Abstract. The purpose of this note is to provide an introduction to several articles concerning k-surfaces [3], [6], and more specially random ones [4]. Recall briefly that a *k-surface* is an immersed surface in M, a Riemannian manifold with curvature less than -1, such that the product of the principal curvatures is k, where $k \in]0,1[$. Following these articles, we explain that k-surfaces possess (like geodesics) a "genuine" laminated phase space which has chaotic properties similar to those of the geodesic flow, and that, furthermore, the dynamics on this space can be coded, hence producing transversal measures.

The phase space of k-surfaces

François LABOURIE[1] *

Université Paris-Sud, Topologie et Dynamique, F-91405 Orsay Cedex, France

1 Presentation

This paper will be mainly expository, and will not contain any new material except in section 7 where we shall see a particular and easy case of the constructions of [4]. The first two sections contain an elementary discussion of rather general nature of what a phase space for a PDE could be. We then concentrate on k-surfaces.

I would like to thank the organizers of the Cambridge March Workshop, M. Burger, S. Dani, A. Eskin and G. Margulis for giving me the opportunity to talk and to learn beautiful mathematics in this very enjoyable conference.

2 The geodesic flow

Let's start with classical Riemannian differential geometry. From this viewpoint, the geodesic flow is associated to a second order differential differential equation. More precisely, geodesics in a Riemannian manifold can be described as curves tangent to a vector field in a *phase space*, the unit tangent bundle. Cauchy-Lipschitz theorem then tells us that through every point in the phase space runs one and exactly one geodesic: *the phase space is foliated by geodesics.*

It is well known, in the negatively curved context, that the geodesic flow is a central geometric object, in studying properties of the fundamental group for instance. It is therefore very desirable to have an object analogous to the geodesic flow in the more general context of metric length spaces such as graphs, simplex. At first glance, the construction explained above does not make sense in our new context. However, it is easy and now classical to bypass this difficulty. The trick is the following. One should not forget that geodesics are locally length minimizing curves. Now, this last notion makes sense in the context of metric length spaces such as graphs, simplices, etc. Then, the *phase space* of the geodesic flow in our length space is going to be the space of pairs (x, γ) where γ is a geodesic and x is a point of γ. We recover our picture: *the phase space is foliated by geodesics.* Indeed each geodesic γ gives a curve in our phase space, namely the set (x, γ) when x moves along γ, and the collection of all these curves (that we may call again geodesics) foliates, or more precisely laminates the phase space. As expected, this phase

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space is the right counterpart of the geodesic flow. One should stress that such spaces are even useful in classical differential geometry: it is known (thanks to the work of Bowen, Ratner) that the geodesic flow of a compact negatively curved Riemannian manifold is efficiently described (or coded) by the (oriented) geodesic flow of a finite oriented graph. Among the many uses of this coding, one should notice the construction of a large family of invariant measures for the geodesic flow, and the thermodynamic formalism.

Before proceeding any further, I should say I have been extremely lousy in describing the phase space, since I said nothing about it's topology. When one is more precise, one has to pay a little attention to multiple covers of closed geodesics in order to obtain a Hausdorff space.

3 One more dimension

Geodesics are one dimensional objects. Now we shall turn to 2-dimensional objects and start again with differential geometry. Differential geometers have always been interested in surfaces solutions of partial differential equations, mainly elliptic, and usually coming from variational problems, such as minimal surfaces, holomorphic curves ... The main focus is to describe solutions with such and such property, for instance compactness. This study can be considered as the analog of the study of closed geodesics. The main difficulty is that there is no such a powerful and versatile theorem as is Cauchy-Lipschitz's for ODE.

Statistical Field Theory and String Theory have made us used to the idea that it is important to study the space of all solutions of a given PDE and to try to obtain statistical informations about them; what is a random minimal surface, or a random holomorphic curve ? In other words to study the *phase space* for the PDE, whatever it could be.

Let's start with an elementary remark. Recall that for an ODE, the phase space is the space of admissible initial solutions. For an elliptic PDE, the analogous of an initial condition is, for instance, a boundary value. It follows that a phase space for a PDE is expected to be infinite dimensional. As a consequence there is no hope that an interesting phase space could be a finite dimensional manifold with an integral plane distribution and the picture of the unit tangent bundle with a vector field is certainly too naive and cannot be reproduced.

I sketch now two approaches aiming at the construction of a phase space, trying to keep in mind our now familiar picture *the phase space is foliated by solutions.*

- As I underlined above, lots of the thermodynamical properties of a hyperbolic dynamical system, such as the geodesic flow of a negatively curved manifold, can be described using only closed orbits. For instance the partition function or the random value of an observable are limit of similar objects on finite dimensional spaces spaces of closed orbits of growing length. Hence, by analogy, a very powerful approach is then to describe the system, and more specifically random variables, by approaching it by finite dimensional moduli spaces, or, roughly saying, spaces of compact solutions of growing complexity. Notice here that the lamination picture is not clear.

- Another idea in order to build a phase space, is to use the same trick as we did to study the geodesic flow of a length space. We can set our phase space to be the set of pairs (x, Σ), where Σ is a solution of the PDE, correctly interpreted as a submanifold in a jet space, and x a point of Σ. Each solution Σ is now interpreted as a leaf in the phase space, namely the set of all the (x, Σ) for x running on Σ. As it was before, the collection of all these leaves give a partition of the phase space.
  This construction is very natural and appealing, but in its full generality, it raises difficulties which cannot be discarded as mere technical problems. Let us state at least one: in some case, like holomorphic curves for instance, singularities and ramified coverings of a solution have to be considered; it is therefore not clear how to make the phase space Hausdorff. Nevertheless, this approach has been used very fruitfully from a measure theoretical point of view by M. Gromov in [2]. What I will explain later follows that line.

To conclude this general discussion, let's move aside from differential geometry and consider combinatorial questions. Start a finite 2-dimensional simplicial complex. The analog of a surface solution of a PDE is a simplicial mapping of a combinatorial surface in the complex, satisfying certain local rules. In the case of geodesics in a finite graph, we required for the local rule: geodesics do not go backwards. In this situation, it is again natural to study the phase spaces as a space of pairs as before. The interest of this construction is that is easy to build transversal measures in that situation by projective constructions, even though the leaves do not have polynomial growth. It follows that if we can code in some way the phase space of a PDE by that of a combinatorial object, we would obtain transversal measures.

So far, the discussion has been very general and very uneffective. However, the aim of this note is not to speculate about the existence of a general theory, but to concentrate on a specific example. This preliminary exposition of well known facts and principles is just meant as a justification of the constructions we shall now concentrate on.

4 A hyperbolic example

The geometric situation I want to discuss is the following. Let M be a compact 3-manifold with curvature less than -1, and $\tilde{M}$ its universal cover. Let k be a fixed number in $]0, 1[$. A k-surface is a surface whose extrinsic curvature ($i.e$ the product of the principal curvatures) is k. I will describe a compact phase space with a laminated structure for this equation, and explain how to code such a space and thus describe transversal measures. The content of this note is as follows:

5 GEOMETRIC PROPERTIES OF k-SURFACES AND EXAMPLES: In particular, we explain how a k-surface is described by a local homeomorphism of a surface in the boundary at infinity of $\tilde{M}$ as is shown in [3].
6 PHASE SPACE: We describe the phase space $\mathcal{M}$ of k-surfaces using the construction alluded above. In this context, this phase space makes sense, and furthermore it has a natural compactification, which has the structure of a lamination. This behavior is specific to a class of elliptic problems, usually associated to Monge-Ampere equations and studied in [6]. Furthermore this phase space has "chaotic" properties as described by the main result of [3].
7 TRANSVERSAL MEASURES AND CODING: In this section, we state the main result of [4] asserting the existence of many transversal ergodic and full support measures on $\mathcal{M}$, when the metric of M is homotopic, through negatively curved metrics, to a hyperbolic one. This is achieved through a "coding" argument. We explain a specific construction of one such measure.
8 QUESTIONS: We list several questions about that construction.

The general conclusion is that the phase space for k-surfaces presents lot's of similarity with the geodesic flow and has characteristics (coding, chaos, and stability) which are usually associated with hyperbolic dynamics.

5 Geometric properties of k-surfaces and examples

Let M be a compact 3-manifold with curvature less than -1. Let $k \in]0,1[$ be a real number. All definitions and results are expanded in [3].

5.1 Definition

If S is an immersed surface in $\tilde{M}$, it carries several natural metrics. By definition, the *u-metric* is the metric induced from the immersion in the unit tangent bundle given by the Gauss map. We shall say a surface is *u-complete* if the u-metric is complete.

A *k-surface* is an immersed u-complete connected surface such that the determinant of the shape operator (*i.e.* the product of the principal curvatures) is constant and equal to k, where $k \in]0,1[$.

Since k-surfaces are solutions of an elliptic problem, the germ of a k-surface determines the k-surface. It follows that a k-surface is determined by its image, up to coverings. More precisely, for every k-surface S immersed by f in N, there exists a unique k-surface Σ, the *representative of* S, immersed by ϕ, such that for every k-surface $\bar{S}$ immersed by $\bar{f}$ satisfying $f(S) = \bar{f}(\bar{S})$, there exists a covering $\pi : \bar{S} \to \Sigma$ such that $\bar{f} = \phi \circ \pi$.

By a slight abuse of language, the expression "k-surface" will generally mean "representative of a k-surface".

5.2 Local convexity

From the inequality $k < 1$ and from Gauss equation, we see that k-surfaces have intrinsic negative curvature. If M has constant curvature k_0, then a k-surface has intrinsic curvature $k + k_0$. On the other hand, the inequality $k > 0$, implies that a k-surface S is *locally convex*, that is: every point in S has a neighborhood in S, which is included in the boundary of a geodesically convex set of $\tilde{M}$.

5.3 Examples

In the hyperbolic 3-space, the surfaces equidistant to a totally geodesic plane are negatively curved and have extrinsic constant curvature. These are the simplest examples of k-surfaces, and justify the idea that k-surfaces are generalisations of totally geodesic surfaces.

More generally, the observation of the previous paragraph 5.2 will allow us to describe various ways of constructing k-surfaces following [3], as solutions of an *asymptotic Plateau problem.*

Let S be a locally convex surface immersed in $\tilde{M}$. Let ν_S be the exterior normal vector field to S. The *Gauss-Minkowski* (Figure 1) map from S to the boundary at infinity $\partial_\infty \tilde{M}$ of $\tilde{M}$ is the local homeomorphism n_S:

$$\begin{cases} S \to \partial_\infty \tilde{M} \\ x \mapsto n_S(x) = \exp(\infty \nu_S(x)). \end{cases}$$

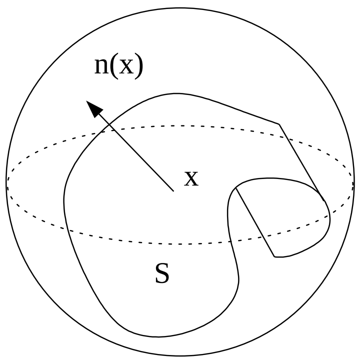

Fig. 1. Gauss-Minkowski map

An *asymptotic Plateau problem* is a pair (i, U) where U is a surface, and i is a local homeomorphism from U to $\partial_\infty \tilde{M}$. A k-solution to an asymptotic Plateau problem (i, U), is a k-surface S immersed in $\tilde{M}$, such that there exists a homeomorphism g from U to S such that $i = n_s \circ g$.

We now state existence and uniqueness results for k-surfaces from [3] using this terminology.

Theorem 1. *There exists at most one solution of a given asymptotic Plateau problem.*

Theorem 2. *Let (i, S) be an asymptotic Plateau problem. Assume $\partial_\infty \tilde{M} \setminus i(S)$ contains at least three points, then (i, S) admits a solution.*

Notice that 2 points in the boundary at infinity are associated with a geodesic, and this last result says more points can be represented by a k-surface.

Theorem 3. *Let Γ be a group acting on S. Assume S/Γ is a compact surface of genus at least 2. Let ρ be a representation of Γ in the isometry group of $\tilde{M}$. If i satisfies*

$$\forall \gamma \in \Gamma, \qquad i \circ \gamma = \rho(\gamma) \circ i,$$

then (i, S) admits a solution.

Theorem 4. *Let (i, U) be an asymptotic Plateau problem. Let S be a relatively compact open subset of U. Then (i, S) admits a solution.*

On the other hand, an asymptotic Plateau problem does not always have a solution:

Theorem 5. *Let S be $\partial_\infty \tilde{M}$ with zero, one or two points removed. Let i be the injection of S in $\partial_\infty \tilde{M}$, then (i, S) does not have a solution.*

Thus, these results provide the existence of many k-surfaces. Next, we will examine the collection of all these objects.

6 Phase space

6.1 Description of the phase space

First, let's introduce a definition. The *tube* of a geodesic is the set of normal vectors to this geodesic. It is a 2-dimensional submanifold of the unit tangent bundle.

The *phase space for k-surfaces* is the space of pairs (Σ, x) where $x \in \Sigma$ and Σ is either the representative of a k-surface or a tube. We denote it by $\mathcal{M}$. It inherits a topology coming from the topology of pointed immersed 2-manifolds in the unit tangent bundle (*cf* section 2.3 of [3]). Each k-surface (or tube) S_0 determines a *leaf* $\mathcal{L}_{S_0}$ defined by

$$\mathcal{L}_{S_0} = \{(S_0, x)/x \in S_0\}.$$

We proved in [6] that $\mathcal{M}$ is compact. Furthermore, the partition of $\mathcal{M}$ into leaves is a lamination, *i.e.* admits a local product structure. Notice that $\mathcal{M}$ has two parts:

(1) a dense set which turns out to be infinite dimensional, and which truly consists of k-surfaces,
(2) a "boundary" consisting of the reunion of tubes, closed, finite dimensional, and which is a S^1 fibre bundle over the geodesic flow.

Therefore, in some sense, $\mathcal{M}$ is an extension of the geodesic flow. To enforce this analogy, one should also notice that the 1-dimensional analogue, namely the space of curves of curvature k in a hyperbolic surface is precisely the geodesic flow.

The fact that the phase space can be compactified in such a way is shared by a class of elliptic equations on surfaces, which I called Monge-Ampère and studied in [6]. Roughly these are holomorphic curves transverse to a totally real foliation.

The existence of this space $\mathcal{M}$ is a fancy way to state a compactness theorem for k-surfaces. The underlying compactness phenomenon is the following: k-surfaces degenerate by rolling around a geodesic as shown in figure 2.

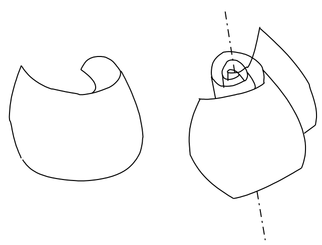

Fig. 2. rolling around a geodesic

6.2 Chaotic properties

The main theorem of [3] which we quote now shows that $\mathcal{M}$, as a dynamical system, enjoys the chaotic properties of the geodesic flow:

Theorem 6. *Let $k \in]0,1[$. Let M be a compact 3-manifold. Let h be a Riemannian metric on M with curvature less than -1. Let $\mathcal{M}_h$ be the space of k-surface of M. Then*

(*i*) *a generic leaf of $\mathcal{M}_h$ is dense,*
(*ii*) *for every positive number g, the union of compact leaves of $\mathcal{M}_h$ of genus greater than g is dense,*
(*iii*) *if $\bar{h}$ is close to h, then there exists a homeomorphism from $\mathcal{M}_h$ to $\mathcal{N}_{\bar{h}}$ sending leaves to leaves.*

This last property will be called the *stability* property.

7 Transversal measure and coding

7.1 First examples

Let's first show some simple examples of natural transversal measures on $\mathcal{M}$. The first three ones are ergodic. They all come from the existence of natural finite dimensional subspaces in $\mathcal{M}$.

- *Dirac measures* supported by closed leaves. By theorem 6 (*ii*), there are plenty of them.
- *Ergodic measures for the geodesic flow.* Indeed, ergodic and invariant measures for the geodesic flow give rise to transversal measures on the space of tubes, hence on the space of k-surfaces.
- *Haar measures for totally geodesic planes.* Assume M has constant curvature. Then, the space of oriented totally geodesic planes carries a transversal measure. Indeed, the Haar measure for $SL(2,\mathbb{C})/\pi_1(N)$ is invariant under the $SL(2,\mathbb{R})$ action. But every oriented totally geodesic plane gives rise to a k-surface, namely the equidistant one to the geodesic plane. This way, we can construct an ergodic transversal measure on $\mathcal{M}$, when M has constant curvature. Its support is finite dimensional.
- *Measures on spaces of ramified coverings.* We sketch briefly here a construction yielding transversal, but non ergodic, measures on $\mathcal{M}$. Let $\partial_\infty \tilde{M}$ be the boundary at infinity of the universal cover $\tilde{M}$ of M. Let Σ be an oriented surface of genus g. Let π be a a topological ramified covering, defined up to homeomorphism of the source, of Σ into $\partial_\infty \mathbb{H}^3$. Let S_π be the set of singular points of π and s_π its cardinal. Let S be a set of extra marked points of cardinal s. Assume $2g + s_\pi + s$. One can show following the ideas of the proof of theorem 7.3.3 of [3] that such a ramified covering can be represented by a k-surface. More precisely, there exists a solution to the asymptotic Plateau problem (as described in paragraph 5.3) represented by $(\pi, \Sigma \setminus (S_\pi \cup S))$. Let now $[\pi]$ be the space of ramified coverings equivalent up to homeomorphisms of the target, to π. The group $\pi_1(N)$ acts properly on $[\pi]$, and explicit invariant measures can be obtained using equivariant family of measures (as defined in [9]) and configuration spaces of finite points. Since $[\pi]/\pi_1(N)$ is a space of leaves of $\mathcal{N}$, this yields transversal measures on this latter space.

None of these examples have full support, and they all have finite dimensional support. So far, apart from these and the construction I will present in the article [4], I do not know of other examples of transversal measures easy to construct.

7.2 Main theorem

We now quote the main theorem of [4].

Theorem 7. *Let M be a compact 3-manifold with curvature less than -1. Assume the metric on M can be deformed, through negatively curved metrics, to a constant curvature one. Then the space of k-surfaces admits infinitely many mutually singular, ergodic transversal finite measures of full support.*

7.3 First remarks

Restriction to the constant curvature case The restriction upon the metric is a severe one. Actually, thanks to the stability property (*iii*) of theorem 6, in order to prove our main result, it suffices to show the existence of transversal ergodic finite measures of full support in the case of constant curvature manifolds.

Choices made in the construction The measure we construct on $\mathcal{M}$ depends on several choices, and various choices lead to mutually singular measures.

We describe now one of the crucial choices needed in the construction.

Let $\tilde{M}$ be the universal cover of M. Let $\partial_\infty\mathbb{H}^3$ be its boundary at infinity. Let $\mathcal{P}(\partial_\infty\mathbb{H}^3)$ be the space of probability Radon measures on $\partial_\infty\mathbb{H}^3$. Let

$$O_3 = \{(x, y, z) \in \partial_\infty\mathbb{H}^3 / x \neq y \neq z \neq x\}.$$

The construction requires a map ν, invariant under the natural action of $\pi_1(N)$,

$$O_3 \xrightarrow{\nu} \mathcal{P}(\partial_\infty\mathbb{H}^3).$$

Here, $\nu(x, y, z)$ is assumed to be of full support, and to fall in the same measure class, independently of (x, y, z). Such maps are easily obtained through *equivariant family of measures* (also described in F. Ledrappier's article [9] as *Gibbs current*, *crossratios etc*) and a barycentric construction.

7.4 Strategy of the proof

As we said in the introduction, the construction is obtained through a coding of the space of k-surfaces. We give now a heuristic, non rigorous, outline of a special case of the construction.

From the stability property, we can assume $\tilde{M}$ has constant curvature.

Pleated and k-surfaces The limit case $k = 0$ corresponds to locally convex pleated surfaces. Here is a more precise statement, which may help in the visualisation: we proved in [8] that a geometrically finite end is foliated by k surfaces for k varying in $]0, 1[$, furthermore when k goes to zero, the corresponding sequence of k-surfaces converge to the pleating locus associated to the convex core of the end.

We shall soon describe an interesting "coded" family of pleated surfaces. Let's first start with some definitions.

The infinite trivalent planar tree Let T be the infinite trivalent planar tree. Let A be the set of edges of T. It is useful now to consider T as the dual tree of a triangulation of $\mathbb{H}^2$ by ideal triangles. Let $B = \mathbb{QP}^1$ be the set of vertices at infinity of this triangulation. The set B can also be understood as the set of connected components of $\mathbb{H}^2 \setminus T$. From this last observation we deduce that each vertex of T corresponds to three points of B (the connected components having this vertex in the closure), and an edge of T corresponds to four points in B. The corresponding sets of three and four points in B are called respectively *tribones* and *quadribones*, as seen in figure 3.

Fig. 3. tribone (a, b, c) and quadribone (a, b, c, d)

Labelled trees and pleated surfaces Let $\mathbb{C}^+$ be the the set of complex numbers with positive imaginary part. Define a *positively labelled tree* to be an element of $(\mathbb{C}^+)^A$, that is a tree such that each edge is labelled by a complex number with positive imaginary part. We claim now that each positively labelled tree represents a locally convex pleated surface, up to an isometry of $\mathbb{H}^3$. For the construction, we need to specify a vertex v_0 in the tree corresponding to a tribone (a, b, c), and three points (a_1, b_1, c_1) in $\partial_\infty \mathbb{H}^3$; different choices leading to the same surface up to an element of $PSL(2, \mathbb{C})$.

Our data leads to the construction by induction of a map from B to $\partial_\infty \mathbb{H}^3$, in the following way: assume we know the images (a_1, b_1, c_1) of the first three points of a quadribone (a, b, c, d) associated to an edge l with label λ, then we map d to the point d_1 such that the crossratio of (a_1, b_1, c_1, d_1) is λ.

Next this map defines a pleated surface: namely, we glue along edges the ideal triangles in $\mathbb{H}^3$ whose vertices at infinity are the images of tribones. The fact that the label of an edge is in $\mathbb{C}^+$ tells us that the pleated surface obtained this way is locally convex.

Construction of the measure So far, we have just coded some locally convex pleated surfaces. We have now to link these pleated surfaces to k-surfaces.

Notice first that, for pleated locally convex surfaces, there exists an analogue of the Gauss-Minkowski map, which maybe described as the boundary at infinity of the end associated to a pleated surface. One can prove now that the asymptotic Plateau problem associated to the pleated surfaces described above indeed as a solution, and that furthermore the corresponding set of k-surfaces obtained this way is dense in the phase space of k-surfaces. We have in some sense "coded"the space of k-surfaces by the combinatorial object $(\mathbb{C}^+)^A$.

Last to obtain an transverse invariant measure ergodic and of full support in $\mathcal{M}$, it suffices to build measures of full support, ergodic and invariant under the action of $PSL(2,\mathbb{Z}) = Aut(T)$ on $(\mathbb{C}^+)^A$, and this is easy to do.

The general construction in [4] generalizes this construction, which corresponds to the Lebesgue measure on the geodesic flow, to other Gibbs measures.

8 Questions

The main theorem of [4] suffers from a severe restriction: the 3-manifold M is in particular assumed to be hyperbolisable. This is of course a hypothesis we would like not to rely on. It could be interesting to understand measures which are weak limits of measures supported on closed leaves. Almost all our questions are concerned with the growth and the measure theoretical repartition of closed leaves.

For any real number A, let $\mathcal{S}(A)$ defined by

$$\mathcal{S}(A) = \{\text{closed leaves } S \ / \ Area(S) \leq A\}.$$

In this definition, the area is computed as the area of the k-surface seen as a surface in the unit tangent bundle. In other words, the area of S is the integral of the mean curvature. Using the techniques of [3] and [6], it is not difficult to show that $N(A) = \#\mathcal{S}(A)$ is finite.

One of the main question, is first to understand the growth of $N(A)$, which I presume exponential, and then to understand the repartition of closed leaves. More specifically, let δ_S be the transversal measure associated with the closed leaf S . Let also

$$\mu_A = \frac{\sum_{S\in\mathcal{S}(A)} \delta_S}{N(A)}.$$

The question is whether or not the sequence of transverse measures $\{\mu_A\}_{A\in\mathbb{R}}$ weakly converges as A goes to infinity.

We may ask these two questions also for various subsets of $\mathcal{S}(A)$.

- *Fixed genus*. We can define

  $$\mathcal{S}_g(A) = \{\text{closed leaves } S \text{ of genus } g \ / \ Area(S) \leq A\}.$$

  The case $g = 1$ is well understood. Indeed closed leaves of genus 1 are tubes, and correspond to closed geodesics. In this situation, the area of the tube is the length of the underlying closed geodesic multiplied by 2π. In this special case, we therefore recover classical problems for closed geodesics solved by Bowen and Margulis. For higher genus, I suspect strongly that all limits measures are supported by the space of tubes, hence corresponding to invariant measures for the geodesic flow, but I have no idea whether or not we recover Bowen-Margulis measure this way.

- *Fixed homotopy type.* We may of course consider fixed homology type but the question concerning fixed homotopy type seems more appealing. Let Γ be a subgroup of $\pi_1(M)$ which is either free, or a surface group. Now we may consider

$$\mathcal{S}_\Gamma(A) = \{\text{closed leaves } S \ /\pi_1(S) = \Gamma \text{ and } Area(S) \leq A\},$$

and

$$\mathcal{S}_{\Gamma,g}(A) = \{\text{closed leaves } S \ \text{ of genus } g \ /\pi_1(S) = \Gamma \text{ and } Area(S) \leq A\}.$$

In this case, I have no feelings of what may happen. A more tractable case seems to be when M has constant curvature and Γ is a Fuchsian group. In this situation all closed leaves whose π_1 is Γ are understood: they correspond to $\mathbb{CP}^1$-structures with Fuchsian holonomy and they have been described by W. Goldman [1].

These were precise questions. A more vague one is the following: can we use some version of the thermodynamical formalism in this situation and produce invariant measures by a variational approach ?

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