QUASICIRCLES AND QUASIPERIODIC SURFACES IN PSEUDO-HYPERBOLIC SPACES

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Abstract. We study in this paper *quasiperiodic maximal surfaces* in pseudo-hyperbolic spaces and show that they are characterised by a curvature condition, Gromov hyperbolicity or conformal hyperbolicity. We show that the limit curves of these surfaces in the Einstein Universe admits a canonical *quasisymmetric* parametrisation, while conversely every quasisymmetric curve in the Einstein Universe bounds a quasiperiodic surface in such a way that the quasisymmetric parametrisation is a continuous extension of the uniformisation; we give applications of these results to asymptotically hyperbolic surfaces, rigidity of Anosov representations and a version of the universal Teichmüller space.

Contents

1. Introduction

We continue our study of complete maximal surfaces in pseudo-hyperbolic spaces of signature $(2, n)$ begun in [LTW20] and [CTT19].

The pseudo-hyperbolic space $\mathbf{H}^{2,n}$ is constructed in the following way: take a vector space E equipped with a quadratic form $\mathbf{q}$ of signature $(2, n + 1)$ – that is admitting a vector space of dimension 2 on which $\mathbf{q}$ is positive definite, while $\mathbf{q}$ is negative definite on the orthogonal of dimension $n + 1$. Then the pseudo-hyperbolic space of signature $(2, n)$ is

$$\mathbf{H}^{2,n} := \{u \in E \mid \mathbf{q}(u) = -1\}/ \pm \mathrm{Id} \ .$$

For $n = 0$, we obtain the Minkowski model of the hyperbolic space. The pseudo-hyperbolic space of $\mathbf{H}^{2,n}$ of signature $(2, n)$ inherits by restriction of the quadratic form $\mathbf{q}$ a pseudo-Riemannian metric of signature $(2, n)$ and constant sectional curvature -1. Let $\mathsf{G} := \mathsf{SO}_0(\mathbf{q})$ be the identity component of $\mathsf{O}(\mathbf{q})$, then G acts transitively by isometries on $\mathbf{H}^{2,n}$, turning $\mathbf{H}^{2,n}$ into a pseudo-Riemannian symmetric space.

The pseudo-hyperbolic space naturally embeds in $\mathbf{P}(E)$ and admits a *boundary at infinity* denoted $\partial_\infty \mathbf{H}^{2,n}$ and called the *Einstein Universe* which is the quadric in $\mathbf{P}(E)$ associated to $\mathbf{q}$. This quadric admits a notion of *positivity* leading to the concept of positive loops and semi-positive loops: for instance, the intersection of the quadric with the projectivisation of a space of type $(2, 1)$ is a positive loop. Positivity in this context has been studied by [BIW10, GW18, Bar15].

One of the concepts introduced and studied in the current paper is that of *quasisymmetric maps* from the projective real line to the Einstein Universe. As for the case of maps into the complex projective line, quasisymmetric maps are defined using a *cross-ratio*. These maps are positive and enjoy similar properties as their counterparts in the complex projective line: they enjoy a uniform

Hölder modulus of continuity – see Theorem 3.17 – while equivariant ones are linked to Anosov representations.

A *maximal surface* in $\mathbf{H}^{2,n}$ is a spacelike surface on which the mean curvature vector vanishes everywhere. We proved in [LTW20] that every complete maximal surface Σ has a limit curve $\partial_\infty\Sigma$ in $\partial_\infty\mathbf{H}^{2,n}$ which is a semi-positive loop and, more importantly, that every such semi-positive loop is the boundary of a unique complete maximal surface.

Among homogeneous complete maximal surfaces, we have the totally geodesic *hyperbolic planes* and *Barbot surfaces* which are flat and orbits of diagonal subgroup of G. Barbot surfaces are characterised by a rigidity theorem. By a result of Cheng [Cheng94, Theorem 2], any complete maximal surface has non positive induced curvature, while Barbot surfaces are the only flat ones. A slight and technical improvement of Cheng's result gives

Theorem 1. [Cheng Curvature rigidity] *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$. If the intrinsic curvature of Σ is zero at a point, it vanishes everywhere and Σ is a Barbot surface.*

For $n = 1$, this result was also obtained by Bonsante and Schlenker in [BS10]. This result sets the background of the main results of this article, which concern *quasiperiodic maximal surfaces*.

Let us introduce the space $\mathcal{M}(n)$ which consists of pairs (x, Σ) where Σ is a complete maximal surface and x a point in Σ. By construction every maximal surface Σ maps into $\mathcal{M}(n)$, by the map which associates (x, Σ) to x. By a compactness theorem proved in [LTW20], G acts cocompactly on $\mathcal{M}(n)$ when the latter is equipped with the topology of smooth convergence on every compact. We will show that Barbot surfaces give rise to a unique point called the *Barbot point* in $\mathcal{M}(n)/\mathsf{G}$.

We then define *quasiperiodic maximal surfaces* – and justify the terminology later – as surfaces whose images in $\mathcal{M}(n)/\mathsf{G}$ have a closure that does not contain the Barbot point. The first goal of this paper is to give different characterisations of quasiperiodic maximal surfaces. Let us state here the most important ones.

Theorem A. [Quasiperiodic surfaces] *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$. The following statements are equivalent:*

   (i) *Σ is quasiperiodic,*
  (ii) *the induced metric on Σ has curvature bounded above by some negative constant,*
 (iii) *the induced metric on Σ is Gromov hyperbolic,*
 (iv) *Σ is of conformal hyperbolic type and any uniformisation is biLipschitz,*
  (v) *the limit curve $\partial_\infty\Sigma$ is the image of a quasisymmetric map.*

These are generalisations of the phenomenon discovered for $\mathsf{PSL}(2, \mathbb{R}) \times \mathsf{PSL}(2, \mathbb{R})$ in [BS10] and $\mathsf{PSL}(3, \mathbb{R})$ in [BH14] and in this last context we see that images of quasisymmetric maps in our sense are also the analogue of the boundary of hyperbolic convex sets in $\mathbf{P}(\mathbb{R}^3)$, as introduced by Benoist in [Ben03].

The quasi-symmetric parametrisation of $\partial_\infty\Sigma$ is explicit and gives the second goal of this paper:

Theorem B. [Extension of uniformisation] *Let Σ be a quasiperiodic maximal surface. Then any uniformisation of Σ by the hyperbolic disk extends continuously to a quasisymmetric homeomorphism between the boundary of the hyperbolic disk to the boundary of Σ in $\partial_\infty\mathbf{H}^{2,n}$.*

An important role is played in the proof by pseudo-Riemannian analogues of functions in hyperbolic spaces. First, a *horofunction* is the function associated to a non-zero lightlike vector z_0, whose projection belongs to $\partial_\infty\Sigma$, and defined by

$$h(x) := \log|\langle x, z_0 \rangle|.$$

When $n = 0$, one recovers usual horofunctions. Similarly, the *spatial hyperbolic distance* between two points in spacelike position is given by

$$\eth(x, y) := \cosh^{-1}(|\langle x, y \rangle|).$$

When $n = 0$, ð is the hyperbolic distance, while in general ð is not a distance in $\mathbf{H}^{2,n}$. We will however prove that the restriction of ð to any complete maximal surface is a distance up to a universal constant – Corollary 7.8 to be compared with [GM18].

Associated to these functions are two other characterisations of quasiperiodic surfaces – Theorems 6.2 and 6.5 as well as the Rigidity Theorem 5.5.

As we already said, a pervasive tool in the paper is the use of the cocompact G action on the laminated space $\mathcal{M}(n)$. Dwelling on that, we also have a description of quasiperiodic surfaces as leaves in a lamination in Theorem 6.6, thus justifying the terminology quasiperiodic surface.

Finally, say a complete maximal surface is *asymptotically hyperbolic* if its curvature converges to -1 at infinity, or equivalently by Gauß equation (Proposition 4.5), if the norm of the second fundamental form goes to zero. Then we have

Theorem C. [ASYMPTOTIC HYPERBOLICITY] *Any C^1 spacelike curve in $\partial_\infty \mathbf{H}^{2,n}$ is quasisymmetric and bounds a complete maximal asymptotically hyperbolic surface.*

As a corollary, we obtain

Theorem D. [C^1 RIGIDITY] *An Anosov representation of a closed surface group in G whose limit curve is C^1 and spacelike factors through $(\mathsf{O}(2,1) \times \mathsf{O}(n))_0$. In particular the limit curve is a circle.*

One should compare with the fact that any Hitchin representation in $\mathsf{SO}(2,3)$ always has a C^1-timelike limit curve [Lab06], while Potrie and Sambarino [PS17] has shown that if the limit map of a Hitchin representation is C^2 then the representation factors though an irreducible $\mathsf{SL}(2,\mathbb{R})$.

This whole set of results represent a natural extension of the theory of uniformisation and quasisymmetric maps, and quite naturally the last section is devoted to a description of consequences of these results in the spirit of Bers' universal Teichmüller space [Ber65].

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1.1. **Description of the paper.**

(i) Section 2 describes the *Einstein Universe* $\partial_\infty \mathbf{H}^{2,n}$, which will be revealed as a bordification of the pseudo-hyperbolic space later. The Einstein Universe admits a conformally flat Minkowski structure, which is described using *Minkowski charts and patches*. This Minkowski structure is associated to a *positive structure* : positive triples, quadruples and loops. Associated to a positive triple are *diamonds and their diamond distance* which play the role of intervals in the real projective line – corresponding to the $n = 0$ case. Critical objects in our study are *Barbot crowns* which are semi-positive loops. Finally we define *quasiperiodic loops*.

(ii) Section 3 describes the *cross-ratio* associated to four generic points in $\partial_\infty \mathbf{H}^{2,n}$ and introduces the important new concept of *quasisymmetric maps*. We describe cross-ratio in Minkowski charts and prove one of the main result of this paper: Theorem 3.17 which shows that quasisymmetric maps admit a Hölder modulus of continuity.

(iii) In section 4 we describe the pseudo-hyperbolic space and maximal surfaces therein. We describe *horofunctions* and *spatial distance*. We spend some time recalling the classical Gauss, Codazzi and Ricci equations governing these maximal surfaces. We finally recall the compactness results of [LTW20] that we express using the *space of pointed maximal surfaces* $\mathcal{M}(n)$ which is a laminated space on which the isometry group G of $\mathbf{H}^{2,n}$ acts cocompactly. We then describe *Barbot surfaces* which are the maximal surfaces associated

to Barbot crowns and show their properties. We finally introduce *quasiperiodic maximal surfaces* which is one of the main object of study of the current paper.

(iv) In section 5, we prove two Rigidity results that characterise Barbot surfaces. The first one, proposition 5.3 stated as Theorem 1 in the introduction, shows that Barbot surfaces are extremal for a pointwise curvature condition, while the second one, Theorem 5.5, shows that Barbot surfaces are extremal for a pointwise condition involving either the gradient of an horofunction or the spatial distance. Proposition 5.3 is proved using a slightly different point of view than Cheng's original proof.

(v) We use these Rigidity Theorems and the compactness results of [LTW20] in section 6 to give six alternative characterisations of quasiperiodic surfaces involving curvature, horofunctions, spatial distance, uniformisation, Gromov hyperbolicity and laminations, some of these being stated in Theorem A.

(vi) We now move in section 7 to prove Theorem 7.1. This result shows that the uniformisation of quasiperiodic surfaces extends in a precise and quantitative way to a quasisymmetric map. The proof involves the bounds on horofunctions given in the Rigidity Theorems as well as the study of *Gromov products* in this setting, that were also considered by Glorieux and Monclair [GM18].

(vii) As another application of our techniques, we study in section 8 asymptotically hyperbolic maximal surfaces and prove Theorem C and D.

(viii) In the final section 9 we show how our results can be used to define an analogue universal Teichmüller space and we discuss briefly the consequence of an announcement by Qiongling Li and Takuro Mochizuki for the $n = 2$ case well as a related conjecture.

(ix) For the sake of completeness, we give in an appendix a proof of a classical Bochner Formula.

2. Einstein Universe and positivity

In this section, we describe the geometry of the Einstein Universe $\partial_\infty \mathbf{H}^{2,n}$ of signature $(1, n)$ and study the notion of positivity in $\partial_\infty \mathbf{H}^{2,n}$. Although it is not relevant at this stage that the Einstein Universe is indeed the boundary of the pseudo-hyperbolic space $\mathbf{H}^{2,n}$, we nevertheless stick (as in [LTW20]) with the notation $\partial_\infty \mathbf{H}^{2,n}$.

The Einstein Universe has been extensively studied and part of the material covered here can be found in [BCD$^+$08, CTT19, DGK18, LTW20].

Although most of what we describe could be described as part of the general theory of parabolic spaces, Bruhat cells *etc.* we keep our approach elementary.

Throughout this paper, E will be a real $(n + 3)$-dimensional vector space equipped with a signature $(2, n + 1)$ quadratic form $\mathbf{q}$ and $\mathsf{G} := \mathsf{SO}_0(\mathbf{q})$ will be the connected component of the identity of the group of linear endomorphisms of E preserving $\mathbf{q}$.

2.1. **The geometry of the Einstein Universe.** The *Einstein Universe* is the quadric associated to $\mathbf{q}$:

$$\partial_\infty \mathbf{H}^{2,n} := \left\{ x \in \mathbf{P}(E), \ \mathbf{q}(x) = 0 \right\}.$$

The group G acts transitively on $\partial_\infty \mathbf{H}^{2,n}$ and the stabiliser of a point in $\partial_\infty \mathbf{H}^{2,n}$ is a (disconnected maximal parabolic) subgroup P.

We denote by $\mathbf{P}_+(E) = (E \setminus \{0\})/\mathbb{R}_{>0}$ the set of linear rays in E. The double cover $\partial_\infty \mathbf{H}^{2,n}_+$ of $\partial_\infty \mathbf{H}^{2,n}$ is defined as the preimage of $\partial_\infty \mathbf{H}^{2,n}$ in the double cover $\mathbf{P}_+(E)$ of $\mathbf{P}(E)$. The stabiliser in G of a point in $\partial_\infty \mathbf{H}^{2,n}_+$ is now isomorphic to the identity component P_0 of P.

Definition 2.1. [Photons and circles]

(i) A *photon* in $\partial_\infty \mathbf{H}^{2,n}$ is the projectivisation of an isotropic 2-plane in E: at most one photon passes through two distinct points.

(ii) A *spacelike circle* is the intersection of $\partial_\infty \mathbf{H}^{2,n}$ with the projectivisation of a space of signature $(2, 1)$: at most one circle passes through three distinct points.

The corresponding subsets of $\partial_\infty \mathbf{H}^{2,n}_+$ are defined in an analogous way.

2.1.1. *The symmetric space of* G. The group G acts transitively on the Grassmannian $\mathrm{Gr}_{2,0}(E)$ of oriented positive definite 2-planes in E. The stabiliser of an element in $\mathrm{Gr}_{2,0}(E)$ is isomorphic to $\mathsf{SO}(2) \times \mathsf{SO}(n + 1)$ and is thus a maximal compact subgroup of G, giving an identification between $\mathrm{Gr}_{2,0}(E)$ and the (Riemannian) symmetric space of G.

Given a plane U in $\mathrm{Gr}_{2,0}(E)$, with orthogonal subspace V, the quadratic form

$$\mathbf{q}_U := \mathbf{q}\big|_U \oplus \left(- \mathbf{q}\big|_V\right) , \tag{1}$$

is positive definite on $E = U \oplus V$. In particular, any element x in $\mathrm{Gr}_{2,0}(E)$ induces an Euclidean scalar product on E, hence on the Lie algebra of G: the (non-invriant) scalar product of tow elements A and B in the Lie algebra of G is $\langle A, B \rangle_x = \mathrm{tr}\,(AB^t)$, where B^t denotes the transpose of B with respect to the Euclidean scalar product on E defined by x.

2.1.2. *Conformally flat structure.* The Einstein Universe $\partial_\infty \mathbf{H}^{2,n}$ is naturally equipped with a (locally) conformally flat structure $[g_{\mathbf{Ein}}]$ of signature $(1, n)$.

Indeed, the tangent space to $\partial_\infty \mathbf{H}^{2,n}$ at x, is identified with the space $\mathrm{Hom}\,(x, x^\perp/x)$. The restriction of $\mathbf{q}$ to $x^\perp/x$ has signature $(1, n)$, providing $\mathrm{Hom}\,(x, x^\perp/x)$ with a conformal class of quadratic form. This structure is conformally flat: any point in $\partial_\infty \mathbf{H}^{2,n}$ has a neighbourhood which is conformal to a Minkowski space.

Observe that the notion of spacelike, timelike and lightlike vectors is meaningful in $\partial_\infty \mathbf{H}^{2,n}$. For details, see [BCD$^+$08, Section 3 and 4] .

2.1.3. *Product structure.* Consider a positive definite 2-plane U in $\mathrm{Gr}_{2,0}(E)$ with orthogonal W, and let $\mathbf{q}_U$ be the corresponding Euclidean quadratic form on E defined in equation (1). Any isotropic ray in $\partial_\infty \mathbf{H}^{2,n}_+$ contains a unique point $(u, v) \in U \oplus W$ with $\mathbf{q}_U(u) = \mathbf{q}_U(v) = 1$. This gives a diffeomorphism

$$\partial_\infty \mathbf{H}^{2,n}_+ \cong \mathbf{S}^1 \times \mathbf{S}^n ,$$

where $\mathbf{S}^1 \subset U$ and $\mathbf{S}^n \subset W$ are the unit spheres. In this coordinate system, the conformal metric of $\partial_\infty \mathbf{H}^{2,n}_+$ is given by

$$[g_{\mathbf{Ein}}] = [g_{\mathbf{S}^1} \oplus (-g_{\mathbf{S}^n})] ,$$

where $g_{\mathbf{S}^i}$ is the canonical metric on $\mathbf{S}^i$ of curvature 1.

Definition 2.2. [Visual distance and metric] The *visual metric* associated to a plane U in $\mathrm{Gr}_{2,0}(E)$ is the Riemannian metric on $\partial_\infty \mathbf{H}^{2,n}_+$ induced by $\mathbf{q}_U$, namely

$$g = g_{\mathbf{S}^1} \oplus g_{\mathbf{S}^n} .$$

This visual metric gives rise to a metric, also called visual, on $\partial_\infty \mathbf{H}^{2,n}$. The corresponding distances are called *visual distances*.

Finally, observe that isotropic 2-planes in E are exactly graphs of linear maps φ from U to V such that for any u in U, we have $\mathbf{q}_U(\varphi(u)) = \mathbf{q}_U(u)$. In particular, photons in the splitting $\partial_\infty \mathbf{H}^{2,n}_+ \cong \mathbf{S}^1 \times \mathbf{S}^n$ correspond to graph of isometries from $\mathbf{S}^1$ to $\mathbf{S}^n$.

2.2. **Positive triples.** We discuss now the important notion of positivity in $\partial_\infty \mathbf{H}^{2,n}$.

2.2.1. *Motivation: the projective line.* Let V be a real vector space of dimension 2. In that context, let us define

  (i) A *positive triple* in $\mathbf{P}(V)$ to be a triple of pairwise distinct points.
  (ii) A *positive quadruple* (a, b, c, d) is a quadruple of pairwise distinct points such that b and d are not in the same connected component of $\mathbf{P}(V) \setminus \{a, c\}$.
  (iii) A map from $\mathbf{P}(V)$ to itself to be *positive* if it is orientation preserving or orientation reversing.

2.2.2. *Higher dimensions.* We now describe the generalisation of positive triples to $\partial_\infty \mathbf{H}^{2,n}$ for $n > 0$ and will do later the same for positive quadruples and positive maps. A pair of points (a, b) in $\partial_\infty \mathbf{H}^{2,n}$ is *transverse*, if $\mathbf{q}(a_0, b_0) \neq 0$ where a_0 and b_0 are non zero vectors in a and b respectively (or equivalently, if a and b do not lie on a same photon).

Definition 2.3. [POSITIVE TRIPLE] A triple of points $\tau = (a, b, c)$ of pairwise distinct points in $\partial_\infty \mathbf{H}^{2,n}$ (or in $\partial_\infty \mathbf{H}^{2,n}_+$) is *positive* if $W_\tau := a \oplus b \oplus c$ is a linear space of signature $(2, 1)$. We denote by $\mathcal{T}(n)$ and $\mathcal{T}_+(n)$ the set of positive triples in $\partial_\infty \mathbf{H}^{2,n}$ and $\partial_\infty \mathbf{H}^{2,n}_+$ respectively.

There is a unique spacelike circle passing through a positive triple. Conversely, any triple of pairwise distinct points in a spacelike circle is positive. We warn the reader that the terminology *positive triples*, though standard, is confusing: being a positive triple is invariant under all permutations.

2.2.3. *Action of* G *on positive triples.* We have the following

Lemma 2.4. *If $n > 0$, then*

  (i) *the action of G on $\mathcal{T}_+(n)$ has two orbits,*
  (ii) *the action of G on $\mathcal{T}(n)$ is transitive and the stabiliser of a point is isomorphic to $\mathsf{SO}(n)$.*

Proof. The group G preserves a space orientation and a time orientation in E. This means that for any positive definite 2-plane P in E, both P and $P^\perp$ carry a natural orientation which is preserved by the action of G.

Recall that an orthonormal basis of E is a basis $(e_1, ..., e_{n+3})$ such that $\langle e_i, e_j \rangle = \varepsilon_i \delta_{i,j}$, where $\varepsilon_i = 1$ for $i = 1, 2$ and $\varepsilon_i = -1$ otherwise. Such a basis will be called *canonical* is moreover (e_1, e_2) and $(e_3, ..., e_{n+3})$ are positively oriented. The group G acts simply transitively on the set of canonical bases.

Given a positive triple (z_1, z_2, z_3) in $\partial_\infty \mathbf{H}^{2,n}$ associated to a vector space V of type $(2, 1)$, one can find an orthonormal set (e_1, e_2, e_3) basis in V so that

$$\begin{cases} e_1 + e_3 & \text{belongs to} \quad z_1 , \\ -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2 + e_3 & \text{belongs to} \quad z_2 , \\ -\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2 + e_3 & \text{belongs to} \quad z_3 , \end{cases}$$

and such a triple can be completed to an orthonormal basis of E. The only obstruction of extending (e_1, e_2, e_3) to a canonical basis is the orientation of (e_1, e_2). This proves item (i).

For item (ii), repeating the construction for a positive triple $\tau = (z_1, z_2, z_3)$ in $\partial_\infty \mathbf{H}^{2,n}$, one notices that we have exactly two choices for (e_1, e_2, e_3) corresponding to the two (amongst the eight possible) lifts of τ in $\partial_\infty \mathbf{H}^{2,n}_+$ which lies on the two lifts of a positive circle. Only one of the two will give the correct orientation on the pair (e_1, e_2) defined above. If $n > 0$, one can find an orthonormal basis $(e_4, ..., e_{n+3})$ of $(z_1 \oplus z_2 \oplus z_3)^\perp$ such that $(e_3, ..., e_{n+3})$ is positively oriented. So G acts transitively on the space of positive triples in $\partial_\infty \mathbf{H}^{2,n}$.

The stabiliser of the positive triple (z_1, z_2, z_3) acts simply transitively on the space of canonical basis having (e_1, e_2, e_3) for the first 3 vectors. This group is thus isomorphic to $\mathsf{SO}(n)$. $\qquad \square$

2.2.4. *Distances.* Let us start with a proposition

Proposition 2.5. *There exists a G equivariant map $\tau \mapsto \beta_\tau$ from the space $\mathcal{T}(n)$ of positive triples in $\partial_\infty \mathbf{H}^{2,n}$ to the symmetric space $\mathrm{Gr}_{2,0}(E)$ of G.*

Proof. This is a direct consequence of Lemma 2.4 since the stabiliser of a positive triple is compact. Let us however give an explicit construction of this map.

Let $\tau = (a, b, c)$ be positive triple and $W_\tau := a \oplus b \oplus c$. Then the intersection of W_τ with $\mathbf{H}^{2,n}$ is a hyperbolic plane H_τ and the points a,b and c are the vertices of an ideal hyperbolic triangle in H_τ. Denote its barycenter by b_τ. The point b_τ corresponds to a negative definite line L in W_τ, and we denote by β_τ its 2-dimensional spacelike orthogonal in W_τ. We define the image of τ in $\mathrm{Gr}_{2,0}(E)$ to be β_τ. □

In particular, since $\mathcal{T}(n)$ is a transitive G space which fibres over $\mathrm{Gr}_{2,0}(E)$, the symmetric space of G by paragraph 2.1.1

Corollary 2.6. [Canonical distance] *The space $\mathcal{T}(n)$ of positive triples in $\partial_\infty \mathbf{H}^{2,n}$ is a Riemannian homogeneous G-space.*

Definition 2.7. [Visual distance] The *visual distance* associated to a positive triple on $\partial_\infty \mathbf{H}^{2,n}$ is the visual distance associated to β_τ according to definition 2.2.

2.3. **Minkowski patch and charts.** In the sequel, $\mathbf{E}^{1,n}$ will be an affine space, whose underlying vector space is equipped with a quadratic form q of signature $(1, n)$.

We explain in this paragraph that $\partial_\infty \mathbf{H}^{2,n}$ is isomorphic to the "conformal compactification" of $\mathbf{E}^{1,n}$. Let us first define

Definition 2.8. [Minkowski patch and light cones] Let a be a point in $\partial_\infty \mathbf{H}^{2,n}$.
  (i) The *light cone* $L(a)$ is the union of all photons passing though a.
  (ii) The *Minkowski patch* of a is $\mathrm{Mink}(a) := \partial_\infty \mathbf{H}^{2,n} \setminus L(a)$.

One may recognise the description of a big Bruhat cell in the parabolic space $\partial_\infty \mathbf{H}^{2,n}$. We now give a natural conformal description of a Minkowski patch:

Proposition 2.9. *Let a be a point in $\partial_\infty \mathbf{H}^{2,n}$ and b be a point in $\mathrm{Mink}(a)$. Consider a_0 and b_0 be non zero vectors in a and b respectively, with $\langle a_0, b_0 \rangle = 1$ and set $F := (a \oplus b)^\perp$. Define ψ_0 as the map from F to E given by the formula*

$$\psi_0(u) = u - b_0 + \frac{1}{2}\mathrm{q}(u)a_0 \,,$$

where $\mathrm{q} = \mathbf{q}|_F$. Then the projectivisation $\psi : F \to \mathbf{P}(E)$ of ψ_0 is a conformal diffeomorphism onto $\mathrm{Mink}(a)$ that we call a canonical Minkowski chart.

Proof. One easily checks that q has signature $(1, n)$. Observe that ψ_0 takes value in the set of null vectors in E whose scalar product with a_0 is non-zero. Thus ψ takes values $\mathrm{Mink}(a)$.

We now claim that ψ is surjective: any point x in $\mathrm{Mink}(a)$ has a unique lift x_0 to E with $\langle x_0, a_0 \rangle = 1$. Thus

$$x_0 = u - b_0 + \alpha a_0 \,,$$

for some α in $\mathbb{R}$ and u in F. The condition $\mathbf{q}(x_0) = 0$ implies $2\alpha = \mathrm{q}(u)$ and so $x_0 = \psi_0(u)$.

Finally, ψ_0 is conformal: given u in F, we have

$$\mathsf{T}_u \psi_0(v) = v + \langle u, v \rangle a_0 \,,$$

so $(\psi_0^* \mathbf{q})(u) = \mathrm{q}(u)$. Since the projection from a non degenerate vector subspace tangent to the null cone to its projectivisation is conformal by definition of the conformal structure on the Einstein Universe, ψ itself is conformal. □

This leads to the following definition

Definition 2.10. [Minkowski chart] Given a point a in $\partial_\infty \mathbf{H}^{2,n}$, a *Minkowski chart for a* is a conformal diffeomorphism ϕ from $\mathbf{E}^{1,n}$ to $\mathrm{Mink}(a)$ so that $\phi = \psi \circ A$ where ψ is the canonical Minkowski chart associated to b in $\mathrm{Mink}(a)$ and A is an affine conformal map from $\mathbf{E}^{1,n}$ to $F = (a \oplus b)^\perp$.

The following describes useful properties of Minkowski charts.

Proposition 2.11. *Let a be a point in $\partial_\infty \mathbf{H}^{2,n}$.*

   (i) *Given a lightlike line in $\mathbf{E}^{1,n}$, the closure of its image by a Minkowski chart is a photon intersecting $L(a)$ in a unique point.*
   (ii) *Given a spacelike line in $\mathbf{E}^{1,n}$, the closure of its image by a Minkowski chart is a spacelike circle passing through a.*

Proof of (i). Let first prove the result for a canonical Minkowski chart (see proposition 2.9 for the notations). Let L be a lightlike line in F parametrised by $c : t \to tw$ with $\mathrm{q}(w) = 0$. Thus

$$\psi_0(c(t)) \quad = \quad tw - b_0 + \frac{1}{2}\mathrm{q}(tw)a_0 = tw - b_0 .$$

Since $\langle b_0, w\rangle = 0$, the 2-plane $U := \mathrm{span}(b_0, w)$ is isotropic. Thus $\psi(c(t))$ is contained in the photon $\mathbf{P}(U)$ Moreover, the intersection of $L(a)$ with U is reduced to the isotropic line spanned by the vector $\frac{w}{\langle w, a_0\rangle} - b_0$.

Given two Minkowski charts ψ and ϕ of $\mathrm{Mink}(a)$, the map $\psi \circ \phi^{-1}$ extends to a conformal transformation of $\partial_\infty \mathbf{H}^{2,n}$ fixing a. Because elements in G maps photons to photons (as well as spacelike circles to spacelike circles), the results proved for ϕ and L extend to all charts and lightlike lines. $\square$

Proof of (ii). Similarly to the proof of item (i), a spacelike linear line in F has the form $c : t \to tw$ with $\mathrm{q}(w) > 0$. Its image by ψ_0 is given by

$$\psi_0(c(t)) = tw - b_0 + \frac{t^2}{2}\mathrm{q}(w)a_0 .$$

Since w is positive definite, the vector space spanned by $\{w, a_0, b_0\}$ has signature $(2, 1)$ and intersects $L(a)$ at a. $\square$

2.3.1. *τ-charts.* Let $\mathbb{R}^{1,n}$ be the vector space $\mathbb{R}^{n+1}$ equipped with the quadratic form $\mathrm{q}_{1,n} = \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \cdots - \mathrm{d}x_n^2$ of signature $(1, n)$. We identify the stabiliser in $\mathsf{O}(1, n)$ of $(1, 0 \ldots, 0)$ with $\mathsf{O}(n)$

Definition 2.12. [τ-chart] Let $\tau = (a, b, c)$ be a positive triple in $\partial_\infty \mathbf{H}^{2,n}$. A *$\tau$-chart* is a Minkowski chart ψ_τ for a whose source is $\mathbb{R}^{1,n}$ and such that $b = \psi_\tau(-1, 0, \ldots, 0)$ and $c = \psi_\tau(1, 0, \ldots, 0)$.

We have the following

Proposition 2.13. *Let τ be a positive triple in $\partial_\infty \mathbf{H}^{2,n}$. Up to the precomposition by an element of $\mathsf{O}(n)$, there exists a unique τ-chart.*

Proof. Let $\tau = (a, b, c)$ a positive triple. Let $\psi : \mathbf{E}^{1,n} \to \mathcal{M}(a)$ be a Minkowski chart. By proposition 2.11, the affine segment δ between $\psi^{-1}(b)$ and $\psi^{-1}(c)$ is spacelike. Let g be the conformal transformation that sends the segment $\delta_0 = [(-1, 0, \ldots, 0), (1, 0, \ldots, 0)]$ to δ. Observe that $\psi \circ g$ is again a Minkowski chart. It follows that $\psi \circ g$ is a τ-chart.

Let ϕ be another τ-chart. From proposition 2.11, the map $g = \psi^{-1} \circ \phi$ is a conformal transformation of $\mathbf{E}^{1,n}$ fixing δ_0 and 0 and so belonging to $\mathsf{O}(n)$. $\square$

2.4. **Diamonds and positive quadruples.** We introduce in this section the notion of positive quadruples which is a special case of a theory developpd in [GLW21] based on [GW18].

2.4.1. *Diamonds and diamond distance.*

Proposition 2.14. *Given two transverse points a and b in $\partial_\infty \mathbf{H}^{2,n}$, the set of points c in $\partial_\infty \mathbf{H}^{2,n}$ so that (a, b, c) is a positive triple has two connected components.*

Proof. Let $\psi : \mathbb{R}^{1,n} \to \mathrm{Mink}(a)$ be a Minkowski chart with $\psi(0) = b$. It follows from proposition 2.11 that the points c in $\partial_\infty \mathbf{H}^{2,n}$ such that (a, b, c) is positive are exactly the image of spacelike vectors in $\mathbb{R}^{1,n}$. This set is thus the union of two convex open cones. □

Definition 2.15. [DIAMOND] Each of these connected components is called a *diamond* defined by a and b. Given a positive triple (a, c, b), the diamond defined by a and b containing c is denoted $\Delta_c(a, b)$, while the diamond not containing c is denoted $\Delta_c^*(a, b)$.

When $n = 0$, diamonds are intervals. We now construct another metric on diamonds than the visual distance – as in definition 2.7 – associated to a positive triple and which is better designed for our purposes.

The *Euclidean metric* on $\mathbb{R}^{1,n}$ is $dx_1^2 + dx_2^2 + ... + dx_{n+1}^2$.

Definition 2.16. [DIAMOND DISTANCE] Let $\tau = (a, b, c)$ be a positive triple in $\partial_\infty \mathbf{H}^{2,n}$, the *diamond distance* δ_τ on the diamond $\Delta_a^*(b, c)$ is the image by (any) τ-chart of the Euclidean metric in $\mathbb{R}^{1,n}$.
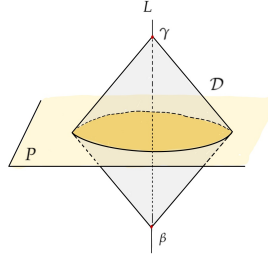
FIGURE 1. Diamonds in in a τ-chart (the point a is at infinity).

Proposition 2.17. *There is a positive constant C so that given a positive triple $\tau = (a, b, c)$, the metrics δ_τ and d_τ are C-biLipschitz on $\Delta_a^*(b, c)$.*

Proof. Indeed a diamond has finite diameter with respect to both the diamond and the visual distances, which are both Riemannian on $\mathbb{R}^{1,n}$. □

2.5. **Positive quadruples and positive maps.**

Definition 2.18. [POSITIVE QUADRUPLE] A quadruple (a, b, c, d) is *positive* if all sub-triples are positive and if furthermore

$$\Delta_b(a, c) = \Delta_d^*(a, c) .$$

We can finally define define positive maps:

Definition 2.19. [POSITIVE MAP AND POSITIVE LOOP] A *positive map* from a subset A of $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$ is a map sending positive quadruples to positive quadruples. A *positive loop* in $\partial_\infty \mathbf{H}^{2,n}$ is the image of $\mathbf{P}(V)$ by a continuous positive map.

Proposition 2.20. *A continuous map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$ sending positive triples to positive triples is positive.*
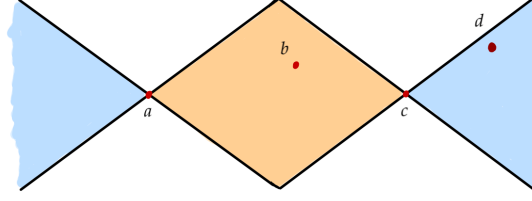
FIGURE 2. A positive quadruple (a, b, c, d).

Proof. Let $\varphi : \mathbf{P}(V) \to \partial_\infty \mathbf{H}^{2,n}$ be a continuous map mapping positive triples to positive triples. Let (a', b', c', d') be a positive quadruple in $\mathbf{P}(V)$ that is mapped to (a, b, c, d) in $\partial_\infty \mathbf{H}^{2,n}$.

Since pairs of distinct points in $\mathbf{P}(V)$ are mapped to transverse pairs in $\partial_\infty \mathbf{H}^{2,n}$, the image by φ of $\mathbf{P}(V) \setminus \{a', c'\}$ consists in two disjoint arcs γ_1 and γ_2 – containing respectively b and d– contained in $\Delta_b(a, c) \sqcup \Delta_b^*(a, c)$ both joining a and c.

Using a τ-chart – with $\tau = (x, b, c)$ – we see that the union of the lightcones $L(x)$ for x in γ_1 covers $\Delta_b(a, c)$. Recall that any point y in γ_2 is transverse to any point x in γ_1 and thus cannot lie in $L(x)$, since no element in $L(x)$ is transverse to x. It follows that γ_2 – and thus d– lies $\Delta_b^*(a, c)$. Thus $\Delta_d(a, c) = \Delta_b^*(a, c)$, and (a, b, c, d) is positive. $\qquad\square$

Thus in an equivalent way, we could have defined a positive loop in $\partial_\infty \mathbf{H}^{2,n}$ as a topological circle for which any triple of pairwise distinct points is positive. This is the definition that we used in [LTW20].

2.6. **Semi-positive loops and Barbot crowns.** For a triple of points in $\partial_\infty \mathbf{H}^{2,n}$, being positive is not a closed condition. As a result, positive loops do not form a closed set. We recall now the notion of semi-positive loops introduced in [LTW20] that are natural degeneration of positive loops.

2.6.1. *Semi-positivity.*

Definition 2.21. [SEMI-POSITIVE LOOP] A triple of points is *non-negative* if the vector space generated is not of type $(1, 2)$.

A *semi-positive loop* is the image of an injective continuous map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$ (or to $\partial_\infty \mathbf{H}_+^{2,n}$) sending positive triples to non-negative triples and sending at least one positive triple to a positive one.

We proved in [LTW20, Lemma 2.8]:

Proposition 2.22. *The preimage in $\partial_\infty \mathbf{H}_+^{2,n}$ of a positive (respectively semi-positive loop) in $\partial_\infty \mathbf{H}^{2,n}$ has two connected components, each of which is a positive (respectively semi-positive) loop.*

As a result, semi-positive loops in $\partial_\infty \mathbf{H}_+^{2,n}$ are exactly lift of semi-positive loops in $\partial_\infty \mathbf{H}^{2,n}$. Positive and semi-positive loops in $\partial_\infty \mathbf{H}_+^{2,n}$ can be characterized in terms of graphs. A map f between two metric spaces (X, d_X) and (Y, d_Y) is *contracting* if $d_Y(f(x), f(y)) < d_X(x, y)$ for any distinct points x and y in X. We proved in [LTW20, Proposition 2.11]:

Proposition 2.23. *Let Λ be a topological circle in $\partial_\infty \mathbf{H}_+^{2,n}$ and consider a splitting $\partial_\infty \mathbf{H}_+^{2,n} \cong \mathbf{S}^1 \times \mathbf{S}^n$ (see Paragraph 2.1.3). Then*

(i) *The set Λ is a semi-positive loop if and only if it is the graph of a 1-Lipschitz map from $\mathbf{S}^1$ to $\mathbf{S}^n$ that is different from a global isometry.*

(ii) *The set Λ is positive if and only if it is the graph of a contracting map from $\mathbf{S}^1$ to $\mathbf{S}^n$.*

We have the following:

Corollary 2.24. *If Λ is a semi-positive loop which is not positive, then it contains an arc with non empty interior which is the intersection of Λ with a photon.*

Proof. Up to taking a lift, we can assume that $\Lambda \subset \partial_\infty \mathbf{H}^{2,n}_+$. By the previous proposition, given a splitting $\partial_\infty \mathbf{H}^{2,n}_+ \cong \mathbf{S}^1 \times \mathbf{S}^n$, the loop Λ is the graph of a 1-Lipschitz map $f : \mathbf{S}^1 \to \mathbf{S}^n$ and since Λ is not positive there exists $x \neq y$ in $\mathbf{S}^1$ such that $d_{\mathbf{S}^1}(x, y) = d_{\mathbf{S}^n}(f(x), f(y))$. The set

$$\sigma = \left\{ z \in \mathbf{S}^1 \ , \ d_{\mathbf{S}^n}(f(x), f(z)) = d_{\mathbf{S}^1}(x, z) \right\},$$

is a closed proper arc in $\mathbf{S}^1$ with non-empty interior. Since photons are graph of isometries, the graph of f above σ is contained in a photon ϕ. By construction, $\Lambda \cap \phi = (\sigma, f(\sigma)) \subset \mathbf{S}^1 \times \mathbf{S}^n$. $\quad\square$

Proposition 2.23 allows us to have a notion of convergence of semi-positive loops – as graph of 1-Lipschitz maps – and we have:

Corollary 2.25. *Let $\{\Lambda_k\}_{k\in\mathbb{N}}$ be a sequence of semi-positive loops in $\partial_\infty \mathbf{H}^{2,n}_+$. Then*
  (i) *The sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ subconverges to either a semi-positive loop or a photon.*
  (ii) *If each Λ_k passes through a given positive triple τ, then the sequence $\{\Lambda_k\}_{k\in\mathbb{N}}$ subconverges to a semi-positive loop passing through τ.*

Proof. The first item comes from the fact that the space M of 1-Lipschitz maps from $\mathbf{S}^1$ to $\mathbf{S}^n$ is compact. From Proposition 2.23, the graph of such a map in $\mathbf{S}^1 \times \mathbf{S}^n \cong \partial_\infty \mathbf{H}^{2,n}_+$ is either a semi-positive loop or a photon.

For the second item, observe that passing through τ is a closed condition and that no photon can pass through a positive triple. $\quad\square$

2.6.2. *Barbot crowns.* We define here an important semi-positive loop in $\partial_\infty \mathbf{H}^{2,n}$, that we call *Barbot crown*, from Barbot's work in [Bar15].

Definition 2.26. A *Barbot crown C* has four cyclically ordered *vertices* in $\partial_\infty \mathbf{H}^{2,n}$, spanning a four dimensional vector space, and four *edges* which are segment of photons joining consecutive vertices.

Let C be a Barbot crown with vertices (v_1, v_2, v_3, v_4) and let $F_C := v_1 \oplus ... \oplus v_4$. Then we denote by A_C the Abelian subgroup of G which, in the splitting $E = v_1 \oplus v_2 \oplus v_3 \oplus v_4 \oplus F_C^\perp$, is defined by the matrices

$$a(\lambda, \mu) = \begin{pmatrix} 1/\lambda & 0 & 0 & 0 & 0 \\ 0 & 1/\mu & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{F_C^\perp} \end{pmatrix}. \tag{2}$$

Then we have

Proposition 2.27. *If (v_1, v_2, v_3, v_4) are the vertices of a Barbot crown C labelled clockwise and $F_C := v_1 \oplus v_2 \oplus v_3 \oplus v_4$, then F_C is of type $(2, 2)$ and the Barbot crown C lies in the projective space of F.*

All Barbot crowns are isomorphic under the action of G. Moreover a Barbot crown is globally invariant under the action of a Cartan subgroup of G.

Proof. The 2-planes $P_i := v_i \oplus v_{i+1}$ (where $i \in \mathbb{Z}/4\mathbb{Z}$) are isotropic and the associated photons $\phi_i := \mathbf{P}(P_i)$ contain the edges of C. The two 2-planes $v_i \oplus v_{i+2}$ have signature $(1, 1)$ and are orthogonal to each other. It follows that F_C has signature $(2, 2)$, and that the Barbot crown C is contained in $\partial_\infty \mathbf{H}^{2,n} \cap \mathbf{P}(F_C)$ which is isomorphic under G to $\partial_\infty \mathbf{H}^{2,1}$.

The subgroup A_C preserves the Barbot crown and is a Cartan subgroup of G. Observe that the Barbot crown is uniquely defined by A_C (up to cyclic transformation). Since all Cartan subgroups are conjugate, it follows that all Barbot crowns are isomorphic under the action of G. $\quad\square$

One of our main result is the following:

Proposition 2.28. *Let Λ be a semi-positive loop in $\partial_\infty \mathbf{H}^{2,n}$. If Λ is not positive, then the closure of its G-orbit contains a Barbot crown.*

Before proving the proposition, let us prove a lemma.

Lemma 2.29. *Let Λ be a semi-positive loop in $\partial_\infty \mathbf{H}^{2,n}$ contained in $M := \bigcup_{i=1}^{4} \phi_i$ where the ϕ_i are the photons spanned by the vertices of a Barbot crown. Then Λ is a Barbot crown.*

Proof. Since ϕ_i and ϕ_{i+2} do not intersect, while ϕ_i and ϕ_{i+i} only intersect at a point and all intersection points are distinct, the set M is homeomorphic to the graph drawn in Figure 3. Then the image of a
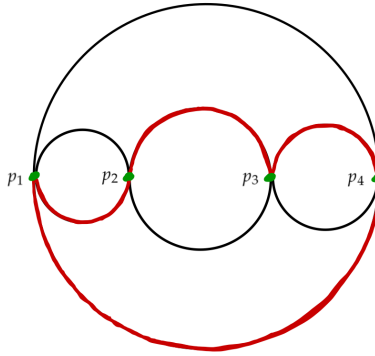
FIGURE 3. The topological space M (in red a Barbot crown)

continuous injective map from $\mathbf{S}^1$ to M is either one of the photons or is made of four arcs, one in each photon. The first case corresponds to a photon, which is not a semi-positive loop, while the second case is a Barbot crown. □

Proof of proposition 2.28. Since Λ is not positive, there is a photon ϕ_1 so that the intersection of ϕ_1 with Λ is a closed non trivial interval σ – see Corollary 2.24. Denote by v_1 and v_2 the extremities of this segment.

Let us pick two other points v_3 and v_4 in $\partial_\infty \mathbf{H}^{2,n}$ and not in ϕ_1 so that (v_1, v_2, v_3, v_4) generates a Barbot crown. Let ϕ_i be the photon passing though v_i and v_{i+1}, and M be the union of this four photons.

In the splitting $E = v_1 \oplus v_2 \oplus v_3 \oplus v_4 \oplus F_C^\perp$, let g be the element in G defined by the matrix $a(4,2)$ using the notation (2). The action of g on $\partial_\infty \mathbf{H}^{2,n}$ has the following dynamics:

  (i) g fixes v_1, v_2, v_3, v_4 and globally fixes σ and M.
  (ii) for any x in $\partial_\infty \mathbf{H}^{2,n}$, any limit values of $\left\{ g^k(x) \right\}_{k \in \mathbb{N}}$ lies in M.
  (iii) for any x in $\partial_\infty \mathbf{H}^{2,n}$ and not in ϕ_1 any limit values of $\left\{ g^k(x) \right\}_{k \in \mathbb{N}}$ is not in ϕ_1.

By Corollary 2.25, up to extracting a subsequence, the sequence $\left\{ g^k(\Lambda) \right\}_{k \in \mathbb{N}}$ converges to Λ_∞ which is either a photon or a semi-positive loop and lies in M.

Since Λ_∞ contains σ by construction and an element not in ϕ_1 by the third item above, Λ_∞ is not a photon. Thus by lemma 2.29 and the second item, Λ_∞ is a Barbot crown. □

2.7. **Quasiperiodic loops and maps.** Let us fix positive triples κ_0 and τ_0 in $\mathbf{P}(V)$ and $\partial_\infty \mathbf{H}^{2,n}$ respectively (recall that V is a 2-dimensional $\mathbb{R}$-vector space). We first introduce the notion of quasiperiodic maps and loops:

Definition 2.30. [QUASIPERIODIC MAPS AND LOOPS]
  (i) Let ξ be a a positive map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$. The map ξ is *quasiperiodic*, if for any sequence $\{\kappa_k\}_{k\in\mathbb{N}}$ of positive triples in $\mathbf{P}(V)$, if
    (a) h_k is the element of $\mathsf{PSL}(V)$ so that $h_k(\kappa_0) = \kappa_k$,
    (b) g_k is an element in G so that $g_k(\xi(\kappa_k)) = \tau_0$,
    Then the sequence $\{g_k \circ \xi \circ h_k\}_{k\in\mathbb{N}}$ subconverges to a positive map.
  (ii) A positive loop Λ in $\partial_\infty \mathbf{H}^{2,n}$ is *quasiperiodic* if for any sequence $\{\tau_k\}_{k\in\mathbb{N}}$ of positive triples in Λ, if g_k is an element in G so that $g_k(\tau_k) = \tau_0$, then the sequence of *renormalised curves* $\{g_k(\Lambda)\}_{k\in\mathbb{N}}$ subconverges as a graph to a positive loop.

Observe that the image of a continuous quasiperiodic map is a quasiperiodic loop. As a byproduct of our constructions, we will show in corollary 7.3 that quasiperiodic loops admits a quasiperiodic parametrisation.

Definition 2.31. [POINTED LOOPS AND SPACES]
  (i) A *pointed semi-positive loop* is a pair (Λ, τ) where Λ is a semi-positive loop and τ is a positive triple in Λ. The space $\mathcal{L}(n)$ of those pairs is the *space of semi-positive loops*.
  (ii) A *pointed positive loop* is a pair (Λ, τ) where Λ is a positive loop and τ is a positive triple in Λ. The space $\mathcal{L}^+(n)$ of those pairs is the *space of positive loops*.
  (iii) The *forgetting map* from $\mathcal{L}(n)$ to $\mathcal{T}(n)$ sends (Λ, τ) to τ.

Since any semi-positive loop Λ is a Lipschitz graph (see proposition 2.23), $\mathcal{L}(n)$ can be embedded as a closed subset of the product of the space of Lipschitz graph with $\mathcal{T}(n)$. This embedding gives $\mathcal{L}(n)$, and hence $\mathcal{L}^+(n)$ the structure of a locally compact topological space on which the group G acts continuously. Corollary 2.25 implies the following

Proposition 2.32. *The forgetting map is G-equivariant and proper. In particular the action of G on $\mathcal{L}(n)$ is proper and $\mathcal{L}(n)/\mathsf{G}$ is compact.*

The space $\mathcal{L}_{\tau_0}(n)$ of pointed semi-positive loops (L, τ_0) is compact and the map π_0 from $\mathcal{L}_{\tau_0}(n)$ to $\mathcal{L}(n)/\mathsf{G}$ is proper and surjective.

2.7.1. *The Barbot locus.* We define the *Barbot locus* in $\mathcal{L}(n)/\mathsf{G}$ as the quotient of all pointed Barbot crowns by the action of G.

Proposition 2.33. *When n equals 1, the Barbot locus is homeomorphic to a disjoint union of two circles. For n larger than 1, the Barbot locus is homeomorphic to a circle.*

Proof. Let us first deal with the case $n = 1$. The action of $\mathsf{G} \cong \mathsf{SO}_0(2,2)$ on the set of photons has two orbits, each corresponding to a ruling of the one sheeted hyperboloid $\partial_\infty \mathbf{H}^{2,1}$. Two distinct photons intersect if and only if they belong to different families, and any point of $\partial_\infty \mathbf{H}^{2,1}$ is the intersection of a unique pair of photons. In particular, a Barbot crown is made of four photons $\phi_1, \ldots, \phi_4$ where ϕ_i and ϕ_{i+1} belong to different families.

We define now a G-invariant map $(\tau, C) \mapsto w_{\tau,C}$ sending a pointed Barbot crown (τ, C) to a point in $\mathbf{P}(\mathbb{R}^2)$ and show that this map is two-to-one .

Let (τ, C) be a Barbot crown where $\tau = (x_1, x_2, x_3)$. By positivity, each edge contains at most one x_i. Let c be the positive circle spanned by τ.

CLAIM: Assume that x_1, x_2 and x_3 are interior points of the edges, then c intersects C in a fourth point $p_\tau \in c \setminus \{x_1, x_2, x_3\}$.

Proof of the claim. It follows from [LTW20, Lemma 2.8] that c is disjoint from $\mathbf{P}(\beta_\tau^\perp)$, where β_τ is the barycenter of τ. So using Subsection 2.1.3, we get an identification between $\partial_\infty \mathbf{H}^{2,1} \setminus \mathbf{P}(\beta_\tau^\perp)$

and $\mathbf{S}^1 \times \mathbb{R}$ in which $c \cong \mathbf{S}^1 \times \{0\}$ and photons in each family give respectively increasing and decreasing maps. Now if x_i, x_j and x_k are points in consecutive edges say ϕ_1, ϕ_2 and ϕ_3, then by the intermediate value theorem, the photon ϕ_4 intersects $\mathbf{S}^1 \times \{0\}$ in a unique point p_τ in the interval (x_k, x_i) of $\mathbf{S}^1 \setminus \{x_i, x_j, x_k\}$. □

Observe in the general case that at most one of the x_i – say x_1 – is a vertex v_k: indeed assuming x_2 is a vertex, then by transversality $x_2 = x_{k+2}$, but then C is included in $L(x_1) \cup L(x_2)$ and thus no element of C can be transverse to both x_1 and x_2. Thus, if one of the x_i is a vertex, we set unambiguously $p_\tau = x_i$.

Taking the unique projective identification between c and $\mathbf{P}(\mathbb{R}^2)$ sending τ to $(0, 1, \infty)$, the map $w : \tau \mapsto p_\tau$ defines a map $\tau \mapsto w_\tau$ from the Barbot locus to $\mathbf{P}(\mathbb{R}^2)$.

Let us now prove that w is two-to-one . The group G acts simply transitively on the positive triples in $\partial_\infty \mathbf{H}^{2,1}$, so if τ_0 is a fixed positive triple, any point in the Barbot locus has a unique representant which is a pointed Barbot crown (τ_0, C). Identify $\mathbf{P}(\mathbb{R}^2)$ with the circle c_0 spanned by τ_0 such that $(0, 1, \infty)$ is mapped to τ_0. The result then follows from the fact that a Barbot crown passing through τ_0 and a fourth point $p \in c_0$ is entirely given by a choice of a photon passing through a given point of τ_0, and that there are exactly 2 such photons (one in each family).

Finally, for the case $n > 1$, we fix a signature $(2, 2)$ subspace of E. Each pointed Barbot crown in $\partial_\infty \mathbf{H}^{2,n}$ can be mapped to $\mathbf{P}(F)$. The stabiliser $\mathrm{Stab}_\mathsf{G}(F) \cong (\mathsf{O}(2, 2) \times \mathsf{O}(n-1))_0$ does not act effectively on a Barbot crown in F and the effective action is given by the index two subgroup of $\mathsf{O}(2, 2)$ whose maximal compact is $\mathsf{SO}(2) \times \mathsf{O}(2)$. The element $(\mathrm{Id}, -\mathrm{Id})$ intertwine the two family of photons in $\partial_\infty \mathbf{H}^{2,1}$ and thus identify the two disjoint circles representing the Barbot locus for $n = 1$. □

Given a semi-positive loop Λ, let T_Λ be the space of pairs (Λ, τ) with τ in Λ. We have a natural map from T_Λ to $\mathcal{L}(n)/\mathsf{G}$.

Now, as a consequence of proposition 2.28, we obtain the following characterisation of quasiperiodic loops:

Proposition 2.34. *A positive loop Λ is quasiperiodic if and only if the closure of the image of T_Λ in $\mathcal{L}(n)/\mathsf{G}$ does not contain a Barbot crown or, equivalently, if the image T_Λ in $\mathcal{L}^+(n)/\mathsf{G}$ is precompact.*

3. Cross-ratio and quasisymmetric maps

We define in this section in definition 3.1 a cross-ratio on $\partial_\infty \mathbf{H}^{2,n}$ which generalises the usual cross-ratio on the real projective line $\mathbf{P}(V)$. We characterise circles using the cross-ratio in proposition 3.4.

This allows us to define quasisymmetric maps in definition 3.5. We then show two properties of quasisymmetric maps: equicontinuity in Theorem 3.12 and the existence of a Hölder modulus of continuity Theorem 3.17. Formally the former is a consequence of the latter but we first have to prove equicontinuity.

3.1. **Cross-ratio and positivity.** Equip the 2-dimensional vector space V with a volume form ω. The usual cross-ratio is defined for a quadruple of pairwise distinct points in $\mathbf{P}(V)$ by

$$[x, y, z, t] := \frac{\omega(x_0, y_0)\omega(z_0, t_0)}{\omega(x_0, t_0)\omega(z_0, y_0)} ,$$

where x_0, y_0, z_0 and t_0 are non-zero vectors in x, y, z and t respectively. Observe that the definition is independent on the choice of x_0, y_0, z_0 and t_0 and on the choice of ω.

Definition 3.1. [Cross-ratio] Given any quadruple (x, y, z, t) of pairwise transverse points in $\partial_\infty \mathbf{H}^{2,n}$, we define the *cross-ratio*

$$\mathrm{b}(x, y, z, t) := \frac{\langle x_0, y_0 \rangle \langle z_0, t_0 \rangle}{\langle x_0, t_0 \rangle \langle z_0, y_0 \rangle} ,$$

where x_0, y_0, z_0 and t_0 are non-zero vectors of x, y, z and t. If ξ is a map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$, we write whenever defined for (x, y, z, t) in $\mathbf{P}(V)$,

$$b_\xi(x, y, z, t) := b(\xi(x), \xi(y), \xi(z), \xi(t)) .$$

We warn the reader that unlike the usual cross-ratio in $\mathbf{P}(V)$, the value of $b(x, y, z, t)$ does not characterise the positivity of a quadruple of pairwise transverse points in $\partial_\infty \mathbf{H}^{2,n}$. For instance, it can be shown – see proposition 3.4 – that under the canonical isomorphism $\mathbf{P}(V) \cong \partial_\infty \mathbf{H}^{2,0}$, we have $b(x, y, z, t) = [x, y, z, t]^2$. Hartnick and Strubel defined in [HS12] a "functorial" cross-ratio for Shilov boundary of Hermitian symmetric spaces, which we suspect could be used to define positivity of quadruples. However, we stick with our definition of cross-ratio which is easier to handle.

The following lemma gives an expression of the cross-ratio in a Minkowski chart:

Lemma 3.2. *Let (a, x, b, y) be a quadruple of pairwise transverse points in $\partial_\infty \mathbf{H}^{2,n}$. Then in any Minkowski chart for a with associated quadratic form q we have*

$$b(a, x, b, y) = \frac{q(b - y)}{q(b - x)} . \tag{3}$$

Proof. Observe that the right hand side of the inequality is invariant by conformal transformation. It follows that it is enough to prove formula (3) in the canonical chart of $\mathrm{Mink}(a)$ associated to b (see proposition 2.9). Let us choose as in proposition 2.9 a_0 in a and b_0 in b with $\langle a_0, b_0 \rangle = 1$. The Minkowski chart from $(a \oplus b)^\perp$ to $\mathrm{Mink}(a)$ is given by

$$\psi_0(u) = u - b_0 - \frac{1}{2}q(u)a_0 .$$

Then a straightforward computations give:

$$\langle \psi_0(u), a_0 \rangle = -1 \text{ and } \langle \psi_0(u), b_0 \rangle = -\frac{1}{2}q(u) .$$

In particular,

$$b(a, \psi_0(u), b, \psi_0(v)) = \frac{\langle \psi_0(u), a_0 \rangle \langle \psi_0(v), b_0 \rangle}{\langle \psi_0(v), a_0 \rangle \langle \psi_0(u), b_0 \rangle} = \frac{q(v)}{q(u)} .$$

This concludes the proof since $b_0 = \psi_0(0)$. $\qquad\square$

Corollary 3.3. *Let (a, b, c) be a positive triple in $\partial_\infty \mathbf{H}^{2,n}$ and let u and v be two points in $\Delta_a^*(b, c)$ such that $b(a, b, v, c) < b(a, b, u, c)$. Then any element x in $\Delta_a^*(u, v)$ satisfies*

$$b(a, b, v, c) < b(a, b, x, c) < b(a, b, u, c) .$$

Proof. Use the notations of the previous lemma with $\tau = (a, b, c)$ and define a causal structure on $\mathbb{R}^{1,n}$ (that is, an orientation of spacelike vectors) such that the vector from b to c is future pointing.

Observe that the function $q(x - b)$ is strictly increasing along future pointing spacelike curves in $\Delta_a^*(b, c)$ while the function $q(x - c)$ is strictly decreasing. Since both are positive, lemma 3.2 implies that the function $f(x) := b(a, b, x, c)$ is strictly decreasing along future pointing spacelike curves.

The result follows from the fact that $\Delta_a^*(u, v)$ is the union of all future pointing spacelike curves from u to v. $\qquad\square$

The proof of the following result follows closely that of [Lab07, Theorem 5.3]

Proposition 3.4. *A continuous map ξ whose image contains a positive triple is a circle map if and only if for all quadruples of pairwise distinct points*

$$b_\xi(x, y, z, t) = [x, y, z, t]^2 .$$

Proof. It follows from the hypothesis that $\xi(x)$ and $\xi(y)$ are transverse whenever x is different from y. In particular ξ is injective.

Let (x_1, x_2, x_3) be a positive triple in $\mathbf{P}(V)$ whose image by ξ is positive. It its enough to show that for any x_4 so that (x_1, x_2, x_3, x_4) is a positive quadruple, then $\xi(x_4)$ belongs to the circle C that passes through $\xi(x_1)$, $\xi(x_2)$ and $\xi(x_3)$. Let y_0 and z_0 be two points in $\mathbf{P}(V)$ different from all the points x_i, let also η be the circle map sending (x_1, x_2, x_3) to $(\xi(x_1), \xi(x_2), \xi(x_3))$. Denote by $\xi_0(x_i)$, $\xi_0(z_0)$ and $\xi_0(y_0)$ some non zero vectors in $\xi(x_i)$, $\xi(z_0)$ and $\xi(y_0)$

Let us consider the 4×4 matrix $A(\xi)$ whose coefficients are $a_{i,j} := b_\xi(y_0, z_0, x_i, x_j)$. The determinant of this matrix is a non zero multiple of the Gram matrix $G(\xi_0)$ whose coefficients are $g_{ij} = \langle \xi_0(x_i), \xi_0(x_j) \rangle$:

$$\det(A(\xi)) = \frac{\langle \xi_0(y_0), \xi_0(z_0) \rangle^4}{\prod_{i=1}^4 \langle \xi(y_0), \xi_0(x_i) \rangle \prod_{j=1}^4 \langle \xi_0(z_0), \xi_0(x_j) \rangle} \det(G(\xi_0)) .$$

Thus, the determinant of $A(\xi)$ is zero if and only if span$\{\xi(x_1), ..., \xi(x_4)\}$ has dimension 3, that is if $\xi(x_4)$ belongs to C. In particular, $\det(A(\eta)) = 0$. Since our hypothesis implies that $A(\xi) = A(\eta)$, it follows that $\det(A(\xi)) = 0$ and that $\xi(x_4)$ belongs to C. □

3.2. **Quasisymmetric maps.** Recall that a homeomorphism φ of $\mathbf{P}(V)$ is *quasisymmetric* if there exists constants A and B greater than 1 such that for any quadruple of pairwise distinct points in $\mathbf{P}(V)$, we have

$$A^{-1} \leqslant \big|[x, y, z, t]\big| \leqslant A \quad \text{implies} \quad B^{-1} \leqslant \big|[\varphi(x), \varphi(y), \varphi(z), \varphi(t)]\big| \leqslant B .$$

Here we do not impose φ to be orientation preserving (this is why we need the absolute value).

We now use our cross-ratio to define quasisymmetric maps from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$. This notion is closely related to both the classical notion of quasicircles for Kleinian groups and the notion of Sullivan maps developed in [KLM18].

After some preliminaries involving the related notion in a Minkowski patch we prove the main results of this section concerning quasisymmetric maps: equicontinuity (Theorem 3.12) and Hölder property (Theorem 3.17).

Definition 3.5. [Quasisymmetric] Let Ω be a subset of $\mathbf{P}(V)$ whose closure is connected, ξ a map from Ω to $\partial_\infty \mathbf{H}^{2,n}$ and A, B be constants greater than 1. The map ξ is (A, B)-*quasisymmetric on* Ω if it is positive and for all quadruple (x, y, z, t) in Ω^4 we have

$$A^{-1} \leqslant \big|[x, y, z, t]\big| \leqslant A \qquad \text{implies} \qquad B^{-1} \leqslant \big|b_\xi(x, y, z, t)\big| \leqslant B . \tag{4}$$

We will refer to Ω as the *defining set* of ξ.

We also say ξ is *quasisymmetric* if it is (A, B)-quasisymmetric for some constants A and B.

Finally, we call *quasicircle* the image of a quasisymmetric map defined on $\mathbf{P}(V)$.

3.3. **Preliminaries.**

3.3.1. *Choice of constants.* The following lemma tells us the choice of A is not relevant:

Lemma 3.6. *For any constants A,B and C greater than 1, there exists a constant D such that any (A, B)-quasisymmetric map is (C, D)-quasisymmetric.*

Proof. If $C \leqslant A$, then we can take $D = B$. Let us thus assume $C > A$. Let (x, y, z, t) be elements of Ω so that

$$C^{-1} \leqslant \big|[x, y, z, t]\big| \leqslant C .$$

Let N be any integer. Using the density of Ω in an interval and the cocycle property of the cross-ratio

$$b(a, c_0, d, c_2) = b(a, c_0, d, c_1) \cdot b(a, c_1, d, c_2) ,$$

we can find $w_0, \cdots, w_N$ in Ω with $w_0 = y$, $w_N = t$, so that

$$C^{-\frac{1}{N}} \leqslant \left| [x, w_i, z, w_{i+1}] \right| \leqslant C^{\frac{1}{N}} .$$

In particular taking

$$\frac{\log C}{\log A} \leqslant N \leqslant 1 + \frac{\log C}{\log A} ,$$

we have

$$C^{\frac{1}{N}} = \exp\left(\frac{\log(C)}{N} \right) \leqslant \exp(\log(A)) = A.$$

Thus

$$A^{-1} \leqslant \left| [x, w_i, z, w_{i+1}] \right| \leqslant A.$$

It follows that

$$B^{-1} \leqslant \left| \mathrm{b}_\xi \left(x, w_i, z, w_{i+1} \right) \right| \leqslant B.$$

Using finally the cocycle property for b_ξ, we have

$$B^{-N} \leqslant \left| \mathrm{b}_\xi(x, y, z, t] \right| \leqslant B^N .$$

In particular, $D := B^{1 + \frac{\log(C)}{\log(A)}} \geqslant B^N$ satisfies the required condition.   □

3.3.2. *Quasisymmetry in a Minkowski patch.* In a manifold equipped with a conformal structure of type $(1, n)$ the set of non zero spacelike vectors is the union of two non intersecting convex cones. A *space orientation* is a continuous choice of one of this cone over the whole manifold.

Let F be a vector space equipped with a quadratic form q of signature $(1, n)$. In this case, the choice of one spacelike vector at one point defines the space orientation.

For $\tau = (a, b, c)$ a positive triple in $\partial_\infty \mathbf{H}^{2,n}$, we choose the space orientation for Mink(c) to be defined by $b - a$. Then we define for any x in Mink(c) the set $I_+(x)$ to be the cone of space like vectors defining the orientation, while $I_-(x)$ is the cone of space like vectors defining the opposite orientation. Similarly $J_+(x)$ and $J_-(x)$ denote the closure of $I_+(x)$ and $I_-(x)$ respectively. Then

$$\Delta^*_c(a, b) = I_+(a) \cap I_-(b) . \tag{5}$$

Definition 3.7. [Positive and semi-positive maps] Let Ω be a dense subset of an interval O of $\mathbb{R}$. A map f from Ω to F is

    (i) *positive* if for all $x > y$, $f(x)$ belongs to $I_+(f(y))$ – equivalently if $f(y)$ belongs to $I_-(f(x))$; in particular $I_+(f(x)) \subset I_+(f(y))$.
   (ii) *Semi-positive* if for all $x > y$, $f(x)$ belongs to $J_+(f(y))$ – equivalently if $f(y)$ belongs to $J_-(f(x))$; in particular $J_+(f(x)) \subset J_+(f(y))$.

We have the following

Lemma 3.8. *Let Ω be a dense subset of an interval O and f be a semi-positive map from Ω to F. Then for any x in O, the following limits exist*

$$f^+(x) := \lim_{y \to x, y > x} f(y) \quad , \quad f^-(x) := \lim_{y \to x, y < x} f(y) .$$

Proof. If z and y belong to Ω and $z > y$, we have $J_+(f(z)) \subset J_+(f(y))$. Let

$$W(x) := \overline{\bigcup_{y > x} J_+(f(y))} ,$$

then $W(x) = J_+(w)$ for some unique element w in F. We then define $f^+(x) := w$. We construct symmetrically f^-. The result follows.   □

Given constants A, B greater than 1, we say that a map f from Ω to F is (A, B)-*quasisymmetric* if it is positive and if (x, y, z) are pairwise distinct in Ω then

$$\frac{1}{A} \leqslant \left| \frac{x - y}{z - x} \right| \leqslant A \quad \text{implies} \quad \frac{1}{B} \leqslant \frac{\mathsf{q}(f(x) - f(y))}{\mathsf{q}(f(z) - f(x))} \leqslant B . \tag{6}$$

This slight abuse of language is justified by lemma 3.2.

Lemma 3.9. *Given a semi-positive map f from a dense subset Ω of an interval O to F,*

   (i) *the maps f^+ and f^- are semi-positive and respectively rightcontinuous and leftcontinuous .*
  (ii) *If f^+ (respectively f^-) is continuous, then f^+ (respectively f^-) is the unique semi-positive continuous extension of f to O.*
 (iii) *Assume furthermore that f is (A, B)-quasisymmetric, then f^+ and f^- are both (A, B)-quasisymmetric.*

Proof. The only non trivial point is item (iii). We first show that if f is positive on Ω, then f^- (or equivalently f^+) is positive on O. Let us first observe that, since f^- is semi-positive, if $y < x < z$ and x is in O, y and z in O, then by density of Ω

$$f(x) \in J_+(f^-(y)) \cap J_-(f^-(z)) =: J(y, z) .$$

Thus $f^-(]y, z[)$ is included in $J(y, z)$. Moreover since Ω is dense, there exists x_0 and w_0 in Ω so that $J(y, z)$ contains

$$J_0 = J_+(f(x_0)) \cap J_-(f(w_0)) ,$$

Since f is positive, J_0 – hence $J(y, z)$ – contains a space like segment. In particular $J(y, z)$ is not a lightlike line. This implies that $f^-(y)$ and $f^-(z)$ are not joined by a lightlike segment, and so f^- is positive. The proof for f^+ is analogous.

We conclude the proof of the last item using that Ω is dense and the cross-ratio is continuous. $\quad\square$

3.3.3. *Quasisymmetry in the Minkowski case and continuity.*

Proposition 3.10. *Let f be a positive map from a dense subset Ω of an interval O to F. If f is (A, B)-quasisymmetric, then f extends uniquely on O to a continuous (A, B)-quasisymmetric map.*

Proof. Let us first assume that f is defined on O and is leftcontinuous. By lemma 3.9, it is enough to show that $f = f^+$, since then f will be both rightcontinuous and leftcontinuous and thus continuous. Let x be in O, $a := f(x)$ and $a^+ := f^+(x)$. Our goal is to show that $a^+ = a$

FIRST STEP: *we have* $\mathsf{q}(a^+ - a) = 0$.

Let us consider $y_k^0 = x + \frac{1}{k}$ and $y_k^1 = x + \frac{2}{k}$, so that both sequences $\left\{ y_k^0 \right\}_{k \in \mathbb{N}}$ and $\left\{ y_k^1 \right\}_{k \in \mathbb{N}}$ converge to x when k goes to infinity and satisfy $x < y_k^0 < y_k^1$. Moreover

$$\frac{|y_k^0 - y_k^1|}{|y_k^0 - x|} = 1 .$$

By the quasisymmetric property, the ratio

$$\frac{\mathsf{q}(f(y_k^0) - f(y_k^1))}{\mathsf{q}(f(y_k^0) - f(x))} ,$$

is bounded from below. Since

$$\lim_{k \to \infty} \mathsf{q}(f(y_k^0) - f(y_k^1)) = \mathsf{q}(a^+ - a^+) = 0 ,$$

it follows that

$$\mathsf{q}(a^+ - a) = \lim_{k \to \infty} (\mathsf{q}(f(y_k^0) - f(x))) = 0 .$$

SECOND STEP. We now prove that $a^+ = a$.

Assume $a \neq a^+$. By the first step the line L through a^+ and a is lightlike. For any $z < x$, since f is positive $\langle f(z), a \rangle$ is non zero. Thus $f(z)$ does not belong to L and moreover if P_z be the plane containing $f(z)$ and L then the restriction of q to P_z is non degenerate and therefore has signature $(1, 1)$. Let L'_z be the unique lightlike line in P_z through $f(z)$ and not parallel to L. Since f is leftcontinuous,

$$\lim_{z \to x, z < x} f(z) = a .$$

Then if $g(z)$ is the intersection of L'_z with L (see Figure 4). We have

$$\lim_{z \to x, z < x} g(z) = a .$$

We now observe that if b is in L'_z and c in L, then
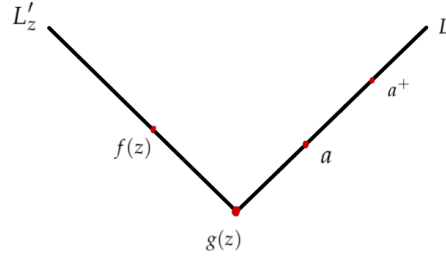
FIGURE 4. Picture drawn in the plane P_z.

$$q(b - c) = 2 \langle b - g(z) \mid g(z) - c \rangle .$$

Thus

$$\frac{q(f(z) - a^+)}{q(f(z) - a)} = \frac{\langle f(z) - g(z) \mid g(z) - a^+ \rangle}{\langle f(z) - g(z) \mid g(z) - a \rangle} = \frac{g(z) - a^+}{g(z) - a} , \tag{7}$$

where the latter ratio is the ratio of three points in an affine line.

Let us now choose a sequence $\{x_k\}_{k \in \mathbb{N}}$ in O converging to x and so that $x_k > x$. Then for all z strictly smaller than x,

$$\lim_{k \to \infty} \frac{x_k - z}{x - z} = 1 .$$

Thus we have

$$\frac{g(z) - a^+}{g(z) - a} = \frac{q(f(z) - a^+)}{q(f(z) - a)} = \lim_{k \to \infty} \frac{q(f(z) - f(x_k))}{q(f(z) - f(x))} \leqslant B .$$

where the last inequality comes from the (A, B)-quasisymmetry. Since $\lim_{z \to x, z < x} g(z) = a$ it follows that $\lim_{z \to x, z < x} g(z) = a^+$. Thus $a = a^+$ and this concludes the proof of the particular case when f is leftcontinuous and defined on O.

Turning to the general case, let f be an (A, B)-quasisymmetric map defined on a dense subset of O. Using item (iii) of lemma 3.9 and the previous discussion we obtain that f^- is continuous. Thus f^- is the unique continuous extension of f to O by the second item of lemma 3.9. This concludes the proof of the general case.   □

3.3.4. *Limit of quasisymmetric maps in the Minkowski case.* We now prove

Proposition 3.11. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of (A,B)-quasisymmetric maps from $\mathbb{R}$ to F. Assume that we have two points y and z so that $\{f_k(y)\}_{k\in\mathbb{N}}$ and $\{f_k(z)\}_{k\in\mathbb{N}}$ converges respectively to a and b so that $q(a-b) > 0$.*

Then after extracting a subsequence $\{f_k\}_{k\in\mathbb{N}}$ converges uniformly to a positive (A,B)-quasisymmetric map f on the interval $[y,z]$.

Proof. Let us choose a dense countable subset Ω of $[y,z]$ containing y and z. For k large enough, we observe that f_k takes values in a compact neighbourhood of $I_+(a)\cap I_-(b)$. Then, using Cantor's diagonal argument and extracting subsequences, we obtain a map f defined on Ω so that $\{f_k\}_{k\in\mathbb{N}}$ converges pointwise to f on Ω. Observe that f is semi-positive. Using the (A,B)-quasisymmetry and lemma 3.6, we deduce that for any x in Ω and $]y,z[$ the ratios

$$\frac{q(f_k(y)-f_k(z))}{q(f_k(y)-f_k(x))} ,$$

are uniformly bounded. Since the numerator converges to a non zero value it follows that $q(f(y)-f(x)) > 0$. Repeating the argument we obtain that for all distinct x and t in Ω, then $q(f(t)-f(x)) > 0$. In other words, f is positive. Obviously f is (A,B)-quasisymmetric. Thus by proposition 3.10, f extends uniquely to a continuous positive (A,B)-quasisymmetric map on $[y,z]$, extension that we also denote f .

Let us prove that $\{f_k\}_{k\in\mathbb{N}}$ converges to f uniformly on $[y,z]$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of points in $[y,z]$ converging to x. Let w and t in Ω and $[y,z]$, with $w < x < t$. Let c be a limit value of $\{f_k(x_k)\}_{k\in\mathbb{N}}$. By semi-positivity of the functions f_k for k large enough $f_k(x_k)$ belongs to

$$J_+(f_k(w))\cap J_-(f_k(t)) .$$

Since $\{f_k(w)\}_{k\in\mathbb{N}}$ and $\{f_k(w)\}_{k\in\mathbb{N}}$ converge to $f(w)$ and $f(t)$ respectively, c belongs to

$$J(w,t) := J_+(f(w))\cap J_-(f(t)) .$$

Since f extends to a continuous function, it follows

$$\lim_{w,t\to x} J(w,t) = \{f(x)\} ,$$

thus $c = f(x)$. We have shown that $\lim_{k\to\infty} f_k(y_k) = f(x)$. This concludes the proof of uniform convergence. $\square$

3.4. **Equicontinuity.** We now come back to quasisymmetric maps from $\mathbf{P}(V)$ to $\partial_\infty\mathbf{H}^{2,n}$. Fix κ_0 and τ_0 positive triples in $\mathbf{P}(V)$ and $\partial_\infty\mathbf{H}^{2,n}$ respectively. Recall that distances between positive triples is given in definition 2.6.

Theorem 3.12. [EQUICONTINUITY] *Every sequence of (A,B)-quasisymmetric maps sending κ_0 to τ_0 possesses a subsequence that converges uniformly to a continuous (A,B)-quasisymmetric map. Equivalently, the action by precomposition and postcomposition of $\mathsf{PSL}(V)\times\mathsf{G}$ on the space of (A,B) quasisymmetric maps is cocompact.*

Here are several corollaries, the first one is immediate, the next ones are proved right after the Equicontinuity Theorem.

Corollary 3.13. *A quasisymmetric map is quasiperiodic.*

Corollary 3.14. *For every constants A, B, C greater than 1, there exists a constant D such that if ξ is (A,B)-quasisymmetric, if (a,b,c,d) is a quadruple in $\mathbf{P}(V)$, then*

$$C^{-1} \leqslant |\mathsf{b}(\xi(a),\xi(b),\xi(c),\xi(d))| \leqslant C \ \ \text{implies} \ \ D^{-1} \leqslant |[a;b;c;d]| \leqslant D .$$

Corollary 3.15. *For every constants A, B, greater than 1, there exists a constant D such that if ϕ is a homeomorphism of $\mathbf{P}(V)$ so that both ξ and $\xi\circ\phi$ are (A,B)-quasisymmetric, then ϕ is (A,D) quasisymmetric.*

Corollary 3.16. *Let A, B be constant greater than 1, then for any positive real a there is a positive real b so that the following holds. If ξ be a (A, B)-quasisymmetric map, then for any positive triples κ_1 and κ_2, we have*

$$d(\kappa_1, \kappa_2) \leqslant a \qquad implies \qquad d(\xi(\kappa_1), \xi(\kappa_2)) \leqslant b \, . \tag{8}$$

Proof of the Equicontinuity Theorem 3.12. Let $\{\xi_k\}_{k \in \mathbb{N}}$ be a sequence of quasisymmetric map sending a triple $\kappa_0 = (x_0, y_0, z_0)$ in $\mathbf{P}(V)$ to a positive triple $\tau_0 = (a_0, b_0, c_0)$.

Let us identify $\mathbf{P}(V) \setminus \{x_0\}$ with $\mathbb{R}$, so that y_0 and z_0 are identified with -1 and 1 respectively.

Using the τ_0-chart of $\text{Mink}(a_0)$, we obtain a sequence $\{f_k\}_{k \in \mathbb{N}}$ of maps from the interval $[-1, 1]$ to $\mathbb{R}^{1,n}$ so that the two sequences $\{f_k(-1)\}_{k \in \mathbb{N}}$ and $\{f_k(1)\}_{k \in \mathbb{N}}$ are constant.

By lemma 3.2, the maps f_k are (A, B)-quasisymmetric as maps from $[-1, 1]$ to $\mathbb{R}^{1,n}$. By proposition 3.11, we can extract a subsequence so that $\{f_k\}_{k \in \mathbb{N}}$ converges uniformly on $[-1, 1]$ to an (A, B) quasi symmetric map in the Minkowski sense. Using lemma 3.2 again, $\{\xi_k\}_{k \in \mathbb{N}}$ converges uniformly on the interval $\overline{\Delta^*_{x_0}[y_0, z_0]}$ to a map which is (A, B)-quasisymmetric. Cycling through (x_0, y_0, z_0) and using the fact that

$$\mathbf{P}(V) = \overline{\Delta^*_{x_0}[y_0, z_0]} \cup \overline{\Delta^*_{y_0}[x_0, z_0]} \cup \overline{\Delta^*_{z_0}[x_0, y_0]} \, ,$$

we complete the proof of the theorem.   □

Proof of corollary 3.14. Let us fix some positive constants A and B, and let

$$\mathcal{B} \quad := \quad \{(\xi, a, b, c, d) \mid \xi \text{ is } (A, B)\text{-quasisymmetric}, (a, b, c, d) \in \mathbf{P}(V)^4\} \, .$$

Then $\mathcal{B}$ is equipped with an action of $\mathsf{H} := \mathsf{PSL}(V) \times \mathsf{G}$ given by

$$(h, g) \cdot (\xi, a, b, c, d)) = (g \circ \xi \circ h^{-1}, h(a), h(b), h(c), h(d)) \, .$$

This action is cocompact by the Equicontinuity Theorem 3.12. Let C be a constant greater than 1 and

$$\mathcal{B}(C) \quad := \quad \{(\xi, a, b, c, d) \in \mathcal{B} \text{ such that } C^{-1} \leqslant |\mathsf{b}(\xi(a), \xi(b), \xi(c), \xi(d))| \leqslant C\} \, .$$

Observe that $\mathcal{B}(C)$ is closed and H-invariant. Thus its projection in $\mathcal{B}/\mathsf{H}$ is compact. Let finally, for any positive integer n,

$$\mathcal{B}_n(C) \quad := \quad \{(\xi, a, b, c, d) \in \mathcal{B}(C) \text{ such that } n^{-1} < |[a; b; c; d]| < n\} \, .$$

Since the union for all n in $\mathbb{N}$ of the sets $\mathcal{B}_n(C)$ is equal to $\mathcal{B}(C)$ and that $\mathcal{B}_n(C)$ is open in $\mathcal{B}(C)$ and H-invariant, it follows by compactness that there exists some p so that $\mathcal{B}_p(C)$ is equal to $\mathcal{B}(C)$. Thus we obtain the corollary by taking $D = p$.   □

Proof of corollary 3.15. Assume that the cross-ratio of the quadruple (a, b, c, d) in $\mathbf{P}(V)^4$ is in $[A, A^{-1}]$ then the cross-ratio of $\xi \circ \phi(a, b, c, d)$ is in $[B, B^{-1}]$, since $\xi \circ \phi$ is (A, B)-quasisymmetric. Hence, by corollary 3.14 and since ξ is (A, B)-quasisymmetric, there exists a constant D only depending on A and B so that the cross-ratio of $\phi(a, b, c, d)$ is in $[D, D^{-1}]$. Thus ϕ is (A, D)-quasisymmetric.   □

Proof of corollary 3.16. Let us fix some constants A and B greater than 1, and let a be positive. Let

$$\mathcal{A}(a) = \{(\xi, \tau_0, \tau_1) \quad | \quad \xi \text{ is } (A, B)\text{-quasisymmetric}$$
$$\tau_0, \tau_1 \text{ are positive triples in } \mathbf{P}(V) \text{ with } d(\tau_0, \tau_1) \leqslant a\} \, .$$

As in the previous corollary $\mathcal{A}(a)$ is equipped with an action of $\mathsf{H} := \mathsf{PSL}(V) \times \mathsf{G}$, and this action is cocompact by the Equicontinuity Theorem. In particular since the function which associates to (ξ, τ_0, τ_1), the real $d(\xi(\tau_0), \xi(\tau_1))$ is continuous and H-invariant, it is bounded. This concludes the proof of the corollary.   □

3.5. **Hölder property.** Recall any positive triple τ in $\partial_\infty \mathbf{H}^{2,n}$ defines a visual distance d_τ on $\partial_\infty \mathbf{H}^{2,n}$ (see definition 2.7).

Theorem 3.17. [Hölder Modulus of Continuity] *For any constants A, B greater than 1, there exist positive constants M and α with the following property: if ξ is an (A, B)-quasisymmetric map from a dense subset of $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$, then ξ extends uniquely to a map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$ such that if τ_0 is any positive triple in $\mathbf{P}(V)$, then for all x and y in $\mathbf{P}(V)$*

$$d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leqslant M \cdot d_{\tau_0}(x, y)^\alpha .$$

Observe that equicontinuity is a formal consequence of this theorem but we actually use equicontinuity in the proof of this theorem.

3.5.1. *A contraction lemma.*

Lemma 3.18. [Contraction Lemma] *For any $B > 1$, if $\tau = (a, b, c)$ and $\tau' = (a, x, y)$ are positive triples in $\partial_\infty \mathbf{H}^{2,n}$ satisfying*

   (i) *the diamond $\Delta_a^*(x, y)$ is included in $\Delta_a^*(b, c)$,*
   (ii) *if we have the inequalities*

$$B^{-1} \leqslant \mathsf{b}(a, b, x, c) \leqslant B \quad , \quad B^{-1} \leqslant \mathsf{b}(a, b, y, c) \leqslant B ,$$

then, on $\Delta_a^*(x, y)$, the diamond distances δ_τ and $\delta_{\tau'}$ (see definition 2.16) satisfy

$$\delta_\tau \leqslant \frac{B - 1}{B + 1} \cdot \delta_{\tau'} .$$

Proof. Let g in G so that $g\tau = \tau'$. Then g defines an isometry

$$(\Delta_a^*(b, c), \delta_\tau) \to (\Delta_a^*(x, y), \delta_{\tau'}) .$$

The statement is thus equivalent to the fact that g is $\frac{B-1}{B+1}$-contracting when seen as a map from $(\Delta_a^*(b, c), \delta_\tau)$ to itself.

Fix a τ-chart $\psi_\tau : \mathbb{R}^{1,n} \to \mathrm{Mink}(a)$ and identify $\mathbb{R}^{1,n}$ with its image by ψ_τ. Denote respectively by q the quadratic form of signature $(1, n)$ and q_τ the Euclidean quadratic form on $\mathbb{R}^{1+n}$. To lighten notations, we write $\Delta = \Delta_a^*(b, c)$.

Since g fixes the point a, it is a conformal transformation of the Minkowski patch $\mathrm{Mink}(a)$ which we identify using the τ-chart as a conformal transformation g' of $\mathbb{R}^{1,n}$. We may thus write

$$g' = g_0 + u ,$$

where u is a translation and g_0 belongs to $\mathsf{H} := \mathbb{R}^* \times \mathsf{SO}(\mathsf{q})$.

The subgroup K of H fixing the first coordinate is both a maximal compact subgroup of H and a subgroup of $\mathsf{SO}(\mathsf{q}_\tau)$. Observe also that K preserves Δ.

Let A be the subgroup of H, fixing the last $n - 1$ coordinates. Then A is a Cartan subgroup of H. Using the Cartan decomposition $\mathsf{H} = \mathsf{KAK}$, we can write $g_0 = k_0 \alpha k_1$, where k_0 and k_1 belongs to K and α to A. In particular,

$$\|g'\| = \|g_0\| = \|\alpha\| ,$$

where the norm is computed with respect to q_τ.

Let $I^\pm$ be the two lightlike lines in the plane defined by the first two coordinates. Then $\|\alpha\| = \max\{\lambda_+, \lambda_-\}$ where $\lambda_\pm$ are the eigenvalues of α on $I^\pm$. Observe now that

$$\lambda^\pm = \frac{\ell\left(\alpha(I^\pm \cap \Delta)\right)}{\ell\left(I^\pm \cap \Delta\right)} = \frac{\ell\left(\alpha(I^\pm \cap \Delta)\right)}{\sqrt{2}} ,$$

where ℓ is the length associated to q_τ, and the last inequality comes from the fact that

$$\ell\left(I^\pm \cap \Delta\right) = \sqrt{2} .$$

Since K globally fixes Δ and preserves ℓ, and the translations preserve ℓ, we have

$$
\begin{aligned}
\ell\left(\alpha(I^\pm \cap \Delta)\right) &= \ell\left((k_0 \alpha k_1)(I^\pm \cap \Delta)\right) \\
&= \ell\left(((k_0 \alpha k_1 + u)\cdot I^\pm) \cap ((k_0 \alpha k_1 + u)\cdot \Delta)\right) \\
&= \ell\left(g'(I^\pm \cap \Delta)\right) .
\end{aligned}
$$

Let $\Delta' := g'\cdot\Delta = \Delta_a^*(x, y)$, we get

$$
\|g'\| \leqslant \sup\left\{ \frac{\ell(\phi \cap \Delta')}{\sqrt{2}} \mid \phi \text{ lightlike line} \right\} . \tag{9}
$$

Corollary 3.3 implies that

$$
\Delta' \subset C_B := \left\{ w \in \Delta, \ B^{-1} \leqslant \mathsf{b}\left(a, b, \psi_\tau(w), c\right) \leqslant B \right\} .
$$

Thus the result follows from the inequality (9) and the following

CLAIM: *any lightlike segment contained in C_B has length less that $\frac{B-1}{B+1}\sqrt{2}$.*
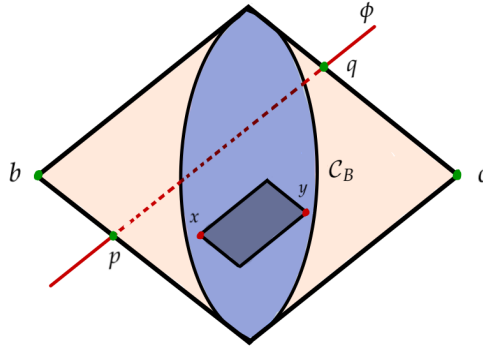
FIGURE 5. The set C_B drawn in $\partial_\infty \mathbf{H}^{2,1}$

Let us prove this claim. Let ϕ be a lightlike ray intersecting the lightcone of b and c in p and q respectively (see Figure 5). For $u = q - p$, the intersection between ϕ and Δ is parametrised by

$$
\phi(t) = p + tu , \ t \in [0, 1] .
$$

From lemma 3.2, we have

$$
\mathsf{b}(a, b, \phi(t), c) = \frac{\mathsf{q}(\phi(t) - c)}{\mathsf{q}(\phi(t) - b)} .
$$

Writing $\phi(t) - b = tu + (p - b)$ and using the fact that u and $(p - b)$ are isotropic (and similarly for $\mathsf{q}(\phi(t) - c)$), we obtain

$$
\begin{cases}
\mathsf{q}(\phi(t) - c) &= -2(1 - t)\langle u, q - c\rangle , \\
\mathsf{q}(\phi(t) - b) &= 2t\langle u, p - b\rangle .
\end{cases}
$$

We get

$$
\mathsf{b}(a, b, \phi(t), c)) = \lambda \cdot \frac{1 - t}{t} , \text{ with } \lambda = \frac{\langle u, q - c\rangle}{\langle u, p - b\rangle} .
$$

Observe that λ is positive. Thus $\phi \cap C_B$ is parametrised by the compact segment J_ϕ of $]0, 1[$, defined by

$$
J_\Phi = \left[\frac{1}{B/\lambda + 1}, \frac{1}{1/B\lambda + 1} \right] ,
$$

Hence

$$\frac{\ell(\phi \cap C_B)}{\ell\left(\phi \cap \Delta\right)} = \left| \frac{1}{B/\lambda + 1} - \frac{1}{1/B\lambda + 1} \right| = \frac{(B^2 - 1)\lambda}{(B + \lambda)(\lambda B + 1)} \leqslant \frac{B - 1}{B + 1} .$$

It follows that

$$\ell(\phi \cap C_B) \leqslant \frac{B - 1}{B + 1} \ell\left(\phi \cap \Delta\right) \leqslant \frac{B - 1}{B + 1} \sqrt{2} .$$

This completes the proof of the claim and thus of the contraction lemma. $\qquad\square$

3.5.2. *Proof of Theorem 3.17.* The proof follows the same scheme as the proof of the corresponding theorem in [KLM18]. We now assume the hypothesis of Theorem 3.17 and prove two preliminary lemmas.

Let us first show that it is enough to prove the statement for x and y so close with respect to τ_0, more precisely

Lemma 3.19. *Assume that there exists positive ε, M_0 and α_0, so that if $d_{\tau_0}(x, y) \leqslant \varepsilon$, then*

$$d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leqslant M_0 \cdot d_{\tau_0}(x, y)^{\alpha_0} .$$

Then there exists positive M and α so that for all x, y and τ_0,

$$d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leqslant M \cdot d_{\tau_0}(x, y)^{\alpha} .$$

Proof. Indeed, let k so that $\varepsilon \geqslant \frac{1}{k} \cdot \operatorname{diam}(\mathbf{P}(V))$. Let x and y in $\mathbf{P}(V)$. Using a geodesic between x and y, let $(x_0, ..., x_k)$ so that $d_{\tau_0}(x_i, x_i + 1) = \frac{1}{k} d_{\tau_0}(x, y)$ and $x_0 = x$, $x_k = y$. It follows that $d_{\tau_0}(x_i, x_{i+1}) \leqslant \frac{1}{k} \operatorname{diam}(\mathbf{P}(V)) \leqslant \varepsilon$. Thus

$$
\begin{aligned}
d_{\xi(\tau_0)}(\xi(x), \xi(y)) &\leqslant \sum_{i=1}^{k} d_{\xi(\tau_0)}(\xi(x_{i-1}), \xi(x_i)) \leqslant M \sum_{i=1}^{k} d_{\tau_0}(x_{i-1}, x_i)^{\alpha} \\
&\leqslant Mk \left(\frac{d_{\tau_0}(x, y)}{k} \right)^{\alpha} = (Mk^{1-\alpha}) d_{\tau_0}(x, y)^{\alpha} .
\end{aligned}
$$

$\qquad\square$

Let $\gamma = \gamma_{\tau_0}$ be the geodesic in $\mathbf{H}^2$ starting at t_0 and orthogonal to the geodesic between x_0 and y_0. Let us now prove an elementary fact from hyperbolic geometry

Lemma 3.20. *Let $A > 1$ be any positive number. There exists a nested sequence of intervals*

$$\Delta_{t_0}^*(x_0, y_0) = \Delta_0 \supset \Delta_1 \supset ... \supset \Delta_N ,$$

with $\Delta_i = \Delta_{t_0}^*(x_i, y_i)$ and such that

  (i) *x and y belong to Δ_N.*
  (ii) *x_{i+1} and y_{i+1} belong to the set of those u so that $\frac{1}{A_0} \leqslant [t_0, x_i, u, y_i] \leqslant A$.*
  (iii) *$N > -C \log(d_{\tau_0}(x, y))$ for some constant C only depending on A.*

Proof. Denote by z_0 the intersection of γ with the geodesic through x_0 and y_0, and by $p \in \gamma$ is the barycenter of (t_0, x, y). As remarked in [KLM18, Equation (43)], there is a universal constant $C_0 > 0$ such that

$$d_{\mathbf{H}^2}(p, z_0) \geqslant -C_1 \log\left(d_{\tau_0}(x, y)\right) .$$

There exists $\delta > 0$, only depending on A, such that if (a, b) and (u, v) are the end points of two geodesics at a distance δ orthogonal to γ, with (t_0, a, u, v, b) going counter-clockwise, then

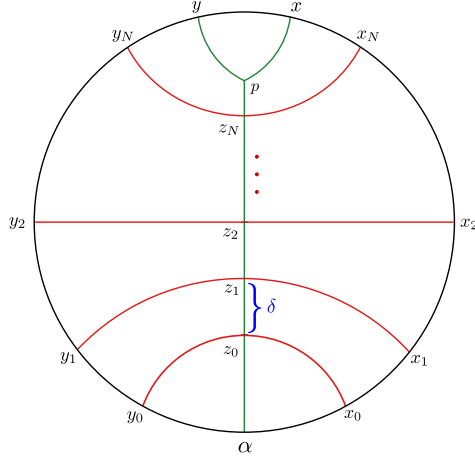$$[t_0, a, u, b] = \frac{1}{A} , \quad [t_0, a, v, b] = A .$$

FIGURE 6.

Let $z_1, ..., z_N$ be the points on γ such that $d_{\mathbf{H}^2}(z_i, z_{i+1}) = \delta$ for all $i = 0, ..., N - 1$ and $d_{\mathbf{H}^2}(z_N, p) < \delta$. Hence

$$d_{\mathbf{H}^2}(z_0, p) < N\delta \ .$$

Thus $N > -C_0 \log(d_{\tau_0}(x, y))$, with C_0 only depending on A. The diamond $\Delta_i = \Delta_{t_0}^*(x_i, y_i)$, with x_i and y_i the intersections of $\partial_\infty \mathbf{H}^2$ with the geodesic orthogonal to γ passing through z_i satisfy the required properties (see Figure 6). □

Proof of Theorem 3.17. We first consider a special case.

SYMMETRIC CASE: We first assume that x and y are symmetric with respect to $\gamma = \gamma_{\tau_0}$ and contained in the diamond $\Delta_{t_0}^*(x_0, y_0)$ – recall that in $\mathbf{P}(V)$ the diamond $\Delta_{t_0}^*(x_0, y_0)$ is the interval with extremities x_0 and y_0 not containing t_0.

Recall that (see remark 2.17) the visual distance $d_{(u,v,w)}$ is uniformly biLipschitz to the diamond distance $\delta_{(u,v,w)}$ on $\Delta_u^*(v, w)$: there exists $F > 1$, so that for x, y in $\Delta_u^*(v, w)$,

$$F^{-1} d_{(u,v,w)}(x, y) \leqslant \delta_{(u,v,w)}(x, y) \leqslant F d_{(u,v,w)}(x, y) \ .$$

Fix $\Delta_i = \Delta_{t_0}^*(x_i, y_i)$ as in lemma 3.20 for A so that ξ is (A, B)-quasisymmetric. Let $\kappa_i := (a_0, b_i, c_i) := \xi(t_0, x_i, y_i)$. Since ξ is (A, B)-quasisymmetric, any pair of diamonds $\Delta_{a_0}^*(b_i, c_i)$ and $\Delta_{a_0}^*(b_{i+1}, c_{i+1})$ satisfies the conditions of lemma 3.18, so we have $\delta_{\kappa_i} \leqslant \lambda \delta_{\kappa_{i+1}}$ on $\Delta_{a_0}^*(b_N, c_N)$ with $\lambda = \frac{B-1}{B+1} < 1$. We get

$$\delta_{\kappa_0}\big(\xi(x), \xi(y)\big) \leqslant \lambda^N \delta_{\kappa_N}\big(\xi(x), \xi(y)\big) \leqslant 2\lambda^{-C \log(d_{\tau_0}(x,y))} = 2d_{\tau_0}(x, y)^{-C \log(\lambda)}.$$

where we used $\delta_{\kappa_N}(\xi(x), \xi(y)) \leqslant 2$ since $\xi(x)$ and $\xi(y)$ both belongs to $\Delta_{a_0}(b_N, c_N)$. Thus

$$d_{\kappa_0}\big(\xi(x), \xi(y)\big) \leqslant M_0 \left(d_{\tau_0}(x, y)\right)^\alpha \ ,$$

with $M_0 = 2F^{-1}$ and $\alpha = -C \log(\lambda)$.

We have thus proved that if x and y are both elements of the interval $\Delta_{t_0}^*(x_0, y_0)$ and symmetric with respect to γ, then

$$d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leqslant M_1 d_{\tau_0}(x, y)^\alpha \ .$$

GENERAL CASE: Let us move to the general case. Let us first observe that there exists some (universal) ε so that if $d_{\tau_0}(x, y) \leqslant \varepsilon$, then using a rotation R by the barycenter of τ_0, we can find $\tau = R(\tau_0)$ so

that the symmetry condition above is satisfied, for x, y and τ. It follows that

$$d_{\xi(\tau)}(\xi(x), \xi(y)) \leqslant M_1 d_\tau(x, y)^\alpha .$$

Now we notice that there exists a constant L only depending on $\mathbf{H}^2$, so that $d(\tau, \tau_0) \leqslant L$. By Condition (8) of corollary 3.16 $d(\xi(\tau), \xi(\tau_0)) \leqslant M$, it follows, by the continuity of the dependence of the visual distance with the positive triple that there exists constants C and D only depending on A and B, so that

$$C^{-1} d_\tau \leqslant d_{\tau_0} \leqslant C d_\tau \quad , \quad D^{-1} d_{\xi(\tau)} \leqslant d_{\xi(\tau_0)} \leqslant D d_{\xi(\tau)} .$$

Thus for ε as above and x, y, τ_0 so that $d_{\tau_0}(x, y) \leqslant \varepsilon$ we have

$$d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leqslant M_2 d_{\tau_0}(x, y)^\alpha .$$

with $M_2 = M_1 C^\alpha D$. We complete the proof of the theorem by using lemma 3.19. □

3.5.3. *The equivariant case.* In this paragraph, we explain the relation between quasisymmetric maps and the theory of maximal representations of surface groups into G.

Let Γ be a cocompact Fuchsian subgroup of $\mathsf{PSL}(V)$ and ρ a morphism from Γ to G. We first have

Proposition 3.21. *Any positive continuous equivariant map is quasisymmetric.*

Proof. This is a direct consequence of the cocompactness of the action of Γ on the set of positive quadruples in $\mathbf{P}(V)$ whose cross-ratios are in $[A^{-1}, A]$ for $A > 1$. □

Since G is of Hermitian type we can define as in [BIW10, BGPG03] maximal representations. From [BILW05, Corollary 6.3], ρ is maximal if and only there exists an equivariant positive continuous map $\xi : \mathbf{P}(V) \to \partial_\infty \mathbf{H}^{2,n}$. As a corollary

Proposition 3.22. *Let ρ be a representation from Γ into G. There exists a ρ-equivariant quasisymmetric map from $\mathbf{P}(V)$ into $\partial_\infty \mathbf{H}^{2,n}$ if and only if ρ is maximal.*

In particular, the space of G-orbits of quasisymmetric maps from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$ contains the space $\mathrm{Rep}^{\max}(\Gamma, \mathsf{G})$ of maximal representations of any cocompact Fuchsian group Γ.

4. Maximal surfaces in pseudo-hyperbolic spaces

We describe the pseudo-hyperbolic space $\mathbf{H}^{2,n}$ in paragraph 4.1, maximal surfaces in paragraph 4.2, the equations governing them in paragraph 4.3 and finally the space of all maximal surfaces in paragraph 4.4 with its compactness properties coming from [LTW20].

4.1. **The pseudo-hyperbolic space.** Recall that E is a $(n+3)$-dimensional real vector space equipped with a signature $(2, n+1)$ quadratic form $\mathbf{q}$. The *pseudo-hyperbolic space* is defined by

$$\mathbf{H}^{2,n} := \{x \in E , \ \mathbf{q}(x) = -1\}/\{\pm \mathrm{Id}\} .$$

We will in the sequel freely identify $\mathbf{H}^{2,n}$ with its projection in $\mathbf{P}(E)$. The space $\mathbf{H}^{2,n}$ is equipped with a natural pseudo-Riemannian metric $\mathbf{g}$ of signature $(2, n)$ and constant curvature -1. The group G acts transitively on $\mathbf{H}^{2,n}$ preserving this metric, turning $\mathbf{H}^{2,n}$ into a pseudo-Riemannian symmetric space of G. The boundary of $\mathbf{H}^{2,n}$ in $\mathbf{P}(E)$ is the Einstein Universe $\partial_\infty \mathbf{H}^{2,n}$.

4.1.1. *Pointed hyperbolic plane.* Given a 3-dimensional linear subspace F of E of signature $(2, 1)$, the projective plane $\mathbf{P}(F)$ intersects $\mathbf{H}^{2,n}$ in a totally geodesic subspace isometric to the hyperbolic disk, called a *hyperbolic plane*.

A *pointed hyperbolic plane* is then a pair $\mathsf{P} = (q, H)$ where H is a hyperbolic plane and q a point in H. Equivalently, a pointed hyperbolic plane corresponds to an orthogonal decomposition $E = q \oplus U \oplus W$ where U is a positive definite 2-plane such that $H = \mathbf{P}(q \oplus U) \cap \mathbf{H}^{2,n}$ and q is a line.

In [LTW20, Section 3.1] given any pointed hyperbolic plane $\mathsf{P} = (q, H)$ we showed the existence of a *warped projection* π_{P} from $\mathbf{H}^{2,n}$ to H with the following properties

(i) The projection is natural: $\pi_{gP}(gx) = g\pi_P(x)$ for any element g in G.
(ii) For any z in H, $\pi_P(z) = z$.
(iii) The preimage of a point is diffeomorphic to a projective space of dimension n and has type $(0, n)$.

The map sending a pointed hyperbolic plane $\mathsf{P} = (q, H)$ to $\mathsf{T}_q H$ gives an isomorphism between the space of pointed hyperbolic planes and the set $\mathcal{G}(\mathbf{H}^{2,n})$ of pairs (x, P) where $x \in \mathbf{H}^{2,n}$ and P is a positive definite 2-plane in $\mathsf{T}_x \mathbf{H}^{2,n}$. Observe that $\mathcal{G}(\mathbf{H}^{2,n})$ is a G-homogeneous space and the stabiliser of a point in $\mathcal{G}(\mathbf{H}^{2,n})$ is compact. In particular, $\mathcal{G}(\mathbf{H}^{2,n})$ carries a natural Riemannian metric whose associated distance will be denoted by $d_{\mathcal{G}}$ (see paragraph 2.1.1). The map $(x, \mathbf{P}(F)) \mapsto x^{\perp} \cap F$ defines a Riemannian fibration from $\mathcal{G}(\mathbf{H}^{2,n})$ to $\mathrm{Gr}_{2,0}(E)$, the symmetric space of G.

Finally, note that if τ is a positive triple in $\partial_{\infty} \mathbf{H}^{2,n}$, then there exists a unique pointed hyperbolic plane $\mathsf{P}_{\tau} = (q_{\tau}, H_{\tau})$ such that H_{τ} is the hyperbolic plane containing τ in its boundary and q_{τ} is the barycenter of the triple, seen as the ideal vertices of a triangle. The map $\tau \mapsto (q_{\tau}, H_{\tau})$ defines a proper G-equivariant map from the set $\mathcal{T}(n)$ of positive triples in $\partial_{\infty} \mathbf{H}^{2,n}$ to $\mathcal{G}(\mathbf{H}^{2,n})$.

4.1.2. *Spatial distance.* Let x, y be two distinct points in $\mathbf{H}^{2,n}$. The complete geodesic between x and y is given by the intersection of $\mathbf{H}^{2,n}$ and the projective line $\mathbf{P}(x \oplus y)$. We call the pair (x, y) *acausal* if the geodesic between them is spacelike, meaning that the induced metric is positive definite. This happens exactly when $\mathbf{q}$ restricts to a signature $(1, 1)$ quadratic form on $x \oplus y$. The following was proved in [GM16][Proposition 3.2]

Lemma 4.1. *Let x, y be two distinct points in $\mathbf{H}^{2,n}$ and x_0, y_0 be vectors in x and y respectively satisfying $\mathbf{q}(x_0) = \mathbf{q}(y_0) = -1$. The pair (x, y) is acausal if and only if $|\langle x_0, y_0 \rangle|$ is greater than 1. Moreover, in this case, the length of the geodesic between x and y is equal to $\cosh^{-1}(|\langle x_0, y_0 \rangle|)$.*

Then the *spatial distance* is the map $\eth$ from $\mathbf{H}^{2,n} \times \mathbf{H}^{2,n}$ to $\mathbb{R}$ so that

$$\eth(x, y) = \cosh^{-1}(|\langle x_0, y_0 \rangle|) , \tag{10}$$

if the pair (x, y) is acausal and 0 otherwise. By lemma , $\eth$ restricts to the usual hyperbolic distance on any hyperbolic plane but we will see that $\eth$ fails to satisfy the triangular inequality.

4.1.3. *Horofunctions.* Given a non-zero isotropic vector z_0 in E, we define the associated *horofunction*, on $\mathbf{H}^{2,n} \setminus \mathbf{P}(z_0^{\perp})$ is given by

$$h_{z_0}(x) = \log |\langle x_0, z_0 \rangle| ,$$

where x_0 is a vector in x with $\mathrm{q}(x_0) = -1$. Rescaling z_0 by a non-zero multiplicative constant changes h_{z_0} by an additive constant. In particular, the gradient of h_{z_0} only depends on the class of z_0 in $\partial_{\infty} \mathbf{H}^{2,n}$. Observer that, given a point $x \in \mathbf{H}^{2,n} \setminus \mathbf{P}(z_0^{\perp})$, the gradient of h_{z_0} is given by

$$(\nabla h_{z_0})_x = \frac{\pi(z_0)}{\langle x_0, z_0 \rangle} , \tag{11}$$

where π is the orthogonal projection on $\mathsf{T}_x \mathbf{H}^{2,n}$ and x_0 is such that $\mathbf{q}(x_0) = -1$ and $\langle x_0, z_0 \rangle > 0$.

When $n = 0$, the function h_{z_0} is the classical Busemann function of hyperbolic geometry.

4.2. **Spacelike surfaces in $\mathbf{H}^{2,n}$.** Recall that a *spacelike surface* in $\mathbf{H}^{2,n}$ is an immersion of a connected 2-dimensional manifold Σ (without boundary) into $\mathbf{H}^{2,n}$ whose induced metric is positive definite. Such a surface is called *complete* if the induced metric is complete.

The following is proved in [LTW20, Proposition 3.10]

Proposition 4.2. *Let Σ be a complete spacelike surface in $\mathbf{H}^{2,n}$. Then*

(i) *any pair of distinct points on Σ is acausal.*
(ii) *The boundary $\partial_{\infty} \Sigma$ of Σ in $\mathbf{P}(E)$ is a semi-positive loop contained in $\partial_{\infty} \mathbf{H}^{2,n}$.*

As a result, if Σ is a complete spacelike surface in $\mathbf{H}^{2,n}$, the restriction $\eth_\Sigma$ of the spatial distance to Σ is non-zero on any pair of distinct points. Observe however that in general $\eth_\Sigma$ fails to be a distance: the triangle inequality fails on triples (x, y, z) such that $x \oplus y \oplus z$ has signature $(1, 2)$.

Another important aspect of complete spacelike surfaces in $\mathbf{H}^{2,n}$ is the fact that they are entire graphs. More specifically, for any pointed hyperbolic plane $\mathsf{P} = (q, H)$ and complete maximal surface Σ, the warped projection π_P restricts to a diffeomorphism from Σ to H. We refer to [LTW20, Section 3.1] for more details. As a corollary

Corollary 4.3. *Let Σ be a complete spacelike surface and $\mathsf{P} = (q, H)$ be a pointed hyperbolic plane. There exists a unique point in Σ whose warped projection on H is equal to q.*

Observe that in particular, a complete spacelike surface is always properly embedded. In what follows, we will always assume that our spacelike surfaces are embedded.

4.2.1. *Second fundamental forms and shape operators.* Let Σ be a spacelike surface in $\mathbf{H}^{2,n}$. The *normal bundle* $\mathsf{N}\Sigma$ of Σ is the orthogonal to $\mathsf{T}\Sigma$. The Levi-Civita connection ∇ of $\mathbf{H}^{2,n}$ decomposes in the splitting $\mathsf{T}\mathbf{H}^{2,n} = \mathsf{T}\Sigma \oplus \mathsf{N}\Sigma$ as

$$\nabla = \begin{pmatrix} \nabla^T & B \\ \mathrm{II} & \nabla^N \end{pmatrix} . \tag{12}$$

The forms II, seen as an element of $\Omega^1(\Sigma, \mathrm{Hom}(\mathsf{T}\Sigma, \mathsf{N}\Sigma))$, and B, seen as an element of $\Omega^1(\Sigma, \mathrm{Hom}(\mathsf{N}\Sigma, \mathsf{T}\Sigma))$, are called the *second fundamental form* and the *shape operator* respectively. The second fundamental form is symmetric, that is satisfies $\mathrm{II}(x, y) = \mathrm{II}(y, x)$ for any vectors x, y of $\mathsf{T}\Sigma$. In particular, II can be considered as a section of $\mathrm{Sym}^2(\mathsf{T}^*\Sigma) \otimes \mathsf{N}\Sigma$.

The connections ∇^T and ∇^N are unitary with respect to the induced metrics on $\mathsf{T}\Sigma$ and $\mathsf{N}\Sigma$ respectively. Finally, if x, y are sections of $\mathsf{T}\Sigma$ and ξ is a section of $\mathsf{N}\Sigma$, then derivating $\langle y, \xi \rangle = 0$ in the direction x gives the classical relation

$$\langle \mathrm{II}(x, y), \xi \rangle = -\langle y, B(x, \xi) \rangle . \tag{13}$$

4.3. **Maximal surfaces and their fundamental equations.** Below is some standard material that we recall for completeness and to emphasise our sign conventions.

Definition 4.4. A spacelike surface Σ in $\mathbf{H}^{2,n}$ is called *maximal* if $\mathrm{tr}_{g_\mathrm{I}} \mathrm{II} = 0$ where g_I is the induced metric on Σ.

Maximal surfaces are critical points for the area functional (see [LTW20, Section 3.3] for more details).

Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$ and fix an orthonormal framing (e_1, e_2) of $\mathsf{T}\Sigma$. Define the norm of the second fundamental form as

$$\|\mathrm{II}\|^2 = -\sum_{i=1}^{2} \langle \mathrm{II}(e_i, e_j), \mathrm{II}(e_i, e_j) \rangle ,$$

and observe that since the restriction of $\mathbf{g}$ is negative definite on $\mathsf{N}\Sigma$, we have $\|\mathrm{II}\|^2 \geqslant 0$.

Proposition 4.5. [GAUSS EQUATION] *Using the above notations, if Σ is a complete maximal surface in $\mathbf{H}^{2,n}$ and K_Σ is its intrinsic curvature, then*

$$K_\Sigma = -1 + \frac{1}{2}\|\mathrm{II}\|^2 . \tag{14}$$

Proof. Using the decomposition of the Levi-Civita connection ∇ of $\mathbf{H}^{2,n}$ given in equation (12), we have

$$\langle \nabla_x \nabla_y z, t \rangle = \langle \nabla_x^T \nabla_y^T z, t \rangle + \langle B(x, \mathrm{II}(y, z)), t \rangle ,$$

where x, y, z and t are sections of $\mathsf{T}\Sigma$. Using the convention

$$R(a, b)c = \nabla_a \nabla_b c - \nabla_b \nabla_a c - \nabla_{[a,b]} c ,$$

and equation (13), we get

$$\langle R(x,y)z,t \rangle = \langle R^T(x,y)z,t \rangle - \langle \mathrm{II}(y,z), \mathrm{II}(x,t) \rangle + \langle \mathrm{II}(x,z), \mathrm{II}(y,t) \rangle \, ,$$

where R^T is the curvature tensor of ∇^T. The result then follows from applying the equation to $x = t = e_1$ and $y = z = e_2$, using $\mathrm{II}(e_1,e_1) = -\mathrm{II}(e_2,e_2)$ and the fact that ∇ has sectional curvature -1. $\qquad\square$

Denote by ω the area form of Σ (with its induced metric). Observe first that there exists a unique endomorphism φ of $\mathsf{N}\Sigma$ so that for any U and V in $\mathsf{N}\Sigma$, u and v in $\mathsf{T}\Sigma$,

$$\langle B(v,U), B(u,V) \rangle - \langle B(u,U), B(v,V) \rangle = \omega(u,v)\langle \varphi(U), V \rangle \, . \tag{15}$$

Proposition 4.6. [RICCI EQUATION] *Using the above notations, the curvature tensor R_N of ∇^N is given by*

$$R_N = \omega \otimes \varphi \, , \tag{16}$$

Moreover if (e_1,e_2) is an orthonormal basis of $\mathsf{T}\Sigma$ and α and β are the sections of $\mathsf{N}\Sigma$ defined by

$$\alpha := \mathrm{II}(e_1,e_1) \, , \quad \beta := \mathrm{II}(e_1,e_2) \, ,$$

then

$$\langle \varphi(\alpha), \beta \rangle = -2\|\alpha \wedge \beta\|^2 := -2\left(\|\alpha\|^2 \|\beta\|^2 - \langle \alpha, \beta \rangle^2 \right) \, . \tag{17}$$

Proof. Given u and v sections of $\mathsf{T}\Sigma$, equation (12) gives

$$\langle \nabla_u \nabla_v U, V \rangle = \langle \nabla_u^N \nabla_v^N U, V \rangle + \langle \mathrm{II}(u, B(v,U)), V \rangle = \langle \nabla_u^N \nabla_v^N U, V \rangle - \langle B(v,U), B(u,V) \rangle \, .$$

This gives Ricci equation:

$$\begin{aligned}
0 &= \langle R(u,v)U, V \rangle \\
&= \langle R_N(u,v)U, V \rangle - \langle B(v,U), B(u,V) \rangle + \langle B(u,U), B(v,V) \rangle \\
&= \langle R_N(u,v)U, V \rangle - \omega(u,v)\langle \varphi(U), V \rangle \, .
\end{aligned}$$

The second statement follows from using the relation

$$B(e_1, \alpha) = \sum_{i=1}^2 \langle B(e_1,\alpha), e_i \rangle \cdot e_i = -\sum_{i=1}^2 \langle \alpha, \mathrm{II}(e_1, e_i) \rangle \cdot e_i = -\langle \alpha, \alpha \rangle \cdot e_1 - \langle \alpha, \beta \rangle \cdot e_2 \, ,$$

and similar formulas for the other terms. $\qquad\square$

For the last equation, we denote by D the connection on $\mathrm{Hom}(\mathsf{T}\Sigma, \mathsf{N}\Sigma)$ induced by ∇^T and ∇^N. The *D-exterior derivative* is the map

$$\mathrm{d}^D \, : \, \Omega^1(\Sigma, \mathrm{Hom}(\mathsf{T}\Sigma, \mathsf{N}\Sigma)) \to \Omega^2(\Sigma, \mathrm{Hom}(\mathsf{T}\Sigma, \mathsf{N}\Sigma)) \, ,$$

defined by

$$\mathrm{d}^D \theta(x,y) := D_x(\theta(y)) - D_y(\theta(x)) - \theta([x,y]) \, .$$

Proposition 4.7. [CODAZZI EQUATION] *Using the same notations as above, seeing II as an element of $\Omega^1(\Sigma, \mathrm{Hom}(\mathsf{T}\Sigma, \mathsf{N}\Sigma))$, we have $\mathrm{d}^D \mathrm{II} = 0$.*

Proof. This is obtained using that $\langle R(x,y)z, \xi \rangle = 0$ for any sections x, y, z of $\mathsf{T}\Sigma$ and ξ of $\mathsf{N}\Sigma$, together with the formula

$$\langle \nabla_x \nabla_y z, \xi \rangle = \left\langle \mathrm{II}(x, \nabla_y^T z), \xi \right\rangle + \left\langle \nabla_x^N(\mathrm{II}(y,z)), \xi \right\rangle \, . \qquad\square$$

4.4. **The space of all maximal surfaces.** We denote by $\mathcal{M}(n)$ the set of all pointed complete maximal surfaces in $\mathbf{H}^{2,n}$

$$\mathcal{M}(n) := \{(x, \Sigma) \mid x \in \Sigma, \ \Sigma \text{ is maximal complete}\} \, . \tag{18}$$

Convergence of pointed spacelike surfaces on every compact induces a natural topology on those spaces (see [LTW20, Appendix A] for more details).

4.4.1. *Cocompact actions.* Let $\mathcal{G}(\mathbf{H}^{2,n})$ be the Riemannian manifold defined in Paragraph 4.1. As a consequence of results in [LTW20], we have

Theorem 4.8. [Compactness Theorem] *The map from $\mathcal{M}(n)$ to $\mathcal{G}(\mathbf{H}^{2,n})$ sending (x, Σ) to $\mathsf{T}_x\Sigma$ is proper. The group G acts properly, continuously and cocompactly on $\mathcal{M}(n)$.*

The first assertion is a rewriting of the first item in [LTW20, Theorem 6.1] while the second assertion is an immediate application of the first using the fact that G acts properly and transitively on $\mathcal{G}(\mathbf{H}^{2,n})$.

Finally, we remark the cocompactness of the action of G on some spaces related to $\mathcal{M}(n)$, again as a consequence of [LTW20, Theorem 6.1].

Corollary 4.9. *The group G acts cocompactly and continuously on the spaces $\mathcal{N}_p(n)$ given by*

$$\mathcal{N}_p(n) := \{(x, \Sigma, z_1, \ldots, z_p), \text{ with } x \in \Sigma , \ z_i \in \Sigma \cup \partial_\infty \Sigma\} .$$

Proof. Let consider the compactification $\overline{\mathbf{H}}^{2,n} := \mathbf{H}^{2,n} \cup \partial_\infty \mathbf{H}^{2,n}$ of $\mathbf{H}^{2,n}$ in the projective model. Observe first that the diagonal action of G on $\mathcal{M}(n) \times \left(\overline{\mathbf{H}}^{2,n}\right)^p$ is cocompact, since the action of G on $\mathcal{M}(n)$ is cocompact. Thus the corollary follows from the third item in [LTW20, Theorem 6.1] which guarantees that $\mathcal{N}_p(n)$ is a closed set in $\mathcal{M}(n) \times \left(\overline{\mathbf{H}}^{2,n}\right)^p$. $\qquad\qquad \square$

4.4.2. *Laminated space.* We end this section by describing a natural product structure on $\mathcal{M}(n)$ that turns it into a laminated space, or a Riemann surface lamination – see appendix in [Sul92]. This structure will be comes from the graph property of complete maximal surfaces (see Section 4.2).

Fix a pointed hyperbolic plane $\mathsf{P} = (q, H)$ and recall that any complete maximal surface is a graph above H with respect to the warped projection π from $\mathbf{H}^{2,n}$ to H. Let

$$\mathcal{M}_0 = \left\{(y, \Sigma) \in \mathcal{M}(n) \mid \pi(y) = q\right\} .$$

Then the map from $\mathcal{M}(n)$ to $\mathcal{M}_0 \times H$ so that the image of (x, Σ) is $((y, \Sigma), \pi(x))$ – where y is the unique point of Σ projecting to q – is a homeomorphism. Moreover, leaves in $\mathcal{M}(n)$ correspond to sets of the form $\{*\} \times H$, in other words maximal surfaces.

We may thus consider $\mathcal{M}(n)$ as a *laminated space* whose leaves identify with maximal surfaces. Observe also that the natural G action on $\mathcal{M}(n)$ preserves the leaves. In the sequel, we will freely identify maximal complete surface Σ with the leaf it corresponds to in $\mathcal{M}(n)$.

4.5. **Loops and surfaces.** The next theorem is the main result of [LTW20, Theorem B]:

Theorem 4.10. [Asymptotic Plateau problem] *Any semi-positive loop Λ in $\partial_\infty \mathbf{H}^{2,n}$ bounds a unique complete maximal surface $\Sigma(\Lambda)$ in $\mathbf{H}^{2,n}$.*

We will now extend this correspondence to a map from the space $\mathcal{L}(n)$ of pointed semi-positive loops to $\mathcal{M}(n)$. Recall from section 4.1 that any positive triple τ defines a pointed hyperbolic plane $\mathsf{P}_\tau = (q_\tau, H_\tau)$. Moreover, by Corollary 4.3, if Σ is a complete maximal surface, there is a unique point b_τ on Σ that projects to q_τ. This defines

Definition 4.11. [Barycenter map] *Using the above notations, the* barycenter map *from $\mathcal{L}(n)$ to $\mathcal{M}(n)$ is the map B which associates to (Λ, τ) the pointed maximal surface $(b_\tau, \Sigma(\Lambda))$.*

We now prove

Proposition 4.12. [Barycenter map] *The barycenter map is G-equivariant, continuous, proper and surjective.*

First step: proof of G-*equivariance, continuity and properness.* The G-equivariance is obvious and the continuity is a direct consequence of the second item in Theorem [LTW20, Theorem 6.1]. It follows that the induced map B_0 from $\mathcal{L}(n)/G$ to $\mathcal{M}(n)/G$ is continuous. The fact that the map B is proper is now a consequence of some general topology argument, using the following facts

    (i) The group G acts properly on $\mathcal{L}(n)$ and $\mathcal{M}(n)$.
    (ii) The map B_0 from $\mathcal{L}(n)/G$ to $\mathcal{M}/G$ is continuous,
    (iii) The spaces $\mathcal{L}(n)/G$ to $\mathcal{M}(n)/G$ are compact,
    (iv) The projection from π_0 from $\mathcal{L}(n)_{\tau_0}$ to $\mathcal{L}(n)/G$ is proper (see proposition 2.32).

Let us show then that B is proper: let K be a compact in $\mathcal{M}(n)$, let K_1 be the projection of K in $\mathcal{M}(n)/G$, let K_2 its preimage in $\mathcal{L}(n)/G$ and $K_3 = B(\pi_0^{-1}(K_2))$. We have that K_1 is compact by continuity, K_2 is compact since it is a closed set (by continuity of B_0) in the compact space $\mathcal{L}(n)/G$, and finally K_3 is compact by properness and continuity.

Then by properness of the action of G

$$K_4 := \{g \in G \mid gK_3 \cap K \neq \emptyset\},$$

is compact. Then one sees that $B^{-1}(K)$ is a closed subset of $K_4 \cdot K$, and is therefore compact. □

Second step: proof of surjectivity. Assume first that Λ is positive. Then the set T_Λ of positive triples contained in Λ identifies with the set of pairwise distinct triples of points in Λ. The boundary ∂T_Λ of T_Λ in Λ^3 is made of triples such that at least two of the elements of the triple are equal. We have a projection π from ∂T_Λ to Λ given by

$$\pi(t,s,s) = \pi(s,t,s) = \pi(s,s,t) = s.$$

Let us now prove

Lemma 4.13. *Let Λ be a positive loop and $\{\tau_k\}_{k\in\mathbb{N}}$ a sequence in T_Λ converging to τ_∞ in ∂T_Λ. Then the sequence $\{q_{\tau_k}\}_{k\in\mathbb{N}}$ of barycenters in $\mathbf{H}^{2,n}$ converges to $\pi(\tau_\infty)$ in $\partial_\infty \mathbf{H}^{2,n}$.*

Proof. Let p be a vector in E such that $\mathbf{P}(p^\perp)$ is disjoint from Λ. The existence of p is guaranteed by [LTW20, Proposition 2.15 (iii)]. Let $F := p^\perp$. Then in the splitting

$$E = F \oplus \mathbb{R}p,$$

any point in Λ has the form $[(x,1)]$ for a unique x in F. Let $\tau = (t^1, t^2, t^3)$ be a positive triple in Λ. We write write $t^i = [u^i] \in \mathbf{P}(E)$ where $u^i = (s^i, 1) \in E$ with s^i in F. Observe that $\langle u^i, u^i \rangle = 0$ while $\langle u^i, u^{i+1} \rangle > 0$. Let

$$\lambda^i = \sqrt{\frac{\langle u^j, u^k \rangle}{\langle u^i, u^j \rangle \langle u^i, u^k \rangle}}, \text{ where } (i,j,k) \text{ are pairwise distinct.}$$

We claim that the barycenter q_τ of τ is the line spanned by

$$v := \lambda^1 u^1 + \lambda^2 u^2 + \lambda^3 u^3.$$

In fact, the λ^i are defined in such a way that $\langle \lambda^i u^i, \lambda^j u^j \rangle = 1$ for any $i \neq j$. So for (i,j,k) pairwise distinct, the line δ between t^i and $[v]$ intersects the geodesic γ between t^j and t^k along $x = [u^j + u^k]$, and $\mathsf{T}_x \gamma = [u^j - u^k]$ is orthogonal to $\mathsf{T}_x \delta = [-t^i + t^j + t^k]$. So $[v] = q_\tau$.

Observe that for (i,j,k) pairwise distinct, we have

$$\frac{\lambda^i}{\lambda^j} = \frac{\langle u^j, u^k \rangle}{\langle u^i, u^k \rangle}.$$

Assume now that we have a sequence $\{\tau_k\}_{k\in\mathbb{N}}$, where $\tau_k = (t^1_k, t^2_k, t^3_k)$ where

$$\lim_{k\to\infty} t^1_k = \lim_{k\to\infty} t^2_k = t_\infty = [(s_\infty, 1)] \neq w_\infty = \lim_{k\to\infty} t^3_k.$$

Then
$$\lim_{k\to\infty} \langle u_k^1, u_k^2 \rangle = 0 \ , \ \lim_{k\to\infty} \langle u_k^2, u_k^3 \rangle = a \neq 0 \ , \ \lim_{k\to\infty} \langle u_k^1, u_k^3 \rangle = b \neq 0 \ .$$

Thus
$$\lim_{k\to\infty} \lambda_k^3 = 0 \ , \ \lim_{k\to\infty} \lambda_k^2 = \lim \lambda_k^1 = \infty \text{ and } \lim_{k\to\infty} \frac{\lambda_k^2}{\lambda_k^1} = \frac{a}{b} \neq 0 \ .$$

Then the barycenter
$$q_{\tau_k} = \left[\left(\frac{\lambda_k^1 s_k^1 + \lambda_k^2 s_k^2 + \lambda_k^3 s_k^3}{\lambda_k^1 + \lambda_k^2 + \lambda_k^3}, 1 \right) \right] \ ,$$

converges to $t_\infty = [(s_\infty, 1)]$.

The case when $\left\{ t_k^1 \right\}_{k\in\mathbb{N}}$, $\left\{ t_k^2 \right\}_{k\in\mathbb{N}}$ and $\left\{ t_k^3 \right\}_{k\in\mathbb{N}}$ converge to the same point t_∞ is simpler: in this case, the sequence of ideal hyperbolic triangles with vertices t_k^1, t_k^2 and t_k^3 converges to the point t_∞. Since any barycenter b_{τ_k} is contained in the ideal triangle, the result follows. □

We obtain the following corollary:

Corollary 4.14. *Assume Λ is positive. If $\{\tau_k\}_{k\in\mathbb{N}}$ is a sequence in T_Λ converging to τ_∞ in ∂T_Λ, then the sequence $\{b_{\tau_k}\}_{k\in\mathbb{N}}$ in $\Sigma(\Lambda)$ converges to $\pi(\tau_\infty)$.*

Proof. Since we already know that the barycenter map is proper, the sequence $\{b_{\tau_k}\}_{k\in\mathbb{N}}$ leaves every compact set. Let us extract a subsequence so that $\{b_{\tau_k}\}_{k\in\mathbb{N}}$ accumulates at a point y in Λ. Assume that y and $\pi(\tau_\infty)$ are distinct.

Recall that the geodesic joining q_{τ_k} and b_{τ_k} is timelike. Going to the limit, we obtain a timelike or lightlike geodesic joining y to $\pi(\tau_\infty)$. However, Λ is positive, and thus any two distinct points are joined by a spacelike geodesic. This implies that $y = \pi(\tau_\infty)$. □

We now prove the surjectivity of B from T_Λ to $\Sigma(\Lambda)$ when Λ is positive: as a consequence of the previous corollary the barycenter map extends continuously to a map from Λ^3 to $\Sigma(\Lambda) \cup \Lambda$. Fixing x_0 in Λ, let
$$D := \{(x_0, y, z) \mid (x_0, y, z) \text{ is positively oriented}\} \ .$$
The set D is an open disk: indeed let us identify $\mathbf{P}(V) \setminus \{x_0\}$ with $\mathbb{R}$, then
$$D = \{x_0\} \times U \ , \ \ U := \{(x, y) \in \mathbb{R}^2 \mid x > y\} \ .$$

Moreover the injection of D in T_Λ extends to a continuous map f from ∂D to ∂T_λ sending a closed interval I to x_0, and is an homeomorphism from the complementary of I to the complementary of x_0. It follows that the barycenter map restricted to ∂D has degree 1 as a map with values in Λ. Thus the barycenter map from D with values in $\Sigma(\Lambda)$ is surjective.

To prove in general that B is surjective, we first observe that its image is closed since B is continuous and proper. Secondly, the image is dense: by our first discussion, it contains all pairs (x, Σ) so that the boundary of Σ is positive. □

4.6. **Barbot surfaces.** A *Barbot surface* is a maximal surface whose boundary at infinity is a Barbot crown. Some of the results explained here can also be found in [BS10] (where Barbot surfaces are called horospheres). Let us start with an explicit description of a Barbot surface. Let z_0, z_1, z_2, z_3 be a quadruple of vectors so that

$$\langle z_i, z_i \rangle = \langle z_i, z_{i+1} \rangle = 0 \ , \ \langle z_i, z_{i+2} \rangle = -\frac{1}{4} \ . \tag{19}$$

In particular if v_i are the lines generated by z_i, then (v_0, v_1, v_2, v_3) are the vertices of a Barbot crown C. Let also A_C be the abelian subgroup defined by equation (2). Then we have

Proposition 4.15. *The Barbot surface for a barbot crown C is the orbit $V := A_C \cdot x$, where $x = z_0 + z_1 + z_2 + z_3$.*

Proof. Observe first that $\langle x, x \rangle = -1$. Let us now consider the following curves which are drawn on V.

$$\alpha(t) := e^t z_0 + z_1 + e^{-t} z_2 + z_3 \quad , \quad \beta(t) := z_0 + e^t z_1 + z_2 + e^{-t} z_3 .$$

Then

$$\alpha(0) = x , \qquad \dot{\alpha}(0) = z_0 - z_2 , \qquad \ddot{\alpha}(0) = z_0 + z_2 ,$$
$$\beta(0) = x , \qquad \dot{\beta}(0) = z_1 - z_3 , \qquad \ddot{\alpha}(0) = z_1 + z_3 ,$$
$$\langle \dot{\alpha}(0), \dot{\beta}(0) \rangle = 0 , \qquad \ddot{\alpha}(0) + \ddot{\beta}(0) = x , \qquad \|\dot{\alpha}(0)\|^2 = \|\dot{\beta}(0)\|^2 = \frac{1}{2} .$$

Denote by H the mean curvature vector. Then

$$\frac{1}{2} H = \text{II}(\dot{\alpha}(0), \dot{\alpha}(0)) + \text{II}(\dot{\beta}(0), \dot{\beta}(0)) ,$$

is the normal part of $\ddot{\alpha}(0) + \ddot{\beta}(0)$. It follows that $H = 0$, hence by homogeneity that V is a maximal surface. It is then easy to show that the boundary of V in the projective model is the Barbot crown C. $\qquad \square$

The following summarises some properties of Barbot surfaces.

Proposition 4.16. *We have*
   (i) *A Barbot surface is an orbit of a Cartan subgroup in* G.
   (ii) *All Barbot surfaces are equivalent under the action of* G.
   (iii) *The induced metric on a Barbot surface is flat.*
   (iv) *At any point of a Barbot surface, there exists a horofunction whose square of the norm of the gradient is* 2.
   (v) *Let* Σ *be a Barbot surface,* d_Σ *the induced distance on* Σ *and* $\eth$ *the spatial distance defined in equation* (10), *then*

$$\sup_{x,y \in \Sigma} \left(\frac{\eth(x,y)}{d_\Sigma(x,y)} \right) = \sqrt{2} . \tag{20}$$

As a consequence of the first two items, Barbot surfaces define a unique point in $\mathcal{M}(n)/\mathsf{G}$ that we call the *Barbot point* and denote m_0.

Proof. The first item follows by Proposition 4.15. The second follows from the fact that Cartan subgroups are conjugate. The third one from the fact that it is a transitive orbit of $\mathbb{R}^2$ acting by isometries.

For the fourth item, let us consider the horofunction h associated to the lightlike vector $Z := z_0$. Then

$$h(\alpha(t)) = -t + \log\left(\frac{1}{4}\right) , \quad h(\beta(t)) = \log\left(\frac{1}{4}\right) .$$

Thus $\langle \nabla h, \dot{\alpha}(0) \rangle = -1$ and $\langle \nabla h, \dot{\beta}(0) \rangle = 0$. It follows that $\nabla h = -2\dot{\alpha}(0)$ and

$$\|\nabla h\|^2 = 2 .$$

For the last item, by the transitive action of $\mathbb{R}^2$, it suffices to prove the result for a fixed x. Let λ, μ be two constants and define

$$\gamma(t) = e^{\lambda t} z_0 + e^{\mu t} z_1 + e^{-\lambda t} z_2 + e^{-\mu t} z_3 .$$

Observe that $\gamma(t)$ is a geodesic. Let us assume, without loss of generality that $\lambda = \max\{\pm\lambda, \pm\mu\}$. Since $\mathbf{q}\left(\dot{\gamma}(t)\right) = \frac{1}{2}(\lambda^2 + \mu^2)$ we have

$$d_\Sigma(\gamma(0), \gamma(t)) = t \sqrt{\frac{\lambda^2 + \mu^2}{2}} .$$

Moreover we have

$$|\langle \gamma(0), \gamma(t) \rangle| = \frac{1}{2} \left(\cosh(\lambda t) + \cosh(\mu t) \right) \ .$$

It follows that the function

$$\frac{\eth(\gamma(0), \gamma(t))}{d_\Sigma(\gamma(0), \gamma(t))} = \frac{\cosh^{-1} |\langle \gamma(0), \gamma(t) \rangle|}{d_\Sigma(\gamma(0), \gamma(t))} \ ,$$

is increasing and its limit as t goes to infinity is equal to $\sqrt{\frac{2}{1 + \frac{\mu^2}{\lambda^2}}}$. The result follows. □

The following characterisation of Barbot surfaces from [BS10, Lemma 5.5] will be used

Lemma 4.17. [Bonsante–Schlenker] *If Σ is a complete flat maximal surface in $\mathbf{H}^{2,1}$, then Σ is a Barbot surface.*

4.7. **Periodic and quasiperiodic surfaces.** We now follow a path similar to the one we followed in section 2.7 for loops in the Einstein Universe.

Definition 4.18. Let Σ be a complete maximal surface. We say

    (i) The surface Σ is *periodic* if it is invariant by a discrete subgroup of G isomorphic to the fundamental group of a closed surface of genus at least 2.
    (ii) The surface Σ is *quasiperiodic* if the closure of its G-orbit in $\mathcal{M}(n)$ does not contain a Barbot surface.

We have the following

Proposition 4.19. *Periodic surfaces are quasiperiodic.*

Proof. Observe that a maximal surface is quasiperiodic if and only if its image is precompact in $(\mathcal{M}(n)/\mathsf{G}) \setminus \{m_0\}$ where m_0 is the Barbot point. By definition, a periodic surface corresponds to a compact leaf in $\mathcal{M}(n)/\mathsf{G}$ which is different from b (since the only groups acting freely and cocompactly on a Barbot surface are abelian). The result follows. □

Remark 4.20. It was proved in [CTT19] that periodic surfaces are in one-to-one correspondence with maximal representations of (closed) surface groups in G.

As a corollary of proposition 4.12

Corollary 4.21. *A complete maximal surface is quasiperiodic if and only if its boundary at infinity is quasiperiodic.*

5. Rigidity Theorems

In this section we show that Barbot surfaces play a special role in the theory. Cheng [Cheng94] proved that

Theorem 5.1. [Cheng] *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$. Then the intrinsic curvature of Σ is non positive. If Σ is flat then Σ is a Barbot surface.*

As a consequence of this theorem and Gauß equation, we get an *a priori* bound on the norm of the second fundamental form.

Corollary 5.2. *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$. Then the norm of the second fundamental form Σ is no greater than 2.*

The corollary is an improvement over Ishihara bounds [Ish88] in which the norm of the second fundamental form by $2n$ in $\mathbf{H}^{2,n}$. Note that Ishihara's bounds are more general and optimal for maximal complete spacelike submanifolds of dimension p in $\mathbf{H}^{p,q}$ for $q < p$, where he obtains the bounds pq.

We improve the statement of Cheng to a pointwise rigidity result – which is not stated as such in [Cheng94] but should also follow by his techniques;

Proposition 5.3. [Curvature rigidity] *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$. Then if the intrinsic curvature Σ is zero at a point, it vanishes everywhere and Σ is a Barbot surface.*

Our proof relies on our compactness results rather than on Omori maximum principle as well as a holomorphic interpretation/Bochner formula which makes the computations in [Cheng94] more transparent. We have a similar corollary

Corollary 5.4. [Rigidity for the second fundamental form] *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$. Then if this norm is 2 at a point, it is 2 everywhere and Σ is a Barbot surface.*

We have a similar rigidity result in term of the norms of gradient of horofunctions and spatial distance. Observe that although the gradient of the spatial distance to a point z in Σ is not defined at z, we can extend continuously its norm by imposing the value 1 at z.

Theorem 5.5. [Horofunction and spatial distance rigidity theorem] *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$ with intrinsic curvature K_Σ. Let $c = \inf(-K_\Sigma)$ and let z be a point in $\Sigma \sqcup \partial_\infty \Sigma$. Let h_z be*

- *the horofunction associated to z, if z belongs to $\partial_\infty \Sigma$;*
- *the function spatial distance to z, if z belongs to Σ.*

Then

(i) *The square of the norm of the gradient of h_z is not less than 1 and no greater than $2 - c$.*
(ii) *If for some z, the square of the norm of the gradient of h_z is equal to 2 at a point, then Σ is a Barbot surface.*

Recall that by the Cheng's Theorem, c is non negative.

5.1. **Rigidity for the curvature.** We first translate the problem into holomorphic terms. Then in the next subsection, we use Bochner's formula of Proposition A.1 to prove proposition 5.3. Finally, in the last subsection, we prove Theorem 5.5.

5.1.1. *A holomorphic picture.* Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$ with induced metric g_T, normal bundle $(\mathsf{N}\Sigma, g_N)$ and second fundamental form II.

Denote by K the canonical bundle over Σ, that is, the holomorphic line bundle $\mathrm{Hom}_{\mathbb{C}}(\mathsf{T}^{1,0}\Sigma, \mathbb{C})$. Define the complex vector bundle

$$\mathsf{E} := \mathsf{K}^2 \otimes \mathsf{N}^{\mathbb{C}}\Sigma \,,$$

where $\mathsf{N}^{\mathbb{C}}\Sigma = \mathsf{N}\Sigma \otimes \mathbb{C}$.

The metrics g_T on $\mathsf{T}\Sigma$ and $(-g_N)$ on $\mathsf{N}\Sigma$ induce a positive definite metric g on the bundle $\mathrm{Sym}^2(\mathsf{T}^*\Sigma) \otimes \mathsf{N}\Sigma$. The Hermitian extension of g restricts to a Hermitian metric h on E.

Similarly, the connections ∇^T and ∇^N on $\mathsf{T}\Sigma$ and $\mathsf{N}\Sigma$ (see equation (12)) define a unitary connection ∇^E on (E, h) whose $(0,1)$-part is a Dolbeaut operator $\overline{\partial}_\mathsf{E}$ (see Appendix A for definitions). In this setting, we have the following:

Lemma 5.6. *Using the same notations as above, there is a holomorphic section σ of E such that the second fundamental form of Σ is the real part of σ. Moreover, the intrinsic curvature K_Σ of Σ satisfies*

$$K_\Sigma = -1 + \|\sigma\|_h^2 \,.$$

Proof. The complexification $\phi_{\mathbb{C}}$ of a section ϕ of $\mathrm{Sym}^2(\mathsf{T}^*\Sigma) \otimes \mathsf{N}\Sigma$ decomposes under types as

$$\phi_{\mathbb{C}} = \phi^{2,0} + \phi^{1,1} + \overline{\phi^{2,0}} \ .$$

Observe that $\phi^{2,0}$ is a section of E.

Let (e_1, e_2) be a local orthonormal framing of $\mathsf{T}\Sigma$. Let $\partial_z = \frac{1}{2}(e_1 - ie_2)$ the associated section of K^* and dz the section of K dual to ∂_z. We have

$$\phi^{1,1} = \phi(\partial_z, \bar{\partial}_z) \, \mathrm{d}z\mathrm{d}\bar{z} = \frac{1}{2} \, \mathrm{tr}_{g_T}(\phi)\mathrm{d}z\mathrm{d}\bar{z} \ .$$

Applying the previous decomposition to II, one sees that Σ is maximal if and only if $\mathrm{II}^{1,1} = 0$ or in other words denoting J the complex structure on Σ

$$\mathrm{II}(X, Y) + \mathrm{II}(JX, JY) = 0 \ .$$

In particular, II is the real part of the section $\sigma := 2\mathrm{II}^{2,0}$ of E or in other words for X and Y in $\mathsf{T}\Sigma$,

$$\sigma(X, Y) = \mathrm{II}(X, Y) - i\mathrm{II}(JX, Y) \ .$$

The Codazzi equation (proposition 4.7) says that

$$\mathrm{d}^D\mathrm{II} = 0 \ .$$

This last equation is equivalent to

$$\bar{\partial}_{\mathsf{E}}\sigma = 0 \ ,$$

the holomorphicity of σ. Indeed

$$
\begin{aligned}
\left(\bar{\partial}_{\mathsf{E}}\right)_X \sigma(X, Y) &= \nabla^E_{JX}\sigma(X, Y) - i\nabla^E_X\sigma(X, Y) \\
&= \nabla_{JX}\mathrm{II}(X, Y) - i\nabla_{JX}\mathrm{II}(JX, Y) - i\left(\nabla_X\mathrm{II}(X, Y) - i\nabla_X\mathrm{II}(JX, Y)\right) \\
&= \left(\nabla_{JX}\mathrm{II}(X, Y) - \nabla_X\mathrm{II}(JX, Y)\right) - i\left(\nabla_{JX}\mathrm{II}(JX, Y) + \nabla_X\mathrm{II}(X, Y)\right) \\
&= \mathrm{d}^D\mathrm{II}(JX, X)(Y) - i\mathrm{d}^D\mathrm{II}(JX, X)(JY) \ ,
\end{aligned}
$$

using in the last equation that $\mathrm{II}(JX, Y) = \mathrm{II}(X, JY)$.

Using a local frame (e_1, e_2) and defining $\alpha = \mathrm{II}(e_1, e_2)$ and $\beta = \mathrm{II}(e_1, e_2)$, we have

$$\sigma = 2\mathrm{II}(\partial_z, \partial_z)\mathrm{d}z^2 = (\alpha - i\beta)\mathrm{d}z^2 \ ,$$

$$\|\sigma\|_h^2 = -g_N(\alpha, \alpha) - g_N(\beta, \beta) = \frac{1}{2}\|\mathrm{II}\|^2 \ .$$

The result now follows from Gauß equation (proposition 4.5). $\qquad\square$

We now use Bochner formula to have the following corollary

Corollary 5.7. *Let Σ be a complete maximal surface. Assume that x in Σ is a maximum of the curvature. Then $K_\Sigma(x) \leqslant 0$. If furthermore at this point x, $K_\Sigma(x) = 0$, then Σ is a Barbot surface.*

Proof. We use the notation introduced in the previous paragraph. Let $f := \frac{1}{2}\|\sigma\|_h^2$. Bochner formula (see proposition A.1) gives

$$\Delta f = h(R_{\mathsf{E}}(e_1, e_2)\sigma, J\sigma) + \|\nabla^E\sigma\|_h^2 \ .$$

Let (e_1, e_2) be a local orthonormal frame of $\mathsf{T}\Sigma$. Let $\partial_z = \frac{1}{2}(e_1 - ie_2)$ be the local section of the complexified bundle of $\mathsf{T}\Sigma$. Let dz be the section of K dual to the section ∂_z. Let $\alpha = \mathrm{II}(e_1, e_1)$, $\beta = \mathrm{II}(e_1, e_2)$. Then we have

$$
\begin{aligned}
R_{\mathsf{E}}(e_1, e_2)\sigma &= R_{\mathsf{E}}(e_1, e_2)\left((\alpha - i\beta)\mathrm{d}z^2\right) \\
&= 2(\alpha - i\beta)\mathrm{d}z \otimes (R_{\mathsf{K}}(e_1, e_2)\mathrm{d}z) + \left(R_N(e_1, e_2)(\alpha - i\beta)\right)\mathrm{d}z^2 \ .
\end{aligned}
$$

We have $R_T(e_1, e_2)\partial_z = -iK_\Sigma \cdot \partial_z$ and hence $R_K(e_1, e_2)\mathrm{d}z = iK_\Sigma \cdot \mathrm{d}z$. On the other hand, if h_N is the Hermitian extension of the (positive definite) metric $(-g_N)$ to $N^{\mathbb{C}}\Sigma$, and using Ricci equation (proposition 4.6), we have

$$
\begin{aligned}
h_N\big(R_N(e_1, e_2)(\alpha - i\beta), i(\alpha - i\beta)\big) &= -\langle R_N(e_1, e_2)(\alpha - i\beta), \beta - i\alpha\rangle \\
&= -2\langle R_N(e_1, e_2)\alpha, \beta\rangle \quad = \quad 4\|\alpha \wedge \beta\|^2 \ .
\end{aligned}
$$

This gives, using lemma 5.6 for the first equality

$$
\frac{1}{2}\Delta K_\Sigma = \Delta f = 2K_\Sigma \|\sigma\|_h^2 + 4\|\alpha \wedge \beta\|^2 + \|\nabla^E \sigma\|_h^2 \geqslant 2K_\Sigma(1 + K_\Sigma) \ . \tag{21}
$$

At a maximum x of K_Σ, we have $\Delta K_\Sigma \leqslant 0$ and thus get $K_\Sigma(x) \leqslant 0$. This shows the first part of the corollary.

Assume now that $K_\Sigma(x) = 0$. Thus on a neighbourhood of x we have $\Delta K_\Sigma \geqslant 4K_\Sigma$. By the Strong Maximum Principle [GT01, Theorem 3.5], $K_\Sigma = 0$ everywhere. Moreover, by equation (21), we have that $\|\alpha \wedge \beta\|^2 = 0$ and $\nabla^E \sigma = 0$ everywhere.

This implies that α and β are colinear and the line bundle $L := \mathrm{span}\{\alpha, \beta\}$ is a parallel subbundle of $N\Sigma$. Hence, the subbundle V of type $(2, 2)$ of the flat trivial E whose fibre at at a point x is $L_x \oplus \mathsf{T}_x\Sigma \oplus x$ is parallel with respect to the flat trivial connection and in particular constant. Thus Σ is a flat maximal surface contained in the totally geodesic copy of $\mathbf{H}^{2,1}$ given by $\mathbf{P}(V) \cap \mathbf{H}^{2,n}$. By Bonsante–Schlenker Lemma 4.17, Σ is a Barbot surface.   □

5.1.2. *Proof of the Curvature Rigidity proposition 5.3.* Recall that by Theorem 4.8 the space $\mathcal{M}(n)/\mathsf{G}$ is compact. Thus the continuous function $\mathcal{M}(n)/\mathsf{G} \to \mathbb{R}$, $(x, \Sigma) \mapsto K_\Sigma(x)$ reaches its maximum at a point (x_0, Σ_0). By corollary 5.7 $K_{\Sigma_0}(x_0) \leqslant 0$, hence for all (x, Σ), we have $K_\Sigma(x) \leqslant K_{\Sigma_0}(x_0) \leqslant 0$.

5.2. **Horofunction Rigidity.**

Consider a complete maximal surface Σ in $\mathbf{H}^{2,n}$, z a point in $\Sigma \sqcup \partial_\infty\Sigma$ and z_0 a non zero vector in Σ. Then any point x in Σ has a unique lift to E, that we still denote by x, such that $\mathbf{q}(x) = -1$ and $\langle z_0, x\rangle < 0$. In the sequel, we will implicitly use this canonical lift to define scalar products. We prove the next proposition in paragraph 5.2.3. Then the Horofunction Rigidity Theorem in 5.2.4.

Proposition 5.8. [GRADIENT BOUNDS] *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$ and z be a point in $\Sigma \sqcup \partial_\infty\Sigma$. Assume a point x in Σ is a critical point of $\|\nabla h_z\|$. Then we have the following inequality*

$$
\|\nabla h_z\|^2 \leqslant 2 + K_\Sigma \ . \tag{22}
$$

In particular $\|\nabla h_z\|^2 \leqslant 2$ with equality and only if $K_\Sigma = 0$.

5.2.1. *The gradient.* Let z be a point in $\Sigma \sqcup \partial_\infty\Sigma$, seen as a line in E, and let z_0 be a non zero vector in the line z. Then

Lemma 5.9. *Let x be a point in Σ not equal to z, then for any u in $\mathsf{T}_x\Sigma$,*

$$
\mathrm{d}_x h_z(u) = -\frac{\langle u, z_0\rangle}{\sqrt{\langle x, z_0\rangle^2 + \langle z_0, z_0\rangle}} \ , \quad \|\nabla h_z\| \geqslant 1,
$$

where the gradient is taken along Σ.

Proof. If z is a point of $\partial_\infty\Sigma$ and z_0 is a non zero vector in z , then $h_z(x) = \log(-\langle x, z_0\rangle)$. It follows that

$$
\mathrm{d}_x h_z(u) = \frac{\langle u, z_0\rangle}{\langle x, z_0\rangle} = -\frac{\langle u, z_0\rangle}{\sqrt{\langle x, z_0\rangle^2 + \langle z_0, z_0\rangle}} \ .
$$

Assume now that z belongs to Σ and let again z_0 be a non zero vector in z. Since the derivative of $\cosh^{-1}$ at a point x is $(x^2 - 1)^{-\frac{1}{2}}$, we get for z in Σ,

$$
\mathrm{d}_x h_z(u) = -\frac{\langle u, z_0\rangle}{\sqrt{\langle x, z_0\rangle^2 - 1}} = -\frac{\langle u, z_0\rangle}{\sqrt{\langle x, z_0\rangle^2 + \langle z_0, z_0\rangle}} \ .
$$

Let us now write

$$z_0 = -\langle x, z_0 \rangle x + z_0^T + z_0^N ,$$

where z_0^T and z_0^N are the tangential and normal component of z_0 at x. Then

$$\mathbf{q}(z_0^T) \geqslant \mathbf{q}(z_0^T) + \mathbf{q}(z_0^N) = \mathbf{q}(z_0^T + z_0^N) = \mathbf{q}(z_0 + \langle x, z_0 \rangle x) = \langle z_0, z_0 \rangle + \langle x, z_0 \rangle^2 .$$

It follows that

$$\|\nabla h_z\|^2 = \frac{\mathbf{q}(z_0^T)}{\langle z_0, z_0 \rangle + \langle x, z_0 \rangle^2} \geqslant 1 . \quad \square$$

5.2.2. *Some preliminary lemmas.* For Σ, z_0 and x as above, we define $\beta_x \in \mathrm{Sym}^2(\mathsf{T}_x^*\Sigma)$ by

$$\beta_x(u, v) = \frac{\langle \mathrm{II}(u, v), z_0 \rangle}{\langle x, z_0 \rangle} ,$$

where II is the second fundamental form of Σ. Observe that β_x only depends on the projectivisation z of z_0. In the sequel, we write $h := h_z$.

Lemma 5.10. *Let Σ be a spacelike surface. For any point x in Σ, the Hessian $\nabla^2 h$ of h at x on Σ satisfies*

$$\nabla_x^2 h = \phi_z \cdot (g - \mathrm{d}h \odot \mathrm{d}h + \beta_x) ,$$

where $\odot$ denotes the symmetric tensor product, g is the induced metric on Σ and

$$\phi_z(x) = -\frac{\langle x, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} ,$$

and in particular $\phi_z = 1$ if z is in $\partial_\infty \Sigma$.

Proof. Let $(x_t)_{t \in (-\varepsilon, \varepsilon)}$ be a smooth geodesic in Σ with $x_0 = x$. We have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} h(x_t) = -\frac{\langle D_{\dot{x}}\dot{x}, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} + \frac{\langle \dot{x}, z_0 \rangle^2 \langle x, z_0 \rangle}{\left(\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}\right)^3} ,$$

where D is the Levi-Civita connection of E. The decomposition of D in the splitting $E = \langle x \rangle \oplus \mathsf{T}_x\Sigma \oplus \mathsf{N}_x\Sigma$ gives

$$(D_a b)_x = \langle a, b \rangle x + (\nabla_a b)_x + \mathrm{II}(a, b) ,$$

where ∇ is the Levi-Civita connection of Σ. Thus

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} h(x_t) &= -\frac{\langle \dot{x}, \dot{x} \rangle \langle x, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} - \frac{\langle \mathrm{II}(\dot{x}, \dot{x}), z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} \\
&\quad + \left(\frac{\langle \dot{x}, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}}\right)^2 \frac{\langle x, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} \\
&= \phi(x)\left(g(\dot{x}, \dot{x}) - \mathrm{d}h(\dot{x})^2 + \beta_x(\dot{x}, \dot{x})\right) ,
\end{aligned}$$

which is what we wanted to prove. $\quad \square$

Lemma 5.11. *Let Σ be a spacelike surface. If x is a critical point of the function $\|\nabla h\|^2$ in Σ and x is different from z, then*

$$\beta_x(\nabla h, \nabla h) = \|\nabla h\|^2(\|\nabla h\|^2 - 1) .$$

Proof. Let (e_1, e_2) be an orthonormal framing of $\mathsf{T}\Sigma$. Then for any vector field X on Σ, we have at x

$$0 = \frac{1}{2} X \cdot (\|\nabla h\|^2) = \sum_{i=1}^{2} (X \cdot \mathrm{d}h(e_i)) \, \mathrm{d}h(e_i) = \sum_{i=1}^{2} \nabla^2 h(X, \mathrm{d}h(e_i)e_i) = \nabla^2 h(X, \nabla h) .$$

Applying the previous lemma to $X = \nabla h$ gives

$$0 = \|\nabla h\|^2 - \|\nabla h\|^4 + \beta_x(\nabla h, \nabla h) .$$

And thus the result.   □

Lemma 5.12. *Assume that Σ is maximal and u is a tangent vector in Σ at x.*

$$\beta_x(u, u)^2 \quad \leqslant \quad \|u\|^4 (\|\nabla h\|^2 - 1)(1 + K_\Sigma(x)) . \tag{23}$$

Proof. Let us use the notation $\|u\|^2 = |\langle u, u \rangle|$. In the orthogonal splitting $E = \mathbb{R}.x \oplus \mathsf{T}_x\Sigma \oplus \mathsf{N}_x\Sigma$, write $z_0 = -\langle x, z_0 \rangle x + z_0^T + z_0^N$. It follows that

$$\|z_0^T\|^2 - \|z_0^N\|^2 = \langle z_0, z_0 \rangle + \langle x, z_0 \rangle^2 \leqslant \langle x, z_0 \rangle^2 . \tag{24}$$

This implies that

$$\frac{\|z_0^N\|^2}{\langle x, z_0 \rangle^2} \leqslant \frac{\|z^T\|^2}{\langle z_0, z_0 \rangle + \langle x, z_0 \rangle^2} - 1 = \|\nabla h\|^2 - 1 . \tag{25}$$

Now

$$\beta_x(u, u)^2 = \frac{\langle \mathrm{II}(u, u), z_0^N \rangle^2}{\langle x, z_0 \rangle^2} \leqslant \|\mathrm{II}(u, u)\|^2 \frac{\|z_0^N\|^2}{\langle x, z_0 \rangle^2} \leqslant \|\mathrm{II}(u, u)\|^2 (\|\nabla h\|^2 - 1) .$$

From Gauß equation (proposition 4.5), and since Σ is maximal, we have

$$\|\mathrm{II}(u, u)\|^2 \leqslant \|\mathrm{II}(u, u)\|^2 + \|\mathrm{II}(Ju, u)\|^2 = (1 + K_\Sigma)\|u\|^4 ,$$

where J is the complex structure on Σ. The result follows.   □

5.2.3. *Proof of proposition 5.8.* Applying lemma 5.11 we obtain that

$$\beta_x(\nabla h, \nabla h) = \|\nabla h\|^2 (\|\nabla h\|^2 - 1) .$$

Then using lemma 5.12 for $u = \nabla h$, we obtain

$$\left(\|\nabla h\|^2 (\|\nabla h\|^2 - 1)\right)^2 \leqslant \|\nabla h\|^4 (\|\nabla h\|^2 - 1)(1 + K_\Sigma(x)) . \tag{26}$$

Since $\|\nabla h\| \geqslant 1$ by lemma 5.9, we get

$$(\|\nabla h\|^2 - 1)^2 \leqslant (\|\nabla h\|^2 - 1)(1 + K_\Sigma(x)) . \tag{27}$$

Assuming now that $\|\nabla h\| > 1$, we get

$$\|\nabla h\|^2 \leqslant 2 + K_\Sigma(x) . \tag{28}$$

This gives the result since $K_\Sigma(x) \leqslant 0$ by the Curvature Rigidity Theorem 5.3. Observe that Inequality (28) is also true whenever $\|\nabla h\| = 1$ since $K_\Sigma(x) \geqslant -1$. This concludes the proof of the proposition.

5.2.4. *Proof of the Horofunction and Spatial Distance RigidityTheorem.* By corollary 4.9, the action of G on the space $\mathcal{N}_1(n)$ of triples (x, Σ, z), where (x, Σ) is a complete maximal surface and z a point in $\Sigma \sqcup \partial_\infty \Sigma$, is cocompact. Then the function

$$\Phi \ : \ (x, \Sigma, z) \mapsto \|\nabla h_z(x)\|^2 ,$$

where h is the horofunction associated to z is continuous and G invariant. It follows that Φ is bounded.

Let now (x, Σ, z) be a point in $\mathcal{N}$ at which the maximum of Φ is reached. Observe then that x is a critical point of $\|\nabla h(x)\|^2$ along Σ.

The result now follows from Proposition 5.8 and the Curvature Rigidity proposition 5.3.

6. Characterisations of Quasiperiodicity

In this section, we give different characterisation of quasiperiodic maximal surfaces using our previous results

6.1. **Curvature bounds.**

Theorem 6.1. [CURVATURE CHARACTERISATION] *A complete maximal surface Σ in $\mathbf{H}^{2,n}$ is quasiperiodic if and only if there exists a positive constant c so that the intrinsic curvature of Σ satisfies $K_\Sigma < -c$.*

Proof. Remark that the function $x \mapsto K_\Sigma(x)$ is continuous and G-invariant from $\mathcal{M}(n)$ to $\mathbb{R}$. By compactness of $\mathcal{M}/\mathsf{G}$ (see Theorem 4.8), it reaches its maximum on the closure C of the image of Σ in $\mathcal{M}(n)/\mathsf{G}$. By Theorem 5.5, this maximum is 0 if and only if C contains a Barbot surface. The result follows. $\qquad\square$

6.2. **Horofunction bounds.** Similarly

Theorem 6.2. [HOROFUNCTION CHARACTERISATION] *A surface is quasiperiodic if and only if there exists a positive real h_0 strictly less than 1 so the square of norm of the gradient of any horofunction is bounded by $1 + h_0$*

Proof. Let $\mathcal{N}_1(n)$ be the space of of triples (x, Σ, z) where (x, Σ) is a complete maximal surface and z a point in $\partial_\infty \Sigma$. By corollary 4.9, the action of G on the space $\mathcal{N}_1(n)$ is cocompact. Denoting h_z the horofunction associated to z in $\partial_\infty \Sigma$, the function associating to (x, Σ, z) the value $\|\nabla h_z(x)\|^2$ is continuous and G-invariant. Thus it reaches its maximum on the closure C of the preimage of Σ in $\mathcal{N}_1(n)/\mathsf{G}$. By Theorem 5.5, this maximum is 2 if and only if C contains a Barbot surface. The result follows. $\qquad\square$

6.3. **Conformal characterisation.**

Theorem 6.3. [CONFORMAL CHARACTERISATION] *A complete maximal surface Σ is quasiperiodic if and only if there exists a conformal biLipschitz Φ map from $\mathbf{H}^2$ to Σ.*

Proof. We have already seen that if Σ is quasiperiodic, its curvature K_Σ is less than a negative constant $-c$.

By the classical Ahlfors–Schwarz–Pick Lemma [CTT19, Theorem A.1], since $K_\Sigma \geqslant -1$, any conformal map from $\mathbf{H}^2$ to Σ is length increasing.

Similarly, because $K_\Sigma \leqslant -c$, any conformal map from Σ to $\mathbf{H}_c^2$ is length increasing (here $\mathbf{H}_c^2$ is the disk equipped with the metric $\frac{1}{c} g_{\mathbf{H}^2}$ of constant curvature $-c$). The uniformisation thus gives $g_{\mathbf{H}^2} \leqslant g_\Sigma \leqslant \frac{1}{c} g_{\mathbf{H}^2}$.

Conversely, if Σ were not quasiperiodic, there would be a sequence of point $\{x_k\}_{k \in \mathbb{N}}$ so that the sequence of pointed maximal surface $\{(x_k, \Sigma)\}_{k \in \mathbb{N}}$ converges to a Barbot surface. Then the sequence of Riemannian surface $\{(x_k, \Sigma)\}_{k \in \mathbb{N}}$ would converge uniformly on every compact to a flat metric. In particular the flat metric would be quasiisometric to a hyperbolic metric which is impossible. $\quad\square$

6.4. **Gromov hyperbolicity.**

Theorem 6.4. [GROMOV HYPERBOLICITY] *A complete maximal surface Σ is quasiperiodic if and only if the induced Riemannian metric is Gromov-hyperbolic.*

This result is to be compared to some of Benoist–Hulin results for $\mathsf{SL}(3, \mathbb{R})$ [BH14].

Proof. By the curvature characterisation, we see that a quasiperiodic complete maximal surface has curvature bounded from above by a negative constant and hence is Gromov hyperbolic.

Conversely, if a surface is not quasiperiodic, there would be a sequence of point $\{x_k\}_{k \in \mathbb{N}}$ so that the sequence of pointed maximal surface $\{(x_k, \Sigma)\}_{k \in \mathbb{N}}$ converges to a Barbot surface. Then the sequence of Riemannian surface $\{(x_k, \Sigma)\}_{k \in \mathbb{N}}$ would converge uniformly on every compact to a flat metric. This would guarantee the existence of arbitrarily thick triangles and contradict Gromov hyperbolicity. $\qquad\square$

6.5. **Characterisation using the spatial distance.** Recall that from proposition 4.2, complete maximal surfaces are acausal. In particular, it is natural to compare the induced distance with the spatial distance ð defined in definition 10. Our characterisation is the following

Theorem 6.5. [SPATIAL DISTANCE] *Let Σ be a complete maximal surface, then we have the following inequality*

$$d \leqslant ð \leqslant \sqrt{2}d . \tag{29}$$

Moreover, Σ is quasiperiodic if and only if there exists $c < \sqrt{2}$ so that

$$ð \leqslant cd . \tag{30}$$

Proof of Theorem 6.5. The inequality $d \leqslant ð$ is consequence of lemma 5.9 or equivalently to the fact that the warped projection on a hyperbolic plane is length increasing [LTW20, Lemma 3.7].

The inequality $ð \leqslant \sqrt{2}d$ comes from the Horofunction and Spatial Distance Rigidity Theorem 5.5.

If Σ is not quasiperiodic, then we can find a sequence $\{x_k\}_{k\in\mathbb{N}}$ in Σ so that $\{(x_k, \Sigma)\}$ converges (up to the action of G) to (x, Σ_0) where Σ_0 is a Barbot surface. It follows by proposition 4.16, that we have

$$\sup\left(\frac{ð}{d}\right) \geqslant \sqrt{2} . \tag{31}$$

Conversely assume that Σ is such that inequality (31) is satisfied. Then, for any positive ε, we can find a geodesic γ and a positive t, so that, we have

$$ð(\gamma(0), \gamma(t)) \geqslant \sqrt{2}(1 - \varepsilon)t .$$

Then setting $y = \gamma(0)$ and $f : x \mapsto ð(y, x)$, we have

$$\int_{\varepsilon t}^{t} df(\dot{\gamma}) \, dt = \int_{0}^{t} df(\dot{\gamma}) \, dt - \int_{0}^{\varepsilon t} df(\dot{\gamma}) \, dt \geqslant t\sqrt{2}(1 - \varepsilon) - t\sqrt{2}\varepsilon \geqslant (1 - 2\varepsilon)\sqrt{2}t ,$$

since the gradient of f has norm less than $\sqrt{2}$ by Theorem 5.5. Thus there exists $x = \gamma(s)$, with s in $[\varepsilon t, t]$ so that

$$\|\nabla ð^y(x)\| \geqslant \sqrt{2}\frac{1 - 2\varepsilon}{1 - \varepsilon} .$$

Since ε is arbitrary, we can find sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} \|\nabla ð^{y_k}(x_k)\| \geqslant \sqrt{2} .$$

Let $f_k(x) = ð(x, y_k)$, then after extracting a subsequences, we can assume that (x_k, Σ, y_k) converges to (x, Σ_0, z) in $\mathcal{N}_1(n)$. Thus $\{f_k - f_k(x_k)\}_{k\in\mathbb{N}}$ converges to a function h_z which is, up to some additive constant, either a horofunction or the spatial distance to a point in Σ, and so that $\|\nabla h_z(x)\|^2 \geqslant 2$. It follows by Theorem 5.5 that Σ_0 is a Barbot surface and thus that Σ is not quasiperiodic. □

6.6. **Lamination characterisation.** Here is another interpretation of quasiperiodic leaves.

Theorem 6.6. [LAMINATED INTERPRETATION] *A complete maximal surface Σ is quasiperiodic if and only if there exists*

(i) *a compact space $\mathcal{F}$ laminated by hyperbolic Riemann surfaces.*
(ii) *a $\mathbf{H}^{2,n}$-bundle $\mathcal{H}$ over $\mathcal{F}$ equipped with a flat connection along the leaves whose parallel transport is transversely continuous in the smooth topology along the leaves.*
(iii) *a section of $\mathcal{H}$, lifting all leaves to maximal complete surfaces so that Σ is one of these lifts.*

The case when $\mathcal{F}$ is a compact surface is the periodic case and it is customary to describe the compact laminated case as quasiperiodic.

Proof. Let Γ be a cocompact torsion free lattice in G. The space $\mathcal{M}(n)/\Gamma$ is then laminated and we take as $\mathcal{F}$ the closure of the image of Σ in $\mathcal{M}(n)/\Gamma$. The bundle $\mathcal{H}$ is the induced bundle from the trivial bundle with the Γ-action. All leaves in $\mathcal{F}$ are hyperbolic because they are uniformised by the hyperbolic disk by the Conformal Characterisation Theorem 6.3.

Conversely, such a leaf in a space $\mathcal{F}$ is quasiperiodic since its closure of its G orbit in $\mathcal{M}(n)$ only contains complete surfaces which are hyperbolic, whereas Barbot surfaces are parabolic. □

7. Extension of uniformisation

Let us fix a point x_0 in $\mathbf{H}^2$ and identify $\partial_\infty \mathbf{H}^2$ with $\mathbf{P}(V)$.

Theorem 7.1. [Extension of uniformisation] *For any positive constant c, there exists some constants A and B with the following property:*

Assume that (x, Σ) is a quasiperiodic surface whose curvature is bounded by $-c$. Then there exists a continuous map Ψ from $\mathbf{H}^2 \cup \partial_\infty \mathbf{H}^2$ to $\Sigma \cup \partial_\infty \Sigma$ which respects interiors and boundaries, is conformal in the interior and (A, B)-quasisymmetric on the boundary and so that $\Psi(x_0) = x$.

Such a map Ψ is called *an extension of uniformisation* for (x, Σ). Recall that any quasiperiodic surface has curvature bounded by some negative constant by Theorem 6.1. The next result tells us extensions behave well under limits.

Theorem 7.2. [Boundary compactness] *Let $\{(x_k, \Sigma_k)\}_{k \in \mathbb{N}}$ be a sequence of quasiperiodic surfaces converging to a quasiperiodic surface (x, Σ). Assume that there is a positive constant c so that the curvature of all surfaces Σ_k is bounded from above by $-c$. Let $\{\Psi_k\}_{k \in \mathbb{N}}$ be the corresponding extensions of conformal parametrisation. Then $\{\Psi_k\}_{k \in \mathbb{N}}$ subconverges uniformly to an extension of conformal parametrisation of (x, Σ).*

Here is an interesting corollary of this construction whose statement does not involve maximal surfaces. Note that we have not been able to give a direct proof of this corollary, without using the maximal surface solution to the asymptotic Plateau problem.

Corollary 7.3. *Every quasiperiodic loop admits a quasisymmetric – hence quasiperiodic – parametrisation.*

Proof. Indeed a quasiperiodic loop bounds a quasiperiodic surface, and we have just shown that the boundary at infinity of such a quasiperiodic surface admits a quasisymmetric parametrisation. □

We then define an (A, B)-*quasicircle* to be a quasiperiodic loop such that any extension of unformisation is (A, B)-quasisymmetric. We then have some converse to Theorem 7.1

Corollary 7.4. *For any constants A and B greater than 1, there exists a positive c so that if the boundary of a complete maximal surface has an (A, B)-quasisymmetric parametrisation then the curvature of Σ is bounded by $-c$ from above.*

Proof. The space of pairs (ξ, τ) where ξ is an (A, B)-quasisymmetric map and τ a positive triple in $\mathbf{P}(V)$ is equipped with a cocompact action of $\mathsf{H} := \mathsf{PSL}(V) \times \mathsf{G}$. Denote by $\Sigma(\xi)$ the solution to the asymptotic Plateau problem for $\xi(\mathbf{P}(V))$. The barycenter map B being continuous by proposition 4.12, then map which associates to (ξ, τ) the curvature of $B(\Sigma(\xi), b_\tau)$ is continuous, H-invariant reaches it maximum at a pair (ξ_0, τ_0). This maximum is negative by the Rigidity proposition 5.3. Since B is surjective, the result is proved. □

Corollary 7.5. *For any constants A and B greater than 1, there exists C so that the image of an (A, B)-quasisymmetric map is an (A, C)-quasicircle.*

Proof. We may as well assume that the quasisymmetric map sends a given positive triple to a fixed positive triple. Since the curvature of any complete maximal surface whose boundary admits an (A, B)-quasisymmetric parametrisation bounded by $-c$, the result is a consequence of the Uniformisation Theorem 7.1. □

We also have another corollary that answers a question asked to us by François Guéritaud, showing that quasisymmetric maps are extension of quasiisometries with respect to the spatial distance, thus generalising the corresponding fact in the context of rank 1 symmetric spaces.

Corollary 7.6. *Every quasisymmetric map from $\partial_\infty \mathbf{H}^2$ to $\partial_\infty \mathbf{H}^{2,n}$, is a continuous extension of a map f from $\mathbf{H}^2$ to $\partial_\infty \mathbf{H}^{2,n}$ such that there exists some positive constants A and B so that for all x and y in $\mathbf{H}^2$*

$$A^{-1}d(x,y) - B \leqslant \eth(f(x), f(y)) \leqslant Ad(x,y) + B .$$

However we do not know whether the converse to that corollary is true.

Proof of Corollary 7.6. Let ξ be a quasisymmetric map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$, and let $\Lambda = \xi(\mathbf{P}(V))$. By Theorem 7.1, and identifying $\mathbf{P}(V)$ with $\partial_\infty \mathbf{H}^2$, there is a quasisymmetric parametrisation ξ_0 of Λ which is a continuous extension of a conformal map f_0 from $\mathbf{H}^2$ to the maximal surface $\Sigma(\Lambda)$. By Theorem 6.3, the map f_0 is biLipschitz. Thus by Theorem 6.5, there is a constant C greater that 1 such that for any x, y in $\mathbf{H}^2$

$$C^{-1}d(x,y) \leqslant \eth(f_0(x), f_0(y)) \leqslant Cd(x,y) .$$

By corollary 3.15, there is a quasisymmetric homeomorphism ϕ of $\partial_\infty \mathbf{H}^2$ such that $\xi = \xi_0 \circ \varphi$. By the Douady–Earle extension Theorem [DE86, Theorem 2], φ is a continuous extension of a quasiisometry Φ from $\mathbf{H}^2$ to itself. Thus ξ is the continuous extension of $f := f_0 \circ \Phi$ which satisfies the required condition.   □

7.1. **Control of Gromov products.**

We study quantities – also considered by Glorieux and Monclair [GM16] – related to Gromov products when $n = 0$.

We prove three propositions that provide a control on these quantities.

In the sequel, we will lift any complete maximal surface in $\mathbf{H}^{2,n}$ to $\{x \in E$, $\mathbf{q}(x) = -1\}$. This will allow us to define the scalar product $\langle x, y \rangle$ between two points x, y in Σ. Observe that the quantity $|\langle x, y \rangle|$ is independent on the lift. Similarly for z, w in $\partial_\infty \Sigma$ the quantity – or *Gromov product* –

$$\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| := \left| \frac{\langle z_0, w_0 \rangle}{\langle z_0, x \rangle \langle x, w_0 \rangle} \right| ,$$

is well defined and independent on the lift of Σ and the choice of the non zero vector z_0, w_0 in z and w respectively.

Proposition 7.7. *There exists a positive constant M_1 such that for any pointed complete maximal surface (x, Σ) in $\mathcal{M}(n)$ and points z, w in $\Sigma \cup \partial_\infty \Sigma$, we have*

$$\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| \leqslant M_1 .$$

Proof. Let $\mathcal{N}_2(n)$ of quadruples (x, Σ, z, w), where (x, Σ) is a pointed quasiperiodic maximal surface and z and w are points in $\Sigma \cup \partial_\infty \Sigma$. Let P_x be the affine hyperplane $\{v \mid \langle v, x \rangle = -1\}$. Let π_x be the radial projection on the hyperplane. By [LTW20, Proposition 2.27 and Proposition 2.5 (iii)] the radial projection of $\Sigma \cup \partial\Sigma$ is bounded in P_x. It follows that the function

$$F : (x, \Sigma, z, w) \mapsto \left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| = \left| \langle \pi_x(z), \pi_x(w) \rangle \right| ,$$

is well defined, G-invariant and continuous on $\mathcal{N}_2(n)$. Since the action of G is cocompact on $\mathcal{N}_2(n)$ by corollary 4.9, F is bounded and the result follows.   □

Corollary 7.8. *There exists a constant C only depending on n, so that for all complete maximal surface Σ in $\mathbf{H}^{2,n}$, for all triple of points x, y, z in Σ*

$$\eth(x,y) \leqslant \eth(x,z) + \eth(z,y) + C .$$

in particular, the function $\eth_0$ so that $\eth_0(x,y) = \eth(x,y) + C$ if x is different than y and $\eth_0(x,x) = 0$ is a distance.

This final remark was pointed out to us by Indira Chatterji.

Proof. Since $\cosh(\eth(x,y)) = |\langle x, y \rangle|$ and $\frac{1}{2}e^u \leqslant \cosh(u) \leqslant e^u$ for any positive u, we get

$$\left| \frac{\langle z, w \rangle}{\langle x, z \rangle \langle x, w \rangle} \right| \geqslant \frac{1}{2} \exp\left(\eth(z, w) - \eth(x, w) - \eth(x, z) \right) .$$

Thus by proposition 7.7, we get $\eth(z, w) - \eth(x, w) - \eth(x, z) \leqslant C := \log(2M_1)$. $\qquad \square$

We prove the next two propositions in paragraph 7.1.4

Proposition 7.9. *For any positive R and K, there is positive M_2 such that for any quasiperiodic maximal surface Σ, if η is an oriented K-quasigeodesic of extremities z and w in Σ such that x is at within a distance at most R from η, then*

$$\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| \geqslant M_2 .$$

We also need the converse property

Proposition 7.10. *For any positive c, any positive K and B, then for any quasiperiodic maximal surface Σ with curvature bounded by $-c$ and any K-quasigeodesic arc η in Σ with extremities w and z, then if x in Σ is so that*

$$\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| \geqslant B ,$$

then $d_\Sigma(x, \eta) \leqslant A$.

7.1.1. *Some a priori bounds.* In this paragraph, we do not require the complete maximal surface to be quasiperiodic.

Lemma 7.11. *There exists a constant M_1 so that for any three points (a, b, c) on a complete maximal surface Σ, we have the inequality*

$$\frac{1}{M_1} \exp\left(-\sqrt{2}d_\Sigma(b, c) \right) \leqslant \left| \frac{\langle a, b \rangle}{\langle a, c \rangle} \right| \leqslant M_1 \exp\left(\sqrt{2}d_\Sigma(b, c) \right) . \tag{32}$$

Proof. Indeed, by Proposition 7.7, we have

$$\frac{1}{M_1} \left| \frac{\langle a, b \rangle}{\langle a, c \rangle} \right| \leqslant |\langle b, c \rangle| = \cosh(\eth(b, c)) \leqslant \exp(\eth(b, c)) \leqslant \exp\left(\sqrt{2}d_\Sigma(b, c) \right) .$$

where we used Theorem 6.5 for the last inequality. This proves the right inequality, the second one comes from reversing the role of b and c. $\qquad \square$

Corollary 7.12. *There exists a constant M so that for any three points (a, b, c) on a maximal surface, we have the inequality*

$$\frac{1}{M} \exp\left(-2\sqrt{2}d_\Sigma(b, c) \right) \leqslant \left| \frac{\langle a, b \rangle}{\langle b, c \rangle \langle a, c \rangle} \right| \leqslant M \exp\left(\sqrt{2}d_\Sigma(b, c) \right) . \tag{33}$$

Proof. Indeed

$$\left| \frac{\langle a, b \rangle}{\langle b, c \rangle \langle a, c \rangle} \right| \geqslant \frac{\exp\left(-\sqrt{2}d_\Sigma(b, c) \right)}{M_1 |\langle b, c \rangle|}$$

$$\geqslant \frac{2\exp\left(-\sqrt{2}d_\Sigma(b, c) \right)}{M_1 \exp\left(\sqrt{2}d_\Sigma(b, c) \right)} = \frac{2}{M_1} \exp\left(-2\sqrt{2}d_\Sigma(b, c) \right) ,$$

where we used the previous lemma for the first inequality and Theorem 6.5 for the last. Similarly

$$\left| \frac{\langle a, b \rangle}{\langle b, c \rangle \langle a, c \rangle} \right| \leqslant \frac{M_1 \exp\left(\sqrt{2}d_\Sigma(b, c) \right)}{|\langle b, c \rangle|} \leqslant M_1 \exp\left(\sqrt{2}d_\Sigma(b, c) \right) ,$$

where we used the previous lemma for the first inequality and the inequality $|\langle b, c \rangle| \geqslant 1$ for the second.   □

7.1.2. *Minima of horofunction.*

Lemma 7.13. *Let Σ be a quasiperiodic maximal surface. Then the restriction of any horofunction to a complete geodesic in Σ has no local maximum.*

Proof. By the Horofunction Characterisation Theorem 6.2 there is $h_0 < 1$, such that for any z in $\partial_\infty \Sigma$ we have $\|\nabla h_z\|^2 \leqslant 1 + h_0$ – where h_z is the horofunction associated to z. Consider a geodesic γ in Σ parametrised by arc-length, and let x be a critical point of h_z on γ. The tangent vector $\dot\gamma$ at x satisfies $dh(\dot\gamma) = 0$. Using lemma 5.10 with the same notations, we have

$$\nabla_x^2 h_z(\dot\gamma, \dot\gamma) = 1 + \beta_x(\dot\gamma, \dot\gamma) \quad \geqslant \quad 1 - \sqrt{(1 + K_\Sigma(x))(\|\nabla h_z\|^2 - 1)}$$
$$\geqslant \quad 1 - \sqrt{h_0} > 0 ,$$

where we used lemma 5.12 in the second inequality, our hypothesis in the third and the fact that $0 \leqslant 1 + K_\Sigma \leqslant 1$ by proposition 5.3. Thus the second derivative of h_z at a critical point is positive and the result follows.   □

7.1.3. *Gromov products on geodesics.* Let us concentrate on geodesics for the induced metric.

Lemma 7.14. *There exists a positive M_3 such that for any quasiperiodic maximal surface Σ, any oriented geodesic γ in Σ, if s and t are positive then*

$$\left| \frac{\langle \gamma(-s), \gamma(t) \rangle}{\langle \gamma(-s), \gamma(0) \rangle \langle \gamma(0), \gamma(t) \rangle} \right| \quad \geqslant \quad M_3 . \tag{34}$$

Proof. Suppose the result wrong. Then there exists a sequence $\{(x_k, \gamma_k, \Sigma_k)\}_{k \in \mathbb{N}}$, where (x_k, Σ_k) is a pointed quasiperiodic surface, γ_k is a complete geodesic with $\gamma_k(0) = x_k$, as well as sequences $\{t_k\}_{k \in \mathbb{N}}$ and $\{s_k\}_{k \in \mathbb{N}}$ of real numbers such that

$$\lim_{k \to \infty} \left| \frac{\langle \gamma_k(-s_k), \gamma_k(t_k) \rangle}{\langle \gamma_k(-s_k), x_k \rangle \langle x_k, \gamma_k(t_k) \rangle} \right| = 0 . \tag{35}$$

Observing that by Corollary 7.12,

$$\left| \frac{\langle \gamma_k(-s_k), \gamma_k(t_k) \rangle}{\langle \gamma_k(-s_k), x_k \rangle \langle x_k, \gamma_k(t_k) \rangle} \right| \geqslant \frac{1}{M} \sup \left\{ \exp(-2\sqrt{2}s_k), \exp(-2\sqrt{2}t_k) \right\} ,$$

this implies that $\{s_k\}_{k \in \mathbb{N}}$ and $\{t_k\}_{k \in \mathbb{N}}$ both converge to ∞.

For (x, Σ) in $\mathcal{M}(n)$, an oriented geodesic γ passing through x is given by a linear ray $\mathsf{T}_x \gamma \subset \mathsf{T}_x \Sigma$. In particular, and one can find a sequence $\{g_k\}_{k \in \mathbb{N}}$ in G such that $g_k(x_k, \gamma_k, \Sigma_k)$ subconverges to (x, γ, Σ) with (x, Σ) quasiperiodic.

To make life simpler, we assume the sequence $\{g_k\}_{k \in \mathbb{N}}$ is constant and equal to the identity. Let us then consider

$$u_k = \frac{1}{\langle x_k, \gamma_k(t_k) \rangle} \gamma_k(t_k) , \quad v_k = \frac{1}{\langle x_k, \gamma_k(-s_k) \rangle} \gamma_k(-s_k) .$$

Up to extracting a subsequence, $\{u_k\}_{k \in \mathbb{N}}$ and $\{v_k\}_{k \in \mathbb{N}}$ converge respectively to lightlike vectors u and v. By [LTW20, Theorem 6.1], u and v gives rise in the projective compactification to elements of $\partial_\infty \Sigma$. By assertion (35), $\langle u, v \rangle = 0$. However since Σ is quasiperiodic, $\partial_\infty \Sigma$ is positive, thus $u = v$.

Let $\{w_k\}_{k \in \mathbb{N}}$ be a sequence of lightlike vectors converging to u so that w_k belongs to $\partial_\infty \Sigma_k$ and normalised so that $\langle x_k, w_k \rangle = 1$. Let h_k be the horofunction associated to w_k. Then

$$\lim_{k \to \infty} h_k(\gamma_k(-s_k)) = \lim_{k \to \infty} \log |\langle w_k, v_k \rangle| \quad = \quad -\infty ,$$
$$\lim_{k \to \infty} h_k(\gamma_k(t_k)) = \lim_{k \to \infty} \log |\langle w_k, u_k \rangle| \quad = \quad -\infty ,$$
$$\lim_{k \to \infty} h_k(\gamma_k(0)) = \lim_{k \to \infty} \log |\langle x_k, w_k \rangle| \quad = \quad 0 .$$

It follows that for k large enough, h_k has a local maximum on $\gamma_k([-s_k, t_k])$. This contradicts Lemma 7.13. □

Corollary 7.15. *For any R there is a positive constant M_4 such that if Σ is a quasiperiodic maximal surface and γ is an oriented geodesic in Σ such that x is at within a distance R of γ, both s and t are positive, then*

$$\left| \frac{\langle \gamma(-s), \gamma(t) \rangle}{\langle \gamma(-s), x \rangle \langle x, \gamma(t) \rangle} \right| \geqslant M_4 \ .$$

Proof. Let assume (after a change of time) that $d(x, \gamma(0))$ is less than R. Then

$$\begin{aligned}
\left| \frac{\langle \gamma(-s), \gamma(t) \rangle}{\langle \gamma(-s), x \rangle \langle x, \gamma(t) \rangle} \right| &= \left| \frac{\langle \gamma(-s), \gamma(0) \rangle}{\langle \gamma(-s), x \rangle} \right| \cdot \left| \frac{\langle \gamma(t), \gamma(0) \rangle}{\langle \gamma(t), x \rangle} \right| \cdot \left| \frac{\langle \gamma(-s), \gamma(t) \rangle}{\langle \gamma(-s), \gamma(0) \rangle \langle \gamma(0), \gamma(t) \rangle} \right| \\
&\geqslant M_3 \left| \frac{\langle \gamma(-s), \gamma(0) \rangle}{\langle \gamma(-s), x \rangle} \right| \cdot \left| \frac{\langle \gamma(t), \gamma(0) \rangle}{\langle \gamma(t), x \rangle} \right| \\
&\geqslant \frac{M_1}{M_2^2} \exp\left(-2\sqrt{2} d_\Sigma(x, \gamma(0)) \right) \geqslant \frac{M_1}{M_2^2} \exp\left(-2\sqrt{2}R \right) \ .
\end{aligned}$$

Where we used lemma 7.14 for the first inequality and lemma 7.11 for the second. □

7.1.4. *Proof of propositions 7.9 and 7.10.* .

Proof of Proposition 7.9. Let γ be the geodesic between $\eta(-s)$ and $\eta(t)$ and observe that γ is at distance at most $K + R$ from x. The result now follows from corollary 7.15. □

Proof of Proposition 7.10. By the quasiisometry property, it is enough to prove the proposition whenever η is a geodesic arc. In other words if

$$\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| \geqslant B \ , \tag{36}$$

then $d_\Sigma(x, y) \leqslant A$ for some constant A and some y on the arc η. Since the curvature of Σ is bounded by $-c$, where c is positive, there exists a point y in η which is R-close to both geodesic arcs $[x, w]$ an $[x, z]$, for R only depending on c. In particular, by Corollary 7.15 we have

$$\left| \frac{\langle x, w \rangle}{\langle x, y \rangle \langle y, w \rangle} \right| \geqslant M_4 \quad \text{and} \quad \left| \frac{\langle x, z \rangle}{\langle x, y \rangle \langle y, z \rangle} \right| \geqslant M_4 \ . \tag{37}$$

Multiplying both inequalities and then inequality (36), we get

$$\left| \frac{\langle z, w \rangle}{\langle x, y \rangle^2 \langle y, w \rangle \langle y, z \rangle} \right| \geqslant B M_4^2 \ . \tag{38}$$

Thus

$$|\langle x, y \rangle|^2 \leqslant \frac{1}{B M_2^2} \left| \frac{\langle z, w \rangle}{\langle y, w \rangle \langle y, z \rangle} \right| \leqslant \frac{M_1}{B M_1^2} \ , \tag{39}$$

where we used proposition 7.7 in the last inequality. As a conclusion, using Theorem 6.5

$$d_\Sigma(x, y) \leqslant \sqrt{2} \eth(x, y) = \sqrt{2} \cosh^{-1}(|\langle x, y \rangle|) \leqslant A := \sqrt{2} \cosh^{-1}\left(\frac{\sqrt{B M_1}}{B M_4} \right) \ .$$

The result follows. □

7.2. **Construction of a quasisymmetric map.** We now build in the second paragraph a candidate to be a quasisymmetric map and proves its property.

7.2.1. *A boundary coincidence.* We will need

Proposition 7.16. *Let $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ be two sequences of points on a quasiperiodic maximal surface Σ so that $d_\Sigma(x_k, y_k)$ is uniformly bounded. Assume that $\{x_k\}_{k\in\mathbb{N}}$ converges to u in $\partial_\infty\Sigma$. Then $\{y_k\}_{k\in\mathbb{N}}$ also converges to u.*

Proof. Let h be the horofunction associated to a non zero vector u_0 in u. Since $\|\nabla h\|^2 < 2$ and $d_\Sigma(x_k, y_k)$ is uniformly bounded, then $|h(x_k) - h(y_k)|$ is uniformly bounded. Since $\lim_{k\to\infty} h(x_k) = -\infty$, we have $\lim_{k\to\infty} h(y_k) = -\infty$ hence $\lim_{k\to\infty}\langle u_0, y_k\rangle = 0$. After extracting a subsequence, we may assume that $\{y_k\}_{k\in\mathbb{N}}$ converges to a point v in $\Sigma \cup \partial_\infty\Sigma$. Let us fix a point z_0 in Σ, then

$$\lim_{k\to\infty} \frac{1}{\langle z_0, y_k\rangle} y_k = v_0 \,,$$

where v_0 is the non zero vector in v so that $\langle v_0, z_0\rangle = 1$. Recall that on a maximal surface $|\langle z_0, y_k\rangle| \geqslant 1$, thus

$$|\langle u_0, v_0\rangle| \leqslant \lim_{k\to\infty} |\langle u_0, y_k\rangle| = 0 \,.$$

Since Σ is quasiperiodic maximal, $\partial_\infty\Sigma$ is positive, so $u_0 = v_0$. This proves the result. $\qquad\square$

7.2.2. *A quasisymmetric map.* Assume Σ is a quasiperiodic maximal surface. By Theorem 6.3, the uniformisation is biLipschitz and thus a quasiisometry. In particular there is a constant K so that any geodesic in $\mathbf{H}^2$ is mapped under the uniformisation to a K-quasigeodesic.

Let us choose once and for all a fixed point x_0 in Σ. All the geodesics we consider here are assumed to be complete. Because Σ is properly embedded, any geodesic γ in $\mathbf{H}^2$ has a properly embedded image. Thus for any point x in $\partial_\infty\mathbf{H}^2$, fix a geodesic γ converging to x and let us consider the (non empty) set B_x of limit values in $\partial_\infty\mathbf{H}^{2,n}$ of all sequences converging to x along γ. Observe that B_x is connected: the set of limit values of a function defined on some interval is connected. Hence B_x a closed interval in $\partial_\infty\Sigma$. Observe now

Proposition 7.17. *Fixing x, $B_{\gamma,x} = B_{\eta,x}$.*

Proof. Choose parametrisations of γ and η so that

$$\lim_{s\to\infty} d_{\mathbf{H}^2}(\gamma(s), \eta(s)) = 0 \,.$$

Now if $\gamma(t_k) \to u$, then since $d_\Sigma(\eta(k), \gamma(t_k)) \to 0$, by proposition 7.16, we also have $\eta(t_k) \to u$. This proves the result. $\qquad\square$

Accordingly, we drop the reference γ and write $B_{\gamma,x} =: B_x$. Let us choose a map

$$\Phi : \partial_\infty\mathbf{H}^2 \longrightarrow \partial_\infty\Sigma \,,$$

such that for any point x in $\partial_\infty\mathbf{H}^2$, the point $\Phi(x)$ belongs to B_x.

From the quasiperiodicity of Σ, we know that $\partial_\infty\Sigma$ is positive. In particular, for any pairwise distinct points x, y, z and t in $\partial_\infty\Sigma$, the cross-ratio $\mathrm{b}(x, y, z, t)$ is well-defined (see definition 3.1). Finally, denote by $[., ., ., .]$ the cross-ratio on $\partial_\infty\mathbf{H}^2$ obtained by identifying $\partial_\infty\mathbf{H}^2$ with $\mathbf{P}(V)$.

Proposition 7.18. *Let Σ and Φ be as above. If (a, b, c, d) is a quadruple of pairwise distinct points in $\partial_\infty\mathbf{H}^2$, then $(\Phi(a), \Phi(b), \Phi(c), \Phi(d))$ are pairwise transverse. Moreover, for every A there is a B such that*

$$A^{-1} < |[a, b, c, d]| < A \ \text{ implies } \ B^{-1} < \left|\mathrm{b}\big(\Phi(a), \Phi(b), \Phi(c), \Phi(d)\big)\right| < B \,.$$

Proof. Since Φ takes value in $\partial_\infty\Sigma$ which is positive, transversality of the image of pairwise distinct points in $\partial_\infty\mathbf{H}^2$ is equivalent to the injectivity of Φ.

For any A, there exists a constant R, such that for any any quadruple (a, b, c, d) of points in $\partial_\infty\mathbf{H}^2$ with

$$A^{-1} \leqslant |[a, b, c, d]| \leqslant A \,,$$

we can find a point p in Σ within a distance R from the four geodesics $\gamma_{ab}, \gamma_{cd}, \gamma_{ad}$ and γ_{cb}, where γ_{ij} is the quasigeodesic in Σ which is the image by the uniformisation of the geodesic between i and j in $\partial_\infty \mathbf{H}^2$ (where $i, j \in \{a, b, c, d\}$). Since by proposition 7.17, we have $B_i = B_{\gamma_{ij}, i}$, we can find sequences $\left\{ t_k^{ij} \right\}_{k \in \mathbb{N}}$, for all distinct pairs $i, j \in \{a, b, c, d\}$ so that

$$\lim_{k \to \infty} \gamma_{ij}(t_k^{ij}) = \Phi(j) \text{ and } \lim_{k \to \infty} \gamma_{ij}(-t_k^{ij}) = \Phi(i)$$

Applying Proposition 7.9 we get that for all distinct i, j we get

$$\left| \frac{\langle \Phi(i), \Phi(j) \rangle}{\langle p, \Phi(i) \rangle \langle p, \Phi(j) \rangle} \right| = \lim_{k \to \infty} \left| \frac{\langle \gamma_{ij}(-t_k^{ij}), \gamma_{ab}(t_k^{ij}) \rangle}{\langle p, \gamma_{ij}(-t_k^{ij}) \rangle \langle p, \gamma_{ij}(t_k^{ij}) \rangle} \right| \geqslant M_2 . \tag{40}$$

In particular, Φ is injective. Moreover, from proposition 7.7 we have

$$\left| \frac{\langle \Phi(i), \Phi(j) \rangle}{\langle p, \Phi(i) \rangle \langle p, \Phi(j) \rangle} \right| \leqslant M_1 . \tag{41}$$

Let us write $(\alpha, \beta, \gamma, \delta) = (\Phi(a), \Phi(b), \Phi(c), \Phi(d))$ and observe that

$$\frac{\langle \alpha, \beta \rangle \langle \gamma, \delta \rangle}{\langle \alpha, \delta \rangle \langle \gamma, \beta \rangle} = \frac{\langle \alpha, \beta \rangle}{\langle p, \beta \rangle \langle p, \alpha \rangle} \cdot \frac{\langle \gamma, \delta \rangle}{\langle p, \gamma \rangle \langle p, \delta \rangle} \cdot \frac{\langle p, \gamma \rangle \langle p, \beta \rangle}{\langle \gamma, \beta \rangle} \cdot \frac{\langle p, \alpha \rangle \langle p, \delta \rangle}{\langle \alpha, \delta \rangle} .$$

Then the inequalities (41) and (40) yields

$$\left(\frac{M_2}{M_1} \right)^2 \leqslant \left| \frac{\langle \alpha, \beta \rangle \langle \gamma, \delta \rangle}{\langle \alpha, \delta \rangle \langle \gamma, \beta \rangle} \right| \leqslant \left(\frac{M_1}{M_2} \right)^2 .$$

Thus the result is proved with $B = \left(\frac{M_1}{M_2} \right)^2$. $\qquad \square$

Proposition 7.19. *The map Φ is positive.*

Proof. From proposition 7.18, for any choice of Φ, Φ maps transverse quadruples to transverse quadruples.

Let H_0 be the totally geodesic hyperbolic plane in $\mathbf{H}^{2,n}$ tangent to Σ at our preferred point x_0. Recall that the positive curve $\partial_\infty \Sigma$ is a graph over any $\partial_\infty H_0$ and let p be the warped projection $\partial_\infty \Sigma \to \partial_\infty H_0$ – see proposition 2.23. Observe that a quadruple (x, y, z, t) in $\partial_\infty \Sigma$ is positive if an only if its projection $(p(x), p(y), p(z), p(t))$ is cyclically oriented in $\partial_\infty H_0$.

Recall that $\Phi(x)$ is any point in the connected subset B_x defined in the beginning of paragraph 7.2.2. The set $p(B_x)$ is an interval I_x. By injectivity of Φ - for any choice of Φ – the intervals I_x and I_y are disjoint of x is different from y.

Let (a, b, c, d) be a positive quadruple in $\partial_\infty \mathbf{H}^2$. To prove that

$$(p(\Phi(a)), p(\Phi(b)), p(\Phi(c)), p(\Phi(d))) ,$$

is cyclically oriented – and hence that $(\Phi(a), \Phi(b)), \Phi(c), \Phi(d))$ is positive – it is then enough to show that there exist points $z(a)$, $z(b)$ $z(c)$ and $z(d)$ in B_a, B_b, B_c and B_d respectively so that $(p(z(a)), p(z(b)), p(z(c)), p(z(d)))$ is cyclically oriented. This is what we are going to prove now.

Let α, β, γ and δ be the four geodesics –with respect to the hyperbolic distance – in $\mathbf{H}^2$ joining x_0 to a, b, c and d respectively. These arcs are pairwise disjoint (except at the origin) and cyclically ordered. The projection on H_0 of these four arcs are four non intersecting embedded arcs which are cyclically ordered. Let now take α_k, β_k, γ_k and δ_k be points in α, β, γ and δ respectively whose projection in H_0 has distance to x_0 exactly k. These four points are cyclically oriented on the circle at distance k from x_0. We may extract a subsequence so that $\{\alpha_k\}_{k \in \mathbb{N}}$, $\{\beta_k\}_{k \in \mathbb{N}}$, $\{\gamma_k\}_{k \in \mathbb{N}}$ and $\{\delta_k\}_{k \in \mathbb{N}}$ also converges to points that we denote $z(a)$, $z(b)$, $z(c)$ and $z(d)$ respectively which belong to B_a, B_b, B_c and B_d respectively. Their projections $p(z(a))$, $p(z(b))$, $p(z(c))$ and $p(z(d))$ to $\partial_\infty H_0$ is then cyclically oriented. This completes the proof that the quadruple $(\Phi(a), \Phi(b), \Phi(c), \Phi(d))$ is positive. The proof follows. $\qquad \square$

7.2.3. *Proof of Theorems 7.1 and 7.2.* As a consequence of propositions 7.19 and 7.18, Φ is quasisymmetric. Hence Φ is Hölder, so continuous, and so uniquely defined.

Both Theorem 7.1 – more precisely that Φ is a continuous extension – and 7.2 now follows from the following assertion:

Let $\{z_k\}_{k\in\mathbb{N}}$ a sequence of points in $\mathbf{H}^2$ converging to z in $\partial_\infty\mathbf{H}^2$, then $\{\Psi_k(z_k)\}_{k\in\mathbb{N}}$ converges to $\Phi(z)$, where $\{\Psi_k\}_{k\in\mathbb{N}}$ is a sequence of uniformisation as in Theorem 7.2.

Let then $\{z_k\}_{k\in\mathbb{N}}$ a sequence of points in $\mathbf{H}^2$ converging to z in $\partial_\infty\mathbf{H}^2$, so that $\{\Psi_k(z_k)\}_{k\in\mathbb{N}}$ converges to a point u.

By construction, there exists a sequence $\{y_k\}_{k\in\mathbb{N}}$ of points in $\mathbf{H}^2$, so that $\{\Psi(y_k)\}_{k\in\mathbb{N}}$ converges to $\Phi(z)$. Using the fact $\{\Psi_k\}_{k\in\mathbb{N}}$ converges on every compact of $\mathbf{H}^2$, we can find a sequence n_k going to ∞, so that $\{\Psi_{n_k}(y_k)\}_{k\in\mathbb{N}}$ converges to $\Phi(z)$. Extracting subsequences and renaming, we have constructed two sequences $\{z_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ so that $\{\Psi_k(z_k)\}_{k\in\mathbb{N}}$ converges to a point u, and $\{\Psi(y_k)\}_{k\in\mathbb{N}}$ converges to $\Phi(z)$. The assertion now follows from the following lemma:

Lemma 7.20. *Let $\{z_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ be sequences of points in $\mathbf{H}^2$ converging to the same point z in $\partial_\infty\mathbf{H}^2$. Then $\{\Psi_k(z_k)\}_{k\in\mathbb{N}}$ and $\{\Psi_k(y_k)\}_{k\in\mathbb{N}}$ converge to the same point in $\partial_\infty\Sigma$.*

Proof. We work by contradiction. Then we would have, after extracting subsequences that $\{\Psi_k(x_k)\}_{k\in\mathbb{N}}$ and $\{\Psi_k(y_k)\}_{k\in\mathbb{N}}$ converges to different points u and v in $\partial_\infty\Sigma$. In particular the quantity

$$\left|\frac{\langle\Psi_k(z_k),\Psi_k(x_0)\rangle\,\langle\Psi_k(x_0),\Psi_k(y_k)\rangle}{\langle\Psi_k(z_k),\Psi_k(y_k)\rangle}\right| ,$$

is bounded above by a positive constant. By proposition 7.10, this implies that the distance between x_0 and the geodesic arc between y_k and z_k is uniformly bounded from above. This would imply that $\{y_k\}_{k\in\mathbb{N}}$ and $\{z_k\}_{k\in\mathbb{N}}$ converges to different points. This is our contradiction and concludes the proof of the lemma. $\qquad\square$

8. Smooth spacelike curves and asymptotic hyperbolicity

Let us say a complete maximal surface Σ is *asymptotically hyperbolic* if the norm of the second fundamental form of Σ goes uniformly to zero as one approaches infinity, or equivalently if its intrinsic curvature goes to -1. We may also use the terminology *asymptotically totally geodesic surfaces*.

Using the following theorem, we obtain asymptotically geodesic complete maximal surface.

Theorem 8.1. [Asymptotically Hyperbolic] *Any C^1 spacelike positive loop in $\partial_\infty\mathbf{H}^{2,n}$ is quasiperiodic and bounds a complete maximal asymptotically hyperbolic surface.*

Here is a corollary

Corollary 8.2. [Smooth rigidity] *Let ρ be a maximal representation of a closed surface group in G. The limit curve of ρ is C^1 and spacelike if and only if ρ factors through a subgroup isomorphic to $(\mathsf{O}(2,1)\times\mathsf{O}(n))_0$.*

Proof. The periodicity of the limit curves implies that the associated maximal surface Σ is periodic. If the limit curve is C^1, Σ is asymptotically totally geodesic, thus it is totally geodesic by cocompactness. The representation then preserves a totally geodesic hyperbolic plane, the stabiliser of which is isomorphic to $(\mathsf{O}(2,1)\times\mathsf{O}(n))_0$. The converse is obvious. $\qquad\square$

One should compare with the smooth rigidity obtained by [GM18] in the case of AdS-quasi-Fuchsian groups (which are in general not surface groups). Glorieux–Monclair also have announced a similar result for $\mathbf{H}^{p,q}$-convex cocompact representations that are deformations of cocompact lattices in $\mathsf{SO}(p,1)$.

Remark that limit curves of Hitchin representations in $\mathsf{SO}(2,3)$ have C^1 limit curves that are lightlike everywhere. In particular, the C^1 rigidity fails without the spacelike condition for $n\geqslant 2$.

8.0.1. *Smooth curves and quasisymmetry.* Let us start with a proposition of independent interest

Proposition 8.3. *If ξ is a C^1 spacelike positive map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$, then it is quasisymmetric. Moreover if $\{(x_k^1, x_k^2, x_k^3, x_k^4)\}_{k \in \mathbb{N}}$ is a sequence of positive quadruples in $\partial_\infty \mathbf{H}^{2,n}$, so that there exists a point x_0 so that for any i in $\{1, 2, 3, 4\}$, $\left\{x_k^i\right\}_{k \in \mathbb{N}}$ converges to x_0, then*

$$\lim_{k \to \infty} \left(\frac{\mathbf{b}^\xi(x_k^1, x_k^2, x_k^3, x_k^4)}{[x_k^1, x_k^2, x_k^3, x_k^4]^2} \right) = 1 .$$

Proof. Let us first show that the map

$$\Phi : (x, y, z, t) \mapsto \frac{\left| \mathbf{b}_\xi(x, y, z, t) \right|}{[x, y, z, t]^2} ,$$

defined for a quadruple of pairwise distinct points in $\mathbf{P}(V)$ extends continuously to $(\mathbf{P}(V))^4$ to a nowhere vanishing function. Let $f : V \setminus \{0\} \to E \setminus \{0\}$ be a smooth $\mathbb{R}^*$ equivariant lift of ξ. Consider a smooth path $(x(t))_{t \in (-\varepsilon, \varepsilon)}$ in V with $x(t) = x_0 + t\dot{x}_0$ where $\omega(x_0, \dot{x}_0) = 1$ so that

$$\omega(x(t), x(s)) = s - t .$$

Write $u(t) = f(x(t))$ and $u(0) = u_0$. Then

$$u(t) - u(s) = \int_s^t \dot{u}(x)\mathrm{d}x = (t - s)\dot{u}(s) + o(t - s) .$$

Thus

$$\lim_{(s,t) \to 0, s \neq t} \left(\frac{\langle u(t), u(s) \rangle}{\omega(x(t), x(s))^2} \right) = \lim_{(s,t) \to 0, s \neq t} \left(\frac{\mathsf{q}(u(t) - u(s))}{2\omega(x(t), x(s))^2} \right) = \lim_{s \to 0,} \frac{1}{2}\mathsf{q}(\dot{u}_s) = \frac{1}{2}\mathsf{q}(\dot{u}_0) ,$$

where we used in the last equality that $\dot{u}$ is continuous. Since f is a spacelike immersion, $\dot{u}(t)$ is a non-zero spacelike vector. Thus the function $\psi(x, y) = \langle f(x), f(y) \rangle (\omega(x, y))^{-2}$ defined for $x \neq y$ in $V \setminus \{0\}$ extends continuously when $x = y$ to a nowhere vanishing function. Thus the $(\mathbb{R}^*)^4$-invariant function on $(V \setminus \{0\})^4$

$$\frac{\psi(x, y)\psi(z, t)}{\psi(x, t)\psi(z, y)} ,$$

pushes down to a nowhere vanishing continuous function on $(\mathbf{P}(V))^4$ extending Φ.

It follows by compactness of $\mathbf{P}(V)^4$ that Φ is bounded below and above by positive constants: we have A so that for all quadruples (x, y, z, t)

$$\frac{1}{A} \leqslant \frac{\left| \mathbf{b}_\xi(x, y, z, t) \right|}{[x, y, z, t]^2} \leqslant A .$$

Hence ξ is quasisymmetric. This completes the proof of the first part.

For the second statement, let us use the same notation. Then

$$\left(\frac{\mathbf{b}_\xi(x_k^1, x_k^2, x_k^3, x_k^4))}{[x_k^1, x_k^2, x_k^3, x_k^4]^2} \right) = \frac{\psi(x_k^1, x_k^2)\psi(x_k^3, x_k^4)}{\psi(x_k^1, x_k^4)\psi(x_k^3, x_k^2)} .$$

Thus by continuity of ψ,

$$\lim_{k \to \infty} \left(\frac{\mathbf{b}_\xi(x_k^1, x_k^2, x_k^3, x_k^4)}{[x_k^1, x_k^2, x_k^3, x_k^4]^2} \right) = \frac{\psi(x_0, x_0)\psi(x_0, x_0)}{\psi(x_0, x_0)\psi(x_0, x_0)} = 1 . \quad \square$$

8.0.2. *Collapsing triples.* Let us say a sequence $\{\kappa_k\}_{k \in \mathbb{N}}$ of positive triples in $\mathbf{P}(V)$ is *collapsing* if $\{\kappa_k\}_{k \in \mathbb{N}}$ converges to a point.

Proposition 8.4. *Let ξ be a C^1 spacelike positive map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$, and let $\{\kappa_k\}_{k \in \mathbb{N}}$ be a sequence of collapsing triples in $\mathbf{P}(V)$. Let $\tau_k = \xi(\kappa_k)$ and*

(i) *g_k an element in G so that $g_k(\tau_k) = \tau_0$*
(ii) *h_k an element in $\mathsf{SL}(V)$ so that $h_k(\kappa_0) = \kappa_k$.*
(iii) *Let finally $\xi_k = g_k \circ \xi \circ h_k$.*

Then the sequence of quasisymmetric maps $\{\xi_k\}_{k \in \mathbb{N}}$ subconverges to a circle map ξ_∞.

Proof. Since ξ is C^1 and spacelike, it is quasisymmetric by proposition 8.3. It follows by the Equicontinuity Theorem 3.12, that we may extract a converging subsequence from $\{\xi_k\}_{k \in \mathbb{N}}$ converging to a limit that we denote ξ_∞.

The sequence $\{h_k\}_{k \in \mathbb{N}}$ goes to infinity in $\mathsf{PSL}(V)$ and thus using the Cartan decomposition and may assume after extracting a subsequence that it is proximal: there exists a point y_0, so that for any z different from y_0, the sequence $\{h_k(y_0)\}_{k \in \mathbb{N}}$ converges to some x_0.

In particular, for any quadruple (x, y, z, t) of pairwise distinct points different from y_0

$$\frac{\mathsf{b}_{\xi_\infty}(x, y, z, t))}{[x, y, z, t]^2} = \lim_{k \to \infty} \left(\frac{\mathsf{b}_{\xi_k}(x, y, z, t)}{[x, y, z, t]^2} \right) = \lim_{k \to \infty} \left(\frac{\mathsf{b}_\xi(h_k(x), h_k(y), h_k(z), h_k(t))}{[h_k(x), h_k(y), h_k(z), h_k(t)]^2} \right) = 1 .$$

where in the last equation we used the second assertion of Proposition 8.3, and in the others the invariance of the cross-ratio by $\mathsf{SL}(V)$ and G.

Using the continuity of ξ_∞, we deduce that ξ_∞ is a positive map so that

$$\mathsf{b}_{\xi_\infty}(x, y, z, t) = [x, y, z, t]^2 ,$$

hence that ξ_∞ is a circle map by proposition 3.4.   □

8.1. **Proof of Theorem 8.1.** Consider Λ be a C^1 spacelike positive loop in $\partial_\infty \mathbf{H}^{2,n}$, ξ a C^1 parametrisation of Λ and $\Sigma(\Lambda)$ the solution to the asymptotic Plateau problem. Given a sequence $\{x_k\}_{k \in \mathbb{N}}$ of points in Σ converging to y_0 in Λ, proposition 4.12 implies that we can find a sequence $\{\tau_k\}_{k \in \mathbb{N}}$ of positive triples in Λ so that $B(\Lambda, \tau_k) = (x_k, \Sigma)$, where B is the barycenter map. By corollary 4.14, we can moreover choose $\{\tau_k\}_{k \in \mathbb{N}}$ to be collapsing to y_0.

By proposition 8.4, we can find a sequence $\{g_k\}_{k \in \mathbb{N}}$ in G such that $\{g_k(\Lambda, \tau_k)\}_{k \in \mathbb{N}}$ converges to (Λ_0, τ_0), where Λ_0 is the circle through τ_0. By continuity of B, the sequence $\{g_k(x_k, \Sigma)\}$ converges to the pointed hyperbolic plane $(x_0, \Sigma_0) = B(\tau_0, \Lambda_0)$. As a result, the curvature of Σ goes uniformly to -1 on any compact containing x_k.

9. Relationship with the universal Teichmüller space

In this final section, we construct in the first paragraph the analogue in our setting of the Bers' universal Teichmüller space $\mathcal{T}(\mathbf{H}^2)$, [Ber65]. In the second paragraph, we describe a natural map from this space to $\mathcal{T}(\mathbf{H}^2)$ that generalises the picture for closed surfaces proved in [CTT19]. Finally, in the last two paragraphs, we propose an analogue of the Hitchin map, and state a conjecture related to works of Qiongling Li and Takuro Mochizuki [LM20].

9.1. **An analogue of the universal Teichmüller space.** Let QS_n the space of quasisymmetric maps from $\mathbf{P}(\mathbb{R}^2)$ to $\partial_\infty \mathbf{H}^{2,n}$. The group $\mathsf{PSL}_2(\mathbb{R})$ acts by precomposition on QS_n while G acts by postcomposition. We denote $\mathcal{QS}_n := \mathsf{QS}_n/\mathsf{G}$ and observe now that $\mathsf{PSL}_2(\mathbb{R})$ acts on $\mathcal{QS}_n$ by precomposition. We equip QS_n with the C^0 topology and $\mathcal{QS}_n$ with the quotient topology.

Let also $\mathcal{T}(\mathbf{H}^2)$ be the group of quasisymmetric homeomorphisms of $\mathbf{P}(\mathbb{R}^2)$ preserving $(0, 1, \infty)$ and observe $\mathcal{T}(\mathbf{H}^2)$ acts by precomposition on QS_n/G. In particular that $\mathcal{QS}_0$ is a principal $\mathcal{T}(\mathbf{H}^2)$ space, in otherwords a torsor over $\mathcal{T}(\mathbf{H}^2)$, which is classically called the *universal Teichmüller space* as introduced by Bers in [Ber65].

We refer to [GH02] for a survey on the universal Teichmüller space and [Leh87] for further references, in particular for the complex Banach manifold structure – which is finer than the topology that we just introduced.

Our proposal is to consider QS_n as an analogue of the universal Teichmüller space in our setting

Let us first explain in which sense one wants to consider QS_n as a universal (higher) Teichmüller space and first recall the $n = 0$ case.

Let S be closed surface of hyperbolic type and let Γ be its fundamental group. The choice of a point x in the Teichmüller space $\mathcal{T}(S)$ of S and a triple τ_0 in $\partial_\infty\Gamma$, defines an identification f_x of $\partial_\infty\Gamma$ with $\mathbf{P}(\mathbb{R}^2)$ with $f_x(\tau_0) = (0, 1, \infty)$ extending a holomorphic map ϕ_x from the universal cover of S – equipped with the complex structure coming from x – to $\mathbf{H}^2$. Thus we obtain a map

$$\Phi^{\tau_0} : \begin{cases} \mathcal{T}(S) \times \mathcal{T}(S) & \to & \mathcal{T}(\mathbf{H}^2)\,, \\ (x, y) & \mapsto & f_y \circ f_x^{-1}\,. \end{cases}$$

Fixing a point x in Teichmüller space and a triple in $\partial_\infty\Gamma$, we obtain the construction of an embedding from $\mathcal{T}(S)$ to the Bers universal Teichmüller space $\mathcal{T}(\mathbf{H}^2)$.

More generally if $Q_p(S)$ denotes the vector bundle over the Teichmüller space $\mathcal{T}(S)$ whose fiber over a point x is $H^0((S, x), \mathsf{K}_x^p)$, and $H_b^0\left(\mathbf{H}^2, \mathsf{K}^p\right)$ the vector space of bounded p-holomorphic differentials on $\mathbf{H}^2$, we obtain a map

$$\Phi_p^{\tau_0} : \begin{cases} \mathcal{T}(S) \times Q_p(S) & \to & \mathcal{T}(\mathbf{H}^2) \times H_b^0(\mathbf{H}^2, \mathsf{K}^p)\,, \\ (x, (y, q)) & \mapsto & \left(f_y \circ f_x^{-1}, (\phi^y)_*(q)\right)\,. \end{cases}$$

Recalling that $\mathcal{T}(\mathbf{H}^2) = QS_0$, the picture describing Φ^{τ_0} generalizes for any n. By [BILW05] an element ρ in $\mathrm{Rep}^{\max}(\Gamma, \mathsf{G})$ gives rise to a positive map F_ρ from $\partial_\infty\Gamma$ to $\partial_\infty\mathbf{H}^{2,n}$, which is unique up to postcomposing by an element in G. So we get a map

$$\Psi^{\tau_0} : \begin{cases} \mathcal{T}(S) \times \mathrm{Rep}^{\max}(\Gamma, \mathsf{G}) & \to & QS_n\,, \\ (y, \rho) & \mapsto & F_\rho \circ f_y^{-1}\,. \end{cases}$$

Then the following relation is satisfied:

$$\Psi^{\tau_0}(x, \rho) \circ \Phi^{\tau_0}(y, x) = \Psi^{\tau_0}(y, \rho)\,,$$

that reflects the action of $\mathcal{T}(\mathbf{H}^2)$ on QS_n described in the begining of the section. Hence fixing a point in Teichmüller space and a triple in $\partial_\infty\Gamma$, we obtain an embedding of $\mathrm{Rep}^{\max}(\Gamma, \mathsf{G})$ in QS_n, thus justifying our proposal of considering QS_n as a universal (higher) Teichmüller space.

9.2. **A fibration.** The action of $\mathcal{T}(\mathbf{H}^2)$ is transitive on QS_0. In our setting, we obtain from the action of $\mathcal{T}(\mathbf{H}^2)$ a fibration that we describe now. Let Λ_n be the space of pointed quasicircles in $\partial_\infty\mathbf{H}^{2,n}$ equipped with the C^0 topology. We have a continuous surjective map

$$p : \begin{cases} QS_n & \to & \Lambda_n\,, \\ \xi & \mapsto & \left(\xi(\mathbf{P}(\mathbb{R}^2)), \xi(0, 1, \infty)\right)\,. \end{cases}$$

Observe that Λ_0/G is a point. We have the following:

Proposition 9.1. *The map p from QS_n to Λ_n constructed above is G-equivariant and a principal $\mathcal{T}(\mathbf{H}^2)$-bundle. It carries a natural G-equivariant section μ. This section is continuous by restriction on the set of (A, B)-quasicircles.*

Proof. By corollary 3.15, the action of $\mathcal{T}(\mathbf{H}^2)$ is simply transitive on the fibre of p, giving it the structure of a principal $\mathcal{T}(\mathbf{H}^2)$-bundle.

By Theorem 7.1, any pointed quasicircle (τ, Λ) comes with a preferred quasisymmetric parametrisation $\mu(\tau, \Lambda)$ that we see as an element of QS_n – which is the extension of the uniformisation of the quasiperiodic surface $\Sigma(\Lambda)$ sending $(0, 1, \infty)$ to τ. The G-equivariance is direct from the construction.

By Theorem 7.2 and corollary 7.4, the restriction of the section μ on the set of (A, B)-quasicircles is continuous. $\square$

It follows from the G-equivariance of p and the fact that G acts properly on $\mathcal{L}(n)$ – thus on Λ_n – that the continuous action of G on $\mathcal{QS}_n$ is also proper. In particular, the quotient $\mathcal{QS}_n$ is a Hausdorff topological space. It would be interesting to see whether the fibration p could make sense as a fibration of (conjectural) complex Banach manifolds.

9.3. **A map to the universal Teichmüller space.** Using the principal $\mathcal{T}(\mathbf{H}^2)$ bundle structure on $\mathcal{QS}_n$ and the section μ, we can define a map

$$\pi_{\mathbf{H}^2} : \mathcal{QS}_n \longrightarrow \mathcal{T}(\mathbf{H}^2) \, ,$$

by setting $\pi_{\mathbf{H}^2}(\xi) = \varphi$, where φ is such that $\xi \circ \varphi = \mu(p(\xi))$. For $n = 0$, this map is a homeomorphism. Moreover

Proposition 9.2. *The restriction of the projection $\pi_{\mathbf{H}^2}$ to the space of (A, B)-quasisymmetric maps is continuous. Moreover the image of the set of (A, B)-quasisymmetric maps is included in the space of (A, D)-quasisymmetric homeomorphisms where D only depends on A and B.*

Proof. For any constants A and B greater than 1, there exist positive constants C and D with the following properties. Given an (A, B)-quasisymmetric map ξ, its image is an (A, C) quasicircle by corollary 7.5. Then the reparametrisation $\pi_{\mathbf{H}^2}$ is an (A, D)-quasisymmetric homeomorphism by corollary 3.15. Then the result follows by the Equicontinuity Theorem 3.12. $\square$

This map is an infinite dimensional analogue of the following finite dimensional case. When S is a closed surface of negative Euler characteristic, it was proved in [CTT19] for $n \geqslant 2$ and [BBZ03] for $n = 1$ that the space $\mathrm{Rep}^{\mathrm{max}}(S, \mathsf{G})$ of maximal representations of $\pi_1(S)$ into G parametrises the space of $\pi_1(S)$-invariant maximal surfaces in $\mathbf{H}^{2,n}$. We thus obtain a map

$$\pi_S : \mathrm{Rep}^{\mathrm{max}}(S, \mathsf{G}) \to \mathcal{T}(S) \, , \tag{42}$$

sending a representation to the complex structure of its induced metric. Then we have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{T}(S) \times \mathrm{Rep}^{\mathrm{max}}(S, \mathsf{G}) & \xrightarrow{\Psi^{\tau_0}} & \mathcal{QS}_n \\
\downarrow{\scriptstyle (\mathrm{Id}, \pi_S)} & & \downarrow{\scriptstyle \pi_{\mathbf{H}^2}} \\
\mathcal{T}(S) \times \mathcal{T}(S) & \xrightarrow{\Phi^{\tau_0}} & \mathcal{T}(\mathbf{H}^2)
\end{array}
$$

9.4. **A universal Hitchin map.** The (finite dimensional) Hitchin map for G-Higgs bundles as defined in [GPPNR18], extending the classical picture discovered by Hitchin in [H$^+$87], also has an an infinite dimensional analogue in our setting.

When S is a closed surface of negative Euler characteristic the Hitchin map associates a holomorphic quartic differential q_4 in $H^0(X, \mathsf{K}_X^4)$ to any maximal representation $\rho : \pi_1(S) \to \mathsf{G}$, where $X = \pi_S(\rho)$ (where π_S is defined in equation (42)). The Hitchin map defines a map

$$h_S : \mathrm{Rep}^{\mathrm{max}}(S, \mathsf{G}) \longrightarrow Q_4(S) \, . \tag{43}$$

It is proved in [H$^+$87] that h_S is proper and related to integrable systems.

The quartic differential corresponding to $h_S(\rho)$ has an explicit description in terms of pseudo-hyperbolic geometry. If Σ is the maximal surface in $\mathbf{H}^{2,n}$ preserved by $\rho(\pi_1(S))$, the second fundamental form of Σ is the real part of a holomorphic section σ of $\mathsf{K}^2 \otimes \mathsf{N}^\mathbb{C}\Sigma$, where $\mathsf{N}^\mathbb{C}\Sigma$ is the complexification of the normal bundle of Σ – see Paragraph 5.1.1. The tensor $q_4 = g_N(\sigma, \sigma)$ is the holomorphic quartic differential associated to $h_S(\rho)$. In fact, with the right choice of basis for Ad-invariant homogeneous polynomials, the quartic differential associated to a Higgs bundle

$(\mathcal{E}, \Phi)$ equals $\mathrm{tr}(\Phi^4)$. Given the Higgs bundle description of equivariant maximal surfaces as in [CTT19, Proof of Proposition 3.17], we have that $q_4 = 4\beta^\dagger \beta$. But [CTT19, Remark 3.18] implies that $\sigma = 2\beta$ and so $q_4 = g_N(\sigma, \sigma)$.

In our setting, we can thus associate a holomorphic quartic differential q_4 to any complete maximal surface Σ in $\mathbf{H}^{2,n}$. Since maximal surfaces have bounded geometry, such a quartic differential is bounded with respect to the induced metric. When Σ is quasiperiodic, by Theorem 6.3, the uniformisation gives a biLipschitz map between $\mathbf{H}^2$ and Σ, so q_4 is bounded with respect to the hyperbolic metric. Denoting by $H_b^0(\mathbf{H}^2, \mathsf{K}^4)$ the vector space of holomorphic quartic differential on $\mathbf{H}^2$ that are bounded with respect to the hyperbolic metric, we finally obtain a map

$$h_{\mathbf{H}^2} : \mathcal{QS}_n \longrightarrow \mathcal{T}(\mathbf{H}^2) \times H_b^0(\mathbf{H}^2, \mathsf{K}^4) \,.$$

This map is again related to the map described in equation (43) through the commutative diagram.

$$
\begin{array}{ccc}
\mathcal{T}(S) \times \mathrm{Rep}^{\max}(S, \mathsf{G}) & \xrightarrow{\;\;\Psi^{\tau_0}\;\;} & \mathcal{QS}_n \\
\downarrow{\scriptstyle (\mathrm{Id}, \pi_S)} & & \downarrow{\scriptstyle h_{\mathbf{H}^2}} \\
\mathcal{T}(S) \times Q_4(S) & \xrightarrow{\;\;\Phi^{\tau_0}\;\;} & \mathcal{T}(\mathbf{H}^2) \times H_b^0(\mathbf{H}^2, \mathsf{K}^4)
\end{array}
$$

When we wrote our first draft of the paper, Qiongling Li informed us she has a proof of the fact that $h_{\mathbf{H}^2}$ is proper. Her proof relies on our Theorem A, Theorem 7.2 and her recent work with Takuro Mochizuki [LM20, Proposition 3.12].

9.5. **The Hitchin–Li–Mochizuki section.** The case $n = 2$ is of special interest. In this case, the group $\mathsf{SO}_0(2, 3)$ being $\mathbb{R}$-split, Hitchin result [Hit92] guarantees the existence of a *Hitchin section* σ_S of the Hitchin map h_S whose image consists of Hitchin representations. These Hitchin representations being positive, they are associated to equivariant positive –hence quasisymmetric– loops in $\partial_\infty \mathbf{H}^{2,n}$ by [BILW05].

More precisely, Hitchin constructed in [Hit92] a section σ_S of the map h_S defined in equation (43). His construction, together with the proof of Labourie's conjecture for $\mathsf{SO}_0(2, 3)$ (see [Lab17]) implies that σ_S is a diffeomorphism onto its image, which corresponds to the Hitchin component. Finding the maximal surface corresponding to (x, q_4) in Q_4 is equivalent to solving the corresponding Toda equations – see [Bar10, Section 2] for more details.

In a recent preprint [LM20], Qiongling Li and Takuro Mochizuki solve the Toda equations for a given bounded holomorphic quartic differential on $\mathbf{H}^2$. In particular, they prove in [LM20, Theorem 1.8] that the corresponding maximal surface is complete and conformally biLispchitz to the hyperbolic disc. Thus combining their work with Theorem 6.3, we get

Proposition 9.3. *For $n = 2$, the Hitchin map $h_{\mathbf{H}^2}$ defined above admits a section $\sigma_{\mathbf{H}^2}$ – that we call the Hitchin–Li–Mochizuki section – related to the section σ_S through the following commutative diagram.*

$$
\begin{array}{ccc}
\mathcal{T}(S) \times \mathrm{Rep}^{\max}(S, \mathsf{G}) & \xrightarrow{\;\;\Psi^{\tau_0}\;\;} & \mathcal{QS}_n \\
{\scriptstyle (\mathrm{Id}, \sigma_s)} \upharpoonleft \;\; \downarrow{\scriptstyle (\mathrm{Id}, \pi_S)} & & {\scriptstyle \sigma_{\mathbf{H}^2}} \upharpoonleft \;\; \downarrow{\scriptstyle h_{\mathbf{H}^2}} \\
\mathcal{T}(S) \times Q_4(S) & \xrightarrow{\;\;\Phi^{\tau_0}\;\;} & \mathcal{T}(\mathbf{H}^2) \times H_b^0(\mathbf{H}^2, \mathsf{K}^4)
\end{array}
$$

We finish with the following conjecture

Conjecture 9.4. *The image of the Hitchin–Li–Mochizuki section consists of quasisymmetric maps whose image is C^1 and lightlike everywhere.*

APPENDIX A. HERMITIAN BUNDLES AND BOCHNER FORMULA

In this appendix we recall the basic definitions of Hermitian vector bundles over a Riemann surface and prove the classical Bochner's formula and refer to [Wel, Chapter III] for more details.

A.1. **Hermitian vector bundles.** Consider a Riemann surface X. A *holomorphic vector bundle* over X is a is complex vector bundle E over X equipped with a *Dolbeaut operator*,

$$\overline{\partial}_\mathsf{E} : \Omega^0(X, \mathsf{E}) \longrightarrow \Omega^{0,1}(X, \mathsf{E}) \,,$$

satisfying the Leibniz rule $\overline{\partial}_\mathsf{E}(f\sigma) = \overline{\partial}f \otimes \sigma + f \cdot \overline{\partial}_\mathsf{E}\sigma$ where f is a function and σ a section of E. We say that σ is a *holomorphic section* if $\overline{\partial}_\mathsf{E}\sigma = 0$.

Observe that, if ∇ is a connection on a complex vector bundle, then $\nabla^{0,1}$ (the $(0,1)$-part of ∇) is a Dolbeaut operator. In this case, a holomorphic section is a section σ such that for any vector field x we have

$$\nabla_{jx}\sigma = J\nabla_x\sigma \,, \tag{44}$$

where j and J are respectively the complex structure on $\mathsf{T}X$ and E.

A *Hermitian metric h* on E is a section of the bundle $\mathsf{E}^* \otimes \overline{\mathsf{E}^*}$ (where $\overline{\mathsf{E}^*}$ is the bundle of $\mathbb{C}$-antilinear form on E) which is positive definite and satisfies $h(\sigma_1, \sigma_2) = \overline{h(\sigma_2, \sigma_1)}$.

A connection ∇ on (E, h) is *unitary* if

$$\mathrm{d}h(\sigma_1, \sigma_2) = h(\nabla\sigma_1, \sigma_2) + h(\sigma_1, \nabla\sigma_2) \,.$$

If (E, h) is a holomorphic vector bundle equipped with a Hermitian metric, there is a unique unitary connection ∇ such that $\nabla^{0,1} = \overline{\partial}_\mathsf{E}$. The connection ∇ is called the *Chern connection*.

Finally, if ∇ is a unitary connection, its *curvature R* is the tensor in $\Omega^2(X, \mathrm{End}(\mathsf{E}))$ defined by

$$R(x, y)\sigma = \nabla_x\nabla_y\sigma - \nabla_y\nabla_x\sigma - \nabla_{[x,y]}\sigma \,.$$

Observe that $R(x, y)$ is skew-hermitian with respect to h.

A.2. **Bochner formula.** We now prove the Bochner formula used in Section 5.1.2.

Let E be a holomorphic vector bundle over a Riemann surface X, h a Hermitian metric on E. We denote by $\langle ., . \rangle$ the corresponding pairing and by ∇ the Chern connection.

Fix a Riemannian metric g on X and denote by Δ the corresponding Laplace–Beltrami operator.

Proposition A.1. [BOCHNER FORMULA] *Using the same notations as above, if σ is a holomorphic section of E and $f := \frac{1}{2}\|\sigma\|^2$, then*

$$\Delta f = \langle R(e_1, e_2)\sigma, J\sigma \rangle + \|\nabla\sigma\|^2 \,,$$

where R is the curvature of ∇, and (e_1, e_2) is an orthonormal framing of $\mathsf{T}X$.

Proof. Let u, v be sections of $\mathsf{T}X$, then $\mathrm{d}f(u) = h(\nabla_u\sigma, \sigma)$, and

$$\nabla^2 f(u, v) = \langle \nabla_u\nabla_v\sigma, \sigma \rangle + \langle \nabla_v\sigma, \nabla_u\sigma \rangle \,,$$

$$\Delta f = \mathrm{tr}_g(\nabla^2 f) = \sum_{i=1}^{2} \langle \nabla_{e_i}\nabla_{e_i}\sigma, \sigma \rangle + \|\nabla\sigma\|^2 \,.$$

Using equation (44) and the fact that J is ∇-parallel, we have

$$\langle \nabla_{e_1}\nabla_{e_1}\sigma, \sigma \rangle = -\langle \nabla_{e_1}\nabla_{Je_2}\sigma, \sigma \rangle = -\langle J\nabla_{e_1}\nabla_{e_2}\sigma, \sigma \rangle = \langle \nabla_{e_1}\nabla_{e_2}\sigma, J\sigma \rangle \,.$$

For the last equation, we used the $\langle J\alpha, J\beta \rangle = \langle \alpha, \beta \rangle$. The result follows from a similar computation for $\langle \nabla_{e_2}\nabla_{e_2}\sigma, \sigma \rangle$. $\square$

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