{\varphi'''(x)}{2\varphi'(x)} + o(\varepsilon^2)\right),$$

and similarly

$$\begin{aligned}(\varphi(x)-\varphi(y))^2 &= \varepsilon^2\varphi'(x)^2\left(1 + \varepsilon\frac{\varphi''(x)}{2\varphi'(x)} + \varepsilon^2\frac{\varphi'''(x)}{6\varphi'(x)} + o(\varepsilon^2)\right)^2 \\ &= \varepsilon^2\varphi'(x)^2\left(1 + \varepsilon\frac{\varphi''(x)}{\varphi'(x)} + \varepsilon^2\left(\frac{\varphi''(x)^2}{4\varphi'(x)^2} + \frac{\varphi'''(x)}{3\varphi'(x)}\right) + o(\varepsilon^2)\right),\end{aligned}$$

that is

$$\frac{\varepsilon^2\varphi'(x)^2}{(\varphi(x)-\varphi(y))^2} = 1 - \varepsilon\frac{\varphi''(x)}{\varphi'(x)} + \varepsilon^2\left(\frac{3}{4}\frac{\varphi''(x)^2}{\varphi'(x)^2} - \frac{1}{3}\frac{\varphi''(x)}{\varphi'(x)}\right) + o(\varepsilon^2).$$

This finally gives

$$\begin{aligned}\frac{(x-y)^2\varphi'(x)\varphi'(y)}{(\varphi(x)-\varphi(y))^2} &= 1 + \varepsilon^2\left(\frac{3\varphi''(x)^2}{4\varphi'(x)^2} - \frac{\varphi'''(x)}{3\varphi'(x)} - \frac{\varphi''(x)^2}{\varphi'(x)^2} + \frac{\varphi'''(x)}{2\varphi'(x)}\right) + o(\varepsilon^2) \\ &= 1 + \frac{1}{6}\varepsilon^2\left(\frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\frac{\varphi''(x)^2}{\varphi'(x)^2}\right) + o(\varepsilon^2).\end{aligned}$$

Thus, using

$$S_\varphi(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\frac{\varphi''(x)^2}{\varphi'(x)^2},$$

we obtain that

$$u = \frac{1}{3}\varepsilon^2 S_\varphi(x) + o(\varepsilon^2),$$

and the result follows. □

The above lemma directly implies that u tends to 0 uniformly on $\partial\mathbf{A}$ and thus u belongs to $L^\infty(\mathbf{A})$.

Observe that $K(g_0) = K(h) = 1$, so by Proposition 2.9. Thus $\square_{g_0} u = e^{2u} - 1$ is then in$L^\infty(\mathbf{A})$. Moreover $\square_{g_0} u$ which is in $\mathsf{L}^1(\mathbf{A}, \mathrm{d}a_{g_0})$ by Lemma 6.14. This completes the proof that h is in $\mathcal{S}_0$.

To prove that $\mathcal{S}(h, g_0) = 0$, observe that the split invariance implies that for a 1-parameter subgroup $\{\Phi_t\}_{t\in\mathbb{R}}$ of split diffeomorphisms, we have

$$\mathcal{S}(\Phi^*_{t+s}g_0, \Phi^*_s g_0) = \mathcal{S}(\Phi^*_t g_0, g_0).$$

Hence, it suffices to prove that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathcal{S}(\Phi^*_t g_0, g_0) = 0,$$

for any 1-parameter subgroup. Let u_t be such that $\Phi_t^* g_0 = e^{2u_t} g_0$ and

$$\alpha := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} u_t \ .$$

Let ξ be the vector field that generates ϕ_t, then we get

$$\alpha(x,y) = \frac{1}{2}\left(\xi'(x) + \xi'(y)\right) - \left(\frac{\xi(x)-\xi(y)}{x-y}\right) = O((x-y)^2) \ ,$$

since ξ is C^3. It follows that α is in $L^1(\mathrm{d}a_{g_0})$. In particular, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{S}(\Phi_t^* g_0, g_0) = \int_{\mathbf{A}} \alpha \ \mathrm{d}a_{g_0} \ . \tag{6.1}$$

By Proposition 2.9, and since g_0 and $\Phi_t^* g_0$ have constant curvature κ

$$\mathrm{d}(\mathrm{d}u_t \circ \mathrm{I}) = F_{g_0} - F_{\Phi_t^* g_0} = \kappa(\mathrm{d}a_{g_0} - \mathrm{d}a_{\Phi_t^* g_0}) = \kappa(1 - e^{2u_t})\mathrm{d}a_{g_0} \ .$$

By Lemma 6.14, for every t, the function u_t extends smoothly to $\mathbf{Ein}^{1,1}$ to a C^2-function vanishing on $\mathbf{A}$. Thus, we have

$$\int_{\mathbf{A}} \mathrm{d}\left(\mathrm{d}u_t \circ \mathrm{I}\right) = 0 \ . \tag{6.2}$$

It follows that

$$\int_{\mathbf{A}} (1 - e^{2u_t}) \ \mathrm{d}a_{g_0} = 0.$$

Taking the derivative at $t = 0$ and using equation (6.2) yields

$$\int_{\mathbf{A}} \alpha \ \mathrm{d}a_{g_0} = 0 \ ,$$

and the result follows by equation (6.1). □

7. Positive curves

7.1. **Crossratio.** Denote by $(\mathbf{RP}^1)^{(4)}$ the space of 4-tuple of pairwise distinct points in $\mathbf{RP}^1$.

Definition 7.1. A *crossratio* is a continuous function b from $(\mathbf{RP}^1)^{(4)}$ to $\mathbb{R}$ that satisfies the cocycle relations

$$\mathrm{b}(x,w,X,Y)\mathrm{b}(w,y,X,Y) = \mathrm{b}(x,y,X,Y) \ , \tag{7.1}$$

$$\mathrm{b}(x,y,W,Y)\mathrm{b}(x,y,X,W) = \mathrm{b}(x,y,X,Y) \ . \tag{7.2}$$

We say that a crossratio is *positive* if it satisfies furthermore $\mathrm{b}(x,y,X,Y) > 1$ for any cyclically oriented 4-tuple (x,y,X,Y) in $(\mathbf{RP}^1)^{(4)}$.

Smooth positive crossratios are related to smooth metrics as follows. The *diamond* defined by a cyclically oriented 4-tuple (x, y, X, Y) in $(\mathbf{RP}^1)^{(4)}$ is

$$\delta(x, y, X, Y) := \{(u, v) \in [x, y] \times [X, Y]\} \subset \mathbf{A} ,$$

where we recall that $\mathbf{A} = (\mathbf{RP}^1 \times \mathbf{RP}^1) \setminus \Delta$.

Given a positive crossratio b, define the b-area of a diamond by

$$\mathrm{Area}_{\mathrm{b}} (\delta(x, y, X, Y)) = \log(\mathrm{b}(x, y, X, Y)) .$$

One easily checks that the cocycle relations (7.1) and (7.2) are equivalent to the additivity of $\mathrm{Area}_{\mathrm{b}}$ (see Figure 2). Since a smooth metric is characterized by its area form, any smooth positive crossratio b defines a unique *crossratio metric* g_{b}.

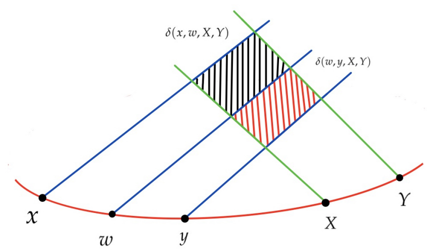

FIGURE 2. Additivity of the crossratio area

7.2. **Positive curves in flag varieties.** An important class of positive crossratios is obtained by considering positive curves in (self dual) flags varieties (see [1]). Here, rather than providing an abstract definition of the theory, we highlight its key aspects and illustrate them with examples. The interested reader can read [12].

Let G be a semi-simple real Lie group with finite center and $\mathbf{F}$ a flag variety of G. The action of an element $g \in \mathsf{G}$ on $\mathbf{F}$ is called *loxodromic* if there is a pair (x_-, x_+) of g-invariant transverse points in $\mathbf{F}$ such that if U is the set of transverse flags to x_-, the sequence $\{g^n\}_{n\in\mathbb{N}}$ converges to the constant map $U \to \{x_+\}$ on any compact set in U. The points x_+ and x_- are respectively called the attracting and repelling fixed point of g.

A *positive structure* on $\mathbf{F}$ is a property of triple and quadruple pairwise transverse flags that satisfy a set of axioms (see [10] for more details). The set of pairs $(\mathsf{G}, \mathbf{F})$ admitting positive structures has been classified by Guichard–Wienhard in [12] and is deeply connected with the theory of higher rank Teichmüller spaces.

As a consequence of the definition of positivity, we have for every transverse pair p and q in $\mathbf{F}$ a finite family of nontrivial open cones $C^i_p(q)$ for i in $\{1, \dots , n\}$ in $\mathsf{T}_p\mathbf{F}$ (which we identify with the Lie algebra of the unipotent radical of the stabilizer in G of q), such that for any u in $C^i_p(q)$, the triple $(p, \exp(u), q)$ is positive. We call such a triple (p, u, q) an *arrow*. By properties of positivity, if (p, u, q) is an arrow, so is $(p, \lambda u, q)$. By positivity, the stabilizer of a positive triple is compact, and so is the stabilizer of an arrow.

This notion of positivity for $(\mathsf{G}, \mathbf{F})$ comes together with:

- A notion of *positive curve*, which is a continuous map γ from $\mathbf{RP}^1$ to $\mathbf{F}$ sending triple and quadruple of cyclically oriented pairwise distinct points in $\mathbf{RP}^1$ to positive triples and quadruples of flags, respectively.
- A *conjugacy class of homomorphisms* $\mathcal{I}$ from $\mathsf{PSL}_2(\mathbb{R})$ to G, such that any semi-simple element is mapped to a loxodromic element and $\iota(\mathsf{PSL}_2(\mathbb{R}))$ has compact centralizer for any ι in $\mathcal{I}$.
- A notion of *circles* that are ι-equivariant positive curves c from $\mathbf{RP}^1$ to $\mathbf{F}$ for ι in $\mathcal{I}$.
- For any dominant weight ω and positive curve γ, a positive crossratio $\mathrm{b}_{\gamma,\omega}$ on γ see [1].

Fix $(\mathsf{G}, \mathbf{F})$ and a dominant weight ω. By uniqueness of the crossratio on $\mathbf{RP}^1$, there exists a positive integer λ such that the positive crossratio associated with a circle has the form

$$\mathrm{b}_{c,\omega}(x, y, X, Y) = [x, y, X, Y]^{\lambda} ,$$

where $[\cdot\,,\cdot\,,\cdot\,,\cdot\,]$ is the standard projective crossratio on $\mathbf{RP}^1$ normalized by

$$[0, 1, x, \infty] = x .$$

So by construction, we can associate a metric $g_{\gamma,\omega}$ with any smooth positive curve γ in $\mathbf{F}$. When γ is a circle, the metric is equal to λg_0.

We define a *super-positive curve* to be a C^1-positive curve f so that for any two distinct points p and q in $\mathbf{RP}^1$, for any u in $\mathsf{T}_p\mathbf{RP}^1$, then $(f(p), \mathsf{T}_p f(u), f(q))$ is an arrow. We leave it to the reader to verify that the requirement that f be positive is, in fact, redundant and that there exist C^1-positive curves that are not super-positive.

Definition 7.2 (Liouville action for positive curves)**.** Let $(\mathsf{G}, \mathbf{F})$, ω and λ be as above, and γ a smooth positive curve in $\mathbf{F}$ such that $g_{\gamma,\omega}$ is in the $\mathcal{S}$-class of λg_0. The *Liouville action of* γ is

$$\mathcal{S}(\gamma) \coloneqq \mathcal{S}(g_{\gamma,\omega}, \lambda g_0) .$$

Observe that the Liouville action of curves is invariant under reparametrization. Indeed, if γ is a smooth positive curve such that $g_{\gamma,\omega}$ is in the $\mathcal{S}$-class of λg_0, and φ is a C^3 orientation preserving diffeomorphism of $\mathbf{RP}^1$, then

$$g_{\delta,\omega} = \Phi^* g_{\gamma,\omega}$$

for $\delta = \gamma \circ \varphi$ and $\Phi(x, y) = (\varphi(x), \varphi(y))$. In particular, by Chasles relation, we have

$$\mathcal{S}(g_{\delta,\omega}, \lambda g_0) = \mathcal{S}(\Phi^* g_{\gamma,\omega}, \lambda \Phi^* g_0) + \mathcal{S}(\Phi^* g_0, \lambda g_0) = \mathcal{S}(g_{\gamma,\omega}, \lambda g_0)$$

where we used the split invariance and Proposition 6.13. Similarly, the Liouville action is invariant under the left action of G.

7.3. **Examples.** We now illustrate the notion of positivity on three different examples.

7.3.1. *The case* $(\mathsf{PSL}_2(\mathbb{R}), \mathbf{RP}^1)$. Consider the pair $(\mathsf{G}, \mathbf{F}) = (\mathsf{PSL}_2(\mathbb{R}), \mathbf{RP}^1)$. In this setting, a positive triple is a triple of pairwise distinct points in $\mathbf{RP}^1$ while a positive quadruple is a cyclically oriented 4-tuple of pairwise distinct points.

A positive curve is then an orientation-preserving homeomorphism, a super-positive one is a C^1-diffeomorphism, and a circle is just a Möbius map.

In this situation, the dominant weight is unique and gives the standard cross-ratio. By invariance of the Liouville action under reparametrization, our invariant is always zero.

7.3.2. *The case* $(\mathsf{PO}(2,2), \mathbf{Ein}^{1,1})$. Consider now $(\mathsf{G}, \mathbf{F}) = (\mathsf{PO}(2,2), \mathbf{Ein}^{1,1})$. A positive triple in $\mathbf{Ein}^{1,1}$ is a triple of points that spans a linear space of signature $(2,1)$. A positive quadruple is a 4-tuple of points (a, b, c, d) such that any subtriple is positive and whose projection on the first factor in $\mathbf{Ein}^{1,1} = \mathbf{RP}^1 \times \mathbf{RP}^1$ is cyclically oriented.

Then a positive curve is of the form

$$\gamma : \begin{cases} \mathbf{RP}^1 & \rightarrow & \mathbf{RP}^1 \times \mathbf{RP}^1 \ , \\ t & \mapsto & (\psi(t), \phi(t)) \ , \end{cases}$$

where φ and ψ are two orientation preserving homeomorphisms. It is super-positive if and only if ψ and ϕ are furthermore C^1-diffeomorphism.

The group homomorphism ι is then the composition of the isomorphism between $\mathsf{PSL}_2(\mathbb{R})$ and $\mathsf{SO}_0(2,1)$ with the reducible embedding into $\mathsf{PO}(2,2)$. In particular, γ is a circle when both ϕ and ψ are Möbius.

Let us now consider the Liouville action. By invariance under reparametrization, it is enough to consider the case $\psi = \mathrm{Id}$. There is a dominant weight ω such that the associated positive crossratio is

$$\mathrm{b}_{\gamma,\omega}(x, y, X, Y) := [x, y, X, Y] \cdot [\varphi(x), \varphi(y), \varphi(X), \varphi(Y)] \ .$$

When φ is C^3, the crossratio metric is equal to $g_0 + \Phi^* g_0$ for $\Phi(x, y) = (\varphi(x), \varphi(y))$ and so is in the $\mathcal{S}$-class of $2g_0$ by Lemma 6.14.

7.3.3. *The case* $(\mathsf{PSL}_3(\mathbb{R}), \mathbf{F}(\mathbb{R}^3))$. Let $\mathbf{F}(\mathbb{R}^3)$ be the space of full flags in $\mathbb{R}^3$ that we see as the set of pointed projective lines in $\mathbf{RP}^2$. A triple (f_1, f_2, f_3) is positive if the points (x_1, x_2, x_3) form a triangle inscribed in a triangle with edges (ℓ_1, ℓ_2, ℓ_3), where $f_i = (x_i, \ell_i)$. Similarly, a quadruple (f_1, f_2, f_3, f_4) is positive if the points (x_1, x_2, x_3, x_4) are the cyclically ordered vertices of a quadrilateral inscribed in a quadrilateral with edges $(\ell_1, \ell_2, \ell_3, \ell_4)$. See Figure 3.

A positive curve in this setting is then a continuous map

$$\gamma : \begin{cases} \mathbf{RP}^1 & \rightarrow & \mathbf{F}(\mathbb{R}^3) \ , \\ t & \mapsto & (x(t), \ell(t)) \ , \end{cases}$$

where the curve defined by x bounds a strictly convex set C in $\mathbf{RP}^2$ and for any t the line $\ell(t)$ is a support line of C (see Figure 3).

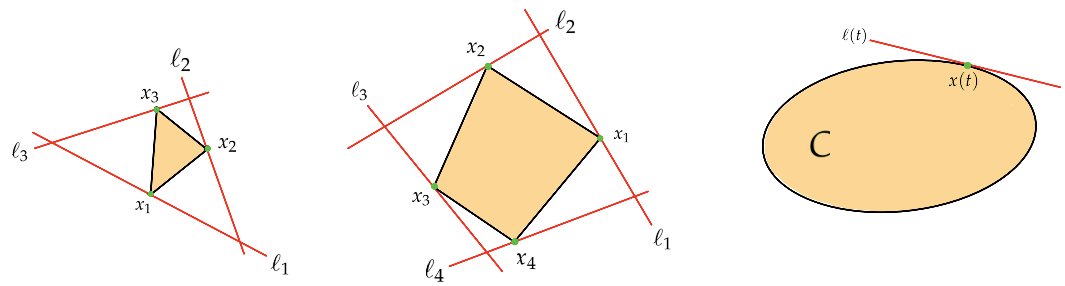

FIGURE 3. From left to right: positive triple, quadruple, and curve

In this case, the group ι is the composition of the isomorphism between $\mathsf{PSL}_2(\mathbb{R})$ and $\mathsf{SO}_0(2,1)$ with the embedding of $\mathsf{SO}_0(2,1)$ into $\mathsf{PSL}(2,\mathbb{R})$. In particular, a circle in $\mathbf{F}(\mathbb{R}^3)$ is the lift of a quadric in $\mathbf{RP}^2$ parametrized projectively.

For the crossratio, there is a dominant weight ω such that

$$\mathrm{b}_{\gamma,\omega}(t_1,t_2,s_1,s_2) = \frac{\langle \ell(s_1)|x(t_1)\rangle\langle \ell(s_2)|x(t_2)\rangle}{\langle \ell(s_1)|x(t_2)\rangle\langle \ell(s_2)|x(t_1)\rangle} ,$$

where to define $\langle \ell|x\rangle$ we choose a non-zero linear form on $\mathbb{R}^3$ with kernel ℓ and apply it to a chosen non-zero vector in x (the above quotient is indeed independent on the choices). In particular, circles have Fuchsian crossratio of weight 2.

7.4. **Piecewise circles and finiteness.** Given a conjugacy class of $\mathsf{SL}_2(\mathbb{R})$ in G, a *circle* (with respect to this class) is a closed orbit of an element of that conjugacy class $\mathsf{PSL}_2(\mathbb{R})$ in G isomorphic to $\mathbf{RP}^1$. This defines a family of circles on which G acts transitively. A *family of super-positive circles* is a family of positive curves such that

(1) every circle in the family is super-positive,
(2) there is a unique circle in the family passing through a given arrow.

Observe that there exist positive circles which are not super-positive: an example comes from the projection of the Veronese embedding (corresponding to the irreducible $\mathsf{SL}_2(\mathbb{R})$ in $\mathsf{SO}(2,3)$) in the Einstein universe as in [5, Section 5.3]. However, one can check that the examples in the previous paragraph are super-positive. Moreover, any positive circle invariant by the Θ-positive subalgebra introduced in [12, Section 7] is super-positive.

Definition 7.3. Let $(\mathsf{G},\mathbf{F})$ be a positive flag variety. A *piecewise circle* is a C^1 positive curve γ from $\mathbf{RP}^1$ to $\mathbf{F}$ that is piecewisely a super-positive circle map. The end of the intervals on which γ is a circle map is called a *turning point*.

Theorem 7.4. *Let* $(\mathsf{G}, \mathbf{F})$ *be a positive flag variety and* ω *a dominant weight such that circles have Fuchsian crossratio of weight* λ*. Then for any piecewise circle* γ*, the crossratio metric* $g_{\gamma,\omega}$ *is in the* $\mathcal{S}$*-class of* λg_0*. In particular,* $\mathcal{S}(\gamma)$ *is finite.*

7.4.1. *Preliminary on super-positive circles.* Let α be the parameterization by $\mathbf{RP}^1$ of a super-positive circle C_0 with $\alpha(0) = p$ and $\alpha(\infty) = q$. Let ι be the embedding of $\mathsf{PSL}_2(\mathbb{R})$ in G associated with α and $H := \iota(h)$, where h is a diagonal element in $\mathsf{PSL}_2(\mathbb{R})$ with attracting fixed point 0 in $\mathbf{RP}^1$. Recall that ι has a compact centralizer denoted by K. Thus, if $\mathbf{C}$ denotes the space of (unparametrized) circles in $\mathbf{F}$ of the same type as C_0, we have

$$\mathsf{C} = \mathsf{G}/\left(\iota(\mathsf{PSL}_2(\mathbb{R})) \times \mathsf{K}\right) .$$

Set $\ell = \mathsf{T}_p C_0$ and define

$$\mathbf{C}(p, \ell) = \{D \in \mathbf{C} \mid p \in D \text{ and } \mathsf{T}_p D = \ell\} .$$

We will need several lemmas in the sequel

Lemma 7.5 (Compactness)**.** *Let* (p_0, u_0, q_0) *be an arrow. Then there is a compact neighborhood* K *of* q_0 *such that*

- *for any* q *in* K*,* (p_0, u_0, q) *is an arrow.*
- *the set of circles in* $\mathbf{C}(p, \ell)$ *that intersects* K *is compact.*

Proof. Observe that being positive for triples is open. Thus, for K a small enough neighborhood of q, the first item holds. Let as before G_1 be the stabilizer in G of (p, u). Observe that G_1 acts algebraically in the algebraic variety $\mathbf{F}$. By a corollary of Rosenlicht (Theorem [9, Corollary 2.2.a]), the orbits of G_1 are embedded. In particular, $K \cap \mathsf{G}_1 \cdot q$ is compact for K small enough. Since $\mathbf{C}(p, \ell) = \mathsf{G}_1/\mathsf{K}_1$, for some compact subgroup K_1, the second item is satisfied. □

Lemma 7.6 (Contraction)**.** *The action of* H *on* $\mathbf{C}$ *preserves* $\mathbf{C}(p, \ell)$*. Moreover, the action of* H^{-1} *on* $\mathbf{C}(p, \ell)$ *has the circle* C_0 *as an attracting fixed point and more precisely*

$$\|\mathsf{T}_{C_0} H^{-1}\| < 1 .$$

Proof. The first statement follows from the definition of $\mathbf{C}(p, \ell)$. The element H^{-1} is loxodromic with an attracting fixed point q in C_0 and repelling fixed point p. Since $\mathbf{F}$ is equal to its opposite, the attraction basin of q is the set of elements of $\mathbf{F}$ that are transverse to p.

Let C be an element of $\mathbf{C}(p, \ell)$. Since C is positive, given x in $C \setminus p$, x is transverse to p.

It follows that for any such x, $\{H^{-n}x\}_{n\in\mathbb{N}}$ converges to q. Then by Lemma 7.5, we have that $\{H^{-n}D\}_{n\in\mathbb{N}}$ converges to a circle in $\mathbf{C}(p, \ell)$ passing through q and p and tangent to ℓ, which thus must be equal to C_0 by the second item in the definition of super-positive circles.

Let G_1 be the subgroup of G preserving (p, ℓ), L_0 the centralizer in G of H, L_1 the centralizer in G_1 of H. Then (p, ℓ) is fixed by L_1 since H has a unique fixed point in $\mathbf{C}(p, \ell)$ near (p, ℓ) by the previous discussion.

Observe now that G_1 acts transitively on $\mathbf{C}(p,\ell)$ and therefore is a G_1-space. Let H_0 be the stabilizer of C_0. Then let $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{h}_0$ be the Lie algebras of G, G_1, and H_0 respectively. Recall that $\mathrm{Ad}(H)$ is real diagonalizable as an endomorphism of $\mathfrak{g}$. Since $\mathrm{Ad}(H)$ preserves both $\mathfrak{g}_1$ and $\mathfrak{h}_0$, it follows that we can write $\mathfrak{g}_1 = \mathfrak{h}_0 \oplus V$, where V is stable by $\mathrm{Ad}(H)$. By the first paragraph, all the eigenvalues of $\mathrm{Ad}(H)$ on V are no greater than 1. Moreover, by the previous paragraph, $\mathrm{Ad}(H)$ does not have 1 as an eigenvalue on V. It follows that all the eigenvalues of $\mathrm{Ad}(H)$ on V are less than 1 and thus $\mathrm{Ad}(H)$ is a contracting endomorphism on $\mathbf{C}(p,\ell)$. □

We observe that as a corollary, $\mathbf{C}(p,\ell)$ is contractible. We will also need

Lemma 7.7. *There is an H-invariant submanifold Σ containing q and transverse to C_0.*

Proof. We choose a representation ρ of G in $\mathsf{SL}_N(R)$, such that there exists an equivariant embedding of $\mathbf{F}$ in $\mathbf{P}(\mathbb{R}^N)$.

Let ι be the corresponding representation of $\mathsf{SL}_2(\mathbb{R})$ in $\mathsf{SL}_N(R)$, let f be a ι-equivariant map from $\mathbf{RP}^1$ to $\mathsf{SL}_N(R)$, h be a generator of the diagonal group in $\mathsf{SL}_2(\mathbb{R})$ and $H := \iota(h)$.

Since H is $\mathbb{R}$-split, there exists a hyperplane invariant by H that does not contain $q = f(\infty)$. Sending this hyperplane to ∞, we have a linear chart of $\mathsf{SL}_N(R)$ in which H is linear and $\mathbb{R}$-split. Then, since the tangent line ℓ_0 to $f(\mathbf{RP}^1)$ at ∞ is H-invariant, there exists a hyperplane V through 0, which is transverse to ℓ and H-invariant. Then, the foliation $\mathcal{F}$ by affine hyperplanes parallel to V satisfies the required properties. This concludes the proof. □

7.4.2. *The conformal factor.* Let $\gamma : \mathbf{RP}^1 \to \mathbf{F}$ be a piecewise circle and write $g_\gamma = e^{2u}(\lambda g_0)$ where λ is the weight of a circle and g_0 is the de Sitter metric on $\mathbf{RP}^1 \times \mathbf{RP}^1 \setminus \Delta$. We now prove several lemmas.

Lemma 7.8. *The function u satisfies the following properties*

1. *The function u is C^0,*
2. *there are finitely many horizontal and vertical lines outside of which u is C^∞,*
3. *u is zero on a neighborhood of $\partial\mathbf{A} \setminus W$, where W is the set of turning points,*
4. *the distribution $\square_g u$ is locally bounded.*

Proof. This is a consequence of the fact that γ is C^1 and smooth outside the turning points. For the last statement, let z be a point in $\mathbf{A}$ and g_{flat} a flat $(1,1)$ metric conformal to g_0 in the neighborhood of z. Since u is piecewise C^2, it follows that $\square_{g_{flat}} u$ is bounded in the neighborhood of z. Since

$$(\square_{g_0} u)\omega_{g_0} = (\square_{g_{flat}} u)\omega_{g_{flat}} ,$$

the same can be said about $\square_g u$. □

Corollary 7.9. *There is a polygonal curve P on which $\mathrm{VB}(u,P)$ is finite.*

Proof. We choose a polygonal curve that is transverse to the horizontal lines and vertical lines on which u ceases to be C^∞. Then u restricted to P is continuous and piecewise C^1. It follows that u is of bounded variation on P and the result follows. □

7.4.3. *The function u on the neighborhood of a turning point.* Thanks to Corollary 7.9 and Lemma 7.8, the theorem reduces to the following lemma.

Lemma 7.10. *We have*

(1) *The functions u and $\square_g(u)$ tend to 0 uniformly on $\partial\mathbf{A}$.*
(2) *The distribution $\square_g(u)$ belongs to $L^\infty(\mathbf{A}, \mathrm{d}a_g)$.*

In fact, it is enough to control the behavior of u and $\square_g u$ in the neighborhood of a turning point w. Since γ is a piecewise C^1-circle, we can find a subdivision of $\mathbf{RP}^1$ into finitely many intervals

$$\mathbf{RP}^1 = \bigcup_{i=0,\ldots,n} I_i \quad , \quad I_i = [a_i, a_{i+1}],$$

such that $\gamma_i := \gamma|_{I_i}$ is a circle parameterization and we have

$$\gamma_i(a_{i+1}) = \gamma_{i+1}(a_{i+1}) \ \ , \ \ \dot{\gamma}_i(a_{i+1}) = \dot{\gamma}_{i+1}(a_{i+1}) \ .$$

Since

$$K := \bigcup_{|i-j|\geqslant 2} I_i \times I_j \ ,$$

is compact in $\mathbf{A}$, by Lemma 7.8, $\square_g u$ belongs to $L^1(K, \mathrm{d}a_g)$ and $L^\infty(K, \mathrm{d}a_g)$. Moreover u is zero on $I_i \times I_i \setminus \Delta$. Thus, the lemma follows from

Lemma 7.11. *We have*

(1) *The function u tends to 0 uniformly when converging to a_i,*
(2) *The distribution $\square_g(u)$ belongs to $L^\infty(\mathbf{A}_i, \mathrm{d}a_g)$ and $L^1(\mathbf{A}_i, \mathrm{d}a_g)$, where $\mathbf{A}_i = I_i \times I_{i+1}$.*

Proof. Since the ordering is meaningless, we can focus on $i = 0$. After some Möbius change of a parametrization of $\mathbf{RP}^1$ we further restrict to the following situation:

(1) let γ_0 be the restriction on an interval $[-c, 0]$, with c positive of the parametrization α_0 of a circle C_0,
(2) let then $q := \alpha_0(\infty)$, h be a diagonal element in $\mathsf{SL}_2(\mathbb{R})$, and $H := \iota_0(h)$ where ι_0 is the embedding of $\mathsf{SL}_2(\mathbb{R})$ associated with α_0,
(3) let Σ be the hypersurface of $\mathbf{F}$, transverse to α_0 and passing through q invariant by H (provided by Lemma 7.7).

Let $\mathcal{O}$ be the open neighborhood of C_0 in $\mathbf{C}(p, \ell)$ of circles that intersect Σ. We saw in Lemma 7.6 that H^{-1} acts on $\mathcal{O}$ and furthermore that C_0 is an attracting point of H^{-1} with

$$\|\mathsf{T}_{C_0} H^{-1}\| \leqslant 1 \ .$$

First step: a function on $\mathcal{O}$

We now parameterize each of the circle C in $\mathcal{O}$ by α_C uniquely defined by

$$\alpha_C(0) = p\ ,\ \dot{\alpha}_C(0) = \dot{\alpha}_0(0)\ ,\ \alpha_C(\infty) \in \Sigma\ .$$

From the uniqueness of the parametrization we have that

$$\alpha_{H^{-1}C} = H^{-1} \circ \alpha_C \circ h\ . \tag{7.3}$$

For each C, we then have a C^1 (piecewise C^2) metric g_C on $[-c, O] \times [0, \infty] \setminus \{(0,0)\}$ and a C^1 (piecewise C^2) function u_C defined by

$$g_C = e^{2u_C} g_0\ .$$

It then follows from equation (7.3), that

$$u_{H^{-1}C} = u_C \circ h\ . \tag{7.4}$$

A final step in our construction is the choice once and for all x and w points in $\mathbf{RP}^1$ with x in $]-c, 0[$ and w in $]0, \infty[$. This allows us to define the L-shape

$$L_k := ([h^k(x), h^{k+1}(x)] \times [0, h^k(z)]) \cup ([h^k(x), 0] \times [h^{k+1}(w), h^{k+1}(w))\ ,$$

and observe that

$$L_k = h(L_0)\ . \tag{7.5}$$

We now define the functions U_L, V_L and W_L on $\mathcal{O}$ by

$$V_L(C) := \int_{L_0} |\,\square_{g_C} u_C|\, \omega_{g_C}\ , \tag{7.6}$$

$$U_L(C) := \max_{L_0}(|u_C|)\ ,\ W_L(C) := \max_{L_0}(|\,\square_{g_C} u_C|)\ . \tag{7.7}$$

Second step: We now prove that the functions V_L, U_L and W_L are Lipschitz on $\mathcal{O}$, and moreover,

$$V_L(H^{-k}C) = \int_{L_k} |\,\square_{g_C} u_C|\, \omega_{g_C}\ , \tag{7.8}$$

$$U_L(H^{-k}C) = \max_{L_k}(|u_C|)\ ,\ \ W_L(H^{-k}C) = \max_{L_k}(|\,\square_{g_C} u_C|)\ . \tag{7.9}$$

The fact that the functions are Lipschitz just follows from the fact that for any compact K in $[-c, 0] \times [0, \infty]$, then $C \mapsto u_C$ is a smooth function with values in $C^\infty(K, \mathbb{R})$. Let us check the final statement. By definition, setting $C_k = H^{-k}(C)$

$$V_L(C_k) = \int_{L_0} |\,\square_{g_{C_k}} u_{C_k}| \cdot \omega_{g_{C_k}} = \int_{L_0} |\mathrm{d}(\mathrm{d}u_{C_k} \circ I)| \tag{7.10}$$

$$= \int_{L_0} |\mathrm{d}(\mathrm{d}u_C \circ h^k \circ I)| = \int_{L_0} |\mathrm{d}(\mathrm{d}u_C \circ I \circ h^k)| = \int_{L_k} |\mathrm{d}(\mathrm{d}u_C \circ I)|\ . \tag{7.11}$$

Here we used equation (7.4) in the third equality and equation (7.5) in the last. The assertion (7.9) that $U_L(H^{-k}C) = \max_{L_k}(|u_C|)$ follows from a similar proof. This concludes the proof of the second step.

FINAL STEP: It follows from the previous step that $U_L(C_k)$ converges to $U_L(C_0) = 0$ when k goes to infinity. This implies that $\max_{L_k}|u|$ converges to zero when k-converges to infinity, and thus, u_C converges uniformly to zero as one approaches 0. The same holds for $\max_{L_k}|\square_{g_C} u_C|$.

Moreover,

$$\int_{A_0} |\mathrm{d}(\mathrm{d}u_C \circ I)| = \sum_{k=0}^{\infty} \int_{L_k} |\mathrm{d}(\mathrm{d}u_C \circ I)| = \sum_{k=0}^{\infty} V(H^{-k}(C)) .$$

But for C in a compact, $|V_L(C_0) - V_L(C)| \leqslant K_1 d(C_0, C)$ since V is Lipschitz. It follows that for k large enough, there exists a constant K_2 such that

$$|V(C_k)| \leqslant K_2 \lambda^k ,$$

since $V_L(C_0) = 0$ and $\|\mathsf{T}_{C_0} H^{-1}\| < \lambda < 1$. The result follows. □

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