POSITIVITY, CROSS-RATIOS, PHOTONS, AND THE COLLAR LEMMA

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ABSTRACT. We prove that Θ-positive representations of fundamental groups of surfaces (possibly cusped or of infinite type) satisfy a collar lemma, and their associated cross-ratios are positive. As a consequence we deduce that Θ-positive representations form closed subsets of the representation variety. Along the way we systematically study a class of curves inside general flag manifolds that we call photons. Using their interplay with transversality in the flag manifold, we construct photon projections, which we use to relate the cross-ratios on the photons to the cross-ratios on the flag manifolds.

CONTENTS

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Introduction

The use of cross-ratios in hyperbolic dynamics was initiated by Otal [34] and notably used by Hamenstädt [23] and Ledrappier [29]. For the purpose of this introduction, let us recall that they consider real valued functions of generic quadruples of points in the boundary at infinity $\partial_\infty\Gamma$ of a hyperbolic group Γ satisfying certain additive (or multiplicative) cocycle identities.

These additive functions arise as logarithms of what we will call cross-ratios in this paper. Given a cross-ratio b, and a non-trivial element γ in Γ, the *period* of γ is $\mathrm{p}(\gamma) := \mathrm{b}(\gamma^+, \gamma^-, x, \gamma(x))$, where γ^- and γ^+ are respectively the repelling and attracting fixed points of γ in $\partial_\infty\Gamma$ and x is any element of $\partial_\infty\Gamma$ not fixed by γ. In the context of plane hyperbolic geometry, when Γ is the fundamental group of a closed hyperbolic surface S, $\partial_\infty\Gamma$ is identified with the projective line over $\mathbb{R}$ and the period of the projective cross-ratio of an element γ in Γ is the exponential of the length of the associated closed geodesic on S. For Anosov representations cross-ratios have been introduced by Labourie in [27] (see Section 2.4); they have since become a standard tool.

In this paper we concern ourselves with special classes of discrete subgroups of semisimple Lie groups, the images of the so-called positive representations [20]. It has been recognized that for special families of Lie groups some classes of representations of fundamental groups of surfaces have a unique behaviour: when the surface is closed, they form entire connected components of the space of homomorphisms; as a corollary one finds components consisting entirely of discrete and faithful representations. For $\mathsf{PSL}_2(\mathbb{R})$, positive representations are precisely the holonomy representations of hyperbolic structures.

More generally, for split real Lie groups, Hitchin representations give rise to connected components consisting entirely of discrete and faithful representations, see [16, 26], and the same is true for maximal representations in groups of Hermitian type [12, 10]. Even though these spaces of representations were introduced and investigated using very different techniques, Guichard and Wienhard unveiled a common structure —called Θ-*positivity*— that underlies them all [21] and generalizes the total positivity à la Lusztig [31], which played a central role in work of Fock and Goncharov [16]. Lie groups admitting a positive structure relative to Θ exist beyond the above mentioned examples, see [21], and Guichard, Labourie, and Wienhard started the study of Θ-positive representations of surface groups in [20]. Θ-Positivity is defined with respect to a subset Θ of simple roots (or equivalently with respect to the choice of a conjugacy class of parabolic groups), and it is shown in [20] that, for closed surface groups, Θ-positive representations are in particular Anosov with respect to Θ. For the purpose of this introduction, we don't recall the precise definition of Θ-positivity, but just recall that it can be described by a subset of quadruples, called *positive quadruples*, in the flag manifold $\mathcal{F}_\Theta$ defined by Θ, and positive representations preserve a cyclically ordered subset of $\mathcal{F}_\Theta$ for which ordered triples and quadruples are positive (see Section 4).

In this paper we explore cross-ratios on general flag manifolds and in particular cross-ratios associated to Θ-positive representations. We show that these cross-ratios are positive, namely the cross-ratio of a cyclically oriented 4-tuple is bigger than 1, so its logarithm is positive. We further prove a Collar Lemma in the spirit of hyperbolic geometry. We use these results in combination with a result from [20], to show that the space of Θ-positive representations of the fundamental group of a closed surface is open and closed in the space of homomorphisms, thus establishing a conjecture of Guichard, Labourie and Wienhard (cf. [20, 21, 22, 38]), extending previous results of the authors in [20] and [6]. Along the way we introduce new objects —called *photons*— that might be useful for the study of surface group representations in wider settings.

Let us describe the results in more detail. Recall that the projective cross-ratio on the projective space associates to a pair (x, y) of lines and a pair (X, Y) of hyperplanes with suitable transversality properties the real number

$$\mathrm{b}(x, y, X, Y) := \frac{\langle \bar{x} \mid \bar{X}\rangle \langle \bar{y} \mid \bar{Y}\rangle}{\langle \bar{y} \mid \bar{X}\rangle \langle \bar{x} \mid \bar{Y}\rangle},$$

where $\bar{X}$, $\bar{Y}$, $\bar{x}$ and $\bar{y}$ are (any) non-zero elements in X, Y, x and y respectively.

Let now Θ be a subset of the set Δ of simple roots of G; we assume that Θ is symmetric with respect to the opposition involution. Let $\mathcal{F}_\Theta$ be the associated generalized flag manifold. For an element θ of Θ, the fundamental weight ω_θ

defines a (projective) representation of G on a vector space V and G-equivariant maps Ξ_θ and Ξ^*_θ from $\mathcal{F}_\Theta$ to the projective space of V and its dual. The *associated cross-ratio* on $\mathcal{F}_\Theta$ is

$$\mathrm{b}^{\omega_\theta}(x, y, z, w) := \mathrm{b}(\Xi_\theta(x), \Xi_\theta(y), \Xi^*_\theta(z), \Xi^*_\theta(w)) \ .$$

Our first result is

Theorem A (Positivity of the cross-ratio)**.** *Let G be a semisimple group admitting a positive structure relative to Θ, $\mathcal{F}_\Theta$ be the generalized flag manifold associated to Θ, ω_θ the fundamental weight associated to θ in Θ. Then for every positive quadruple (x, y, X, Y) in $\mathcal{F}^4_\Theta$*

$$\mathrm{b}^{\omega_\theta}(x, y, X, Y) > 1 \ .$$

In Section 2 we extend the construction of cross-ratios to all positive linear combinations of the fundamental weights ω_Θ for θ in Θ (we call those Θ-compatible dominant weights, cf. Section 1.9) without assuming Θ being invariant under the opposition involution. Theorem A is proved in Section 5.

Special cases of this theorem were known before. This was established in [27, 30] for the cross-ratios of $\mathsf{PSL}_n(\mathbb{R})$, in [10] for maximal representations in $\mathsf{Sp}(2n, \mathbb{R})$, and in [6] for 4-tuples in the limit curve of a Θ-positive representations in $\mathsf{SO}(p, q)$.

In [8] Theorem A is used by Bridgeman and Labourie to obtain the convexity of length functions on moduli spaces of positive representations.

In [28] Labourie and Mc-Shane showed that positive cross-ratios imply the existence of generalized McShane–Mirzakhani identities. In particular, Theorem A implies that Theorem 1.0.1 of [28] holds for all positive representations.

In [32] Martone and Zhang introduced the notion of *positively ratioed* representations with respect to a parabolic subgroup P_Θ, and showed that the set of positively ratioed representations admits appropriate embeddings into the space of geodesic currents. Theorem A implies:

Corollary B. *Let ρ be a Θ-positive representation of a closed surface group, then ρ is P_Θ-positively ratioed.*

Theorem A is also a crucial ingredient for the following Collar Lemma.

For every η in G with attracting and repelling fixed points η^+ and η^- in $\mathcal{F}_\Theta$, the *period* of η with respect to the cross-ratio $\mathrm{b}^{\omega_\theta}$ is

$$\mathrm{p}^{\omega_\theta}(\eta) := \mathrm{b}^{\omega_\theta}(\eta^+, \eta^-, y, \eta(y)),$$

where one checks that the right hand term does not depend on the choice of y in $\mathcal{F}_\Theta$ transverse to η^+ and η^-.

Any linear form λ in $\mathfrak{a}^*$ gives rise to a character $\chi_\lambda \colon \mathsf{G} \to \mathbb{R}$ given by $\chi_\lambda(\eta) := \exp(\langle h \mid \lambda \rangle)$ where h is the Jordan projection of η in the Weyl chamber $\mathfrak{a}^+$ of G. When λ is a weight, periods and characters are related by the following formula:

$$\mathrm{p}^\lambda(g) = \chi_\lambda(g)\chi_\lambda(g^{-1}) \ .$$

Theorem C (Collar Lemma in the Lie group)**.** *Let G be a semisimple Lie group admitting a positive structure relative to Θ. Let A and B be Θ-loxodromic elements of G.*

Denote by (a^+, a^-) and (b^+, b^-) the pair of attracting and repelling fixed points of A and B respectively in the flag variety $\mathcal{F}_\Theta$. Assume that the sextuple

$$(a^+, b^-, a^-, b^+, B(a^+), A(b^+))$$

is positive. Then for any θ in Θ, the following holds

$$\frac{1}{\mathrm{p}^{\omega_\theta}(B)} + \frac{1}{\chi_\theta(A)} < 1 \ .$$

When S is an oriented surface of negative Euler characteristic (not necessarily of finite type) we obtain the following consequence:

Corollary D (Collar Lemma)**.** *Let G a semisimple Lie group admitting a positive structure relative to Θ. Let $\rho\colon \pi_1(S) \to \mathsf{G}$ be a Θ-positive homomorphism. Let γ_0 and γ_1 be elements of $\pi_1(S)$ whose associated free homotopy classes intersect geometrically. Let θ be in Θ, then*

$$\frac{1}{\mathrm{p}^{\omega_\theta}(\rho(\gamma_0))} + \frac{1}{\chi_\theta(\rho(\gamma_1))} < 1 \ .$$

The first Collar Lemma for representations of fundamental groups of closed surfaces in groups of higher rank is a generalization of the Hyperbolic Collar Lemma (cf. [24]) and is due to Lee and Zhang [30]. Since this seminal work, the subject of Collar Lemmas has attracted a lot of attention. We discuss in Section 9.1 the relation of our work with the works of Burger and Pozzetti [13], Beyrer and Pozzetti [4, 6], Tholozan [36] and Collier, Tholozan and Toulisse [15].

Combining Corollary D with a result of [20], we deduce the closedness of positive representations of surface groups into Lie groups admitting a positive structure relative to Θ.

Corollary E. *Let G be a semisimple Lie group admitting a positive structure relative to Θ. If $\{\rho_m\}_{m\in\mathbb{N}}$ is a sequence of Θ-positive homomorphisms from a surface group to G converging to a homomorphism ρ, then ρ is Θ-positive.*

We insist that in this corollary as well as in Corollary D, we do not restrict ourselves to closed surfaces nor to surfaces of finite type.

As a corollary, combining with [20, Corollary C], we obtain a solution to the mentioned conjecture of Guichard, Labourie, and Wienhard [21] refining results in [20] and generalizing results of [6].

Corollary F. *The set of Θ-positive representations is a union of connected components of the space of all representations of a closed surface group.*

In order to prove Theorem A, we investigate the symplectic geometry of products of flag manifolds. The proof of the Collar Lemma itself relies on the positivity of the cross-ratio as well as a new tool: the study of *θ-photons* for θ in Θ, a study that we hope will be useful in future research.

We summarize briefly the construction of θ-photons given in details in Section 3. Associated to a root θ in Θ is a conjugacy class of subgroups H_θ in G isogenic to $\mathsf{PSL}_2(\mathbb{R})$. A θ-photon is a closed orbit of a group H_θ, hence isomorphic to $\mathbf{P}^1(\mathbb{R})$ (Proposition 3.6). In the special case of the Hermitian group $\mathsf{G} = \mathsf{SO}(2, n)$, Θ is reduced to one element, the Θ-flag manifold is the *Einstein universe* equipped with a conformal structure of type $(1, n-1)$ and the θ-photons are the closed light-like geodesics as described for instance in [1] and [15]. Given a θ-photon Φ_θ, we construct a projection from an open set of the flag manifold to Φ_θ and are able to use this projection to relate the classical (projective) cross-ratio on the photon to the cross-ratio introduced in Theorem A and from this deduce bounds on the character χ_θ in terms of cross-ratios (Theorem 6.1).

In the Appendix A, we show how our results extend when $\mathbb{R}$ is replaced by any real closed field.

1. Preliminaries

In this section we recall some facts on semisimple Lie groups and Lie algebras and introduce some notation.

1.1. **Roots.**

Let G_0 be a semisimple Lie group[1] (by this we mean a connected Lie group whose Lie algebra is semisimple) with finite center. Let $\mathfrak{g}_0$ be its Lie algebra and $\mathfrak{k}$ the Lie algebra of a maximal compact subgroup K. The associated Cartan involution σ is the Lie algebra involution of $\mathfrak{g}_0$ whose fixed point set is equal to $\mathfrak{k}$. We fix a Cartan subspace $\mathfrak{a}$ inside the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form on $\mathfrak{g}_0$. Throughout this article, scalar products, and in particular the Killing form, as well as the induced forms on $\mathfrak{a}$ and on $\mathfrak{a}^*$ will be denoted by $\langle\cdot,\cdot\rangle$.

Let Σ be the subset of $\mathfrak{a}^*$ consisting of the restricted roots of G_0: Σ is the set of non-zero weights for the adjoint action of $\mathfrak{a}$ on the Lie algebra $\mathfrak{g}_0$ of G_0; explicitly, β belongs to Σ if and only if $\beta \neq 0$ and

$$\mathfrak{g}_\beta := \{v \in \mathfrak{g}_0 \mid \forall u \in \mathfrak{a},\ [u, v] = \beta(u)\, v\}$$

is not reduced to $\{0\}$. We will often denote the quantity $\beta(u)$ by $\langle u \mid \beta\rangle$ and use a similar notation for every duality.

Let Σ^+ be a fixed choice of positive roots and Δ the corresponding set of simple roots. Later on, we will need to distinguish between the "long roots" and the "short roots", the understood notion of length behind this comes from the Euclidean structure on $\mathfrak{a}^*$ induced by the Killing form.

1.2. **Weyl group.**

The (closed) *Weyl chamber* is the cone $\mathfrak{a}^+$ in $\mathfrak{a}$ defined by the equations $\alpha(a) \geqslant 0$ for all α in Σ^+ (equivalently, for all α in Δ).

The (restricted) *Weyl group* W of G_0 is the quotient of the normalizer of $\mathfrak{a}$ in K by the centralizer of $\mathfrak{a}$ in K; it identifies with the subgroup of $\mathsf{GL}(\mathfrak{a})$ of automorphisms of Σ. The Weyl group is a Coxeter group generated by hyperplane reflections $\{s_\alpha\}_{\alpha\in\Delta}$ characterized (among hyperplane reflections that induce a permutation of Σ) by $s_\alpha(\alpha) = -\alpha$. We will sometimes use representatives in K of elements s of W and we shall often denote them $\dot{s}$.

[1]In Section 1.6 we will consider a group G isomorphic to G_0 to have a treatment of flag manifolds suited to our purposes.

The longest element w in the Weyl group W (with respect to the generating family $\{s_\alpha\}_{\alpha\in\Delta}$) sends Σ^+ to $\Sigma^- := \Sigma \smallsetminus \Sigma^+ = -\Sigma^+$ and the mapping $\iota\colon \alpha \mapsto \iota(\alpha) := -w\cdot\alpha$ induces a permutation of Σ^+ and a permutation of Δ, called the *opposition involution*.

1.3. $\mathfrak{sl}_2$**-triples.** For any positive root β, choose $(x_\beta, x_{-\beta}, h_\beta)$ an associated $\mathfrak{sl}_2$-triple. This means that h_β belongs to $\mathfrak{a}$, that $x_{\pm\beta}$ belongs to $\mathfrak{g}_{\pm\beta}$, and that the relations $[h_\beta, x_{\pm\beta}] = \pm 2x_{\pm\beta}$ and $[x_\beta, x_{-\beta}] = h_\beta$ hold. These elements can be used to construct representatives in K of the reflections s_α: indeed $\dot{s}_\alpha = \exp(\frac{\pi}{2}(x_\alpha - x_{-\alpha}))$ represents the element s_α of the Weyl group.

The family $\{h_\theta\}_{\theta\in\Delta}$ is a basis of the Cartan subspace $\mathfrak{a}$, the elements of the dual basis $\{\omega_\theta\}_{\theta\in\Delta}$ are called the *fundamental weights*.

1.4. **Parabolic subgroups.** Every subset Θ of Δ defines a parabolic subgroup P_Θ in the following manner. First we consider

$$\Sigma^+_\Theta := \Sigma^+ \smallsetminus \mathrm{span}(\Delta \smallsetminus \Theta)\ .$$

This is the set of positive roots whose decomposition as a sum of simple roots contains at least one element of Θ. Equivalently, Σ^+_Θ is the smallest subset of Σ containing Θ and invariant by $\beta \mapsto \beta + \alpha$ for every α in Δ. In particular Θ itself is a subset of Σ^+_Θ. We set

$$\mathfrak{u}_\Theta := \bigoplus_{\alpha\in\Sigma^+_\Theta} \mathfrak{g}_\alpha\ , \quad \mathfrak{u}^{\mathrm{opp}}_\Theta := \bigoplus_{\alpha\in\Sigma^+_\Theta} \mathfrak{g}_{-\alpha}\ .$$

The parabolic group P_Θ is the normalizer of $\mathfrak{u}_\Theta$ in G_0. The unipotent radical of P_Θ is the group $\mathsf{U}_\Theta = \exp(\mathfrak{u}_\Theta)$. In this convention $\mathsf{P}_\emptyset = \mathsf{G}_0$, while P_Δ is the minimal parabolic. Similarly we define the opposite parabolic subgroup $\mathsf{P}^{\mathrm{opp}}_\Theta$. Let $\mathsf{L}_\Theta = \mathsf{P}_\Theta \cap \mathsf{P}^{\mathrm{opp}}_\Theta$ be the reductive part in the Levi decomposition of P_Θ and $\mathsf{S}_\Theta := [\mathsf{L}^\circ_\Theta, \mathsf{L}^\circ_\Theta]$ be the semisimple part of L°_Θ, the connected component of the identity of L_Θ. The opposite parabolic group $\mathsf{P}^{\mathrm{opp}}_\Theta$ is conjugate to $\mathsf{P}_{\iota(\Theta)}$: for every representative $\dot{w}$ of the longest element w of W, one has $\dot{w}\mathsf{P}^{\mathrm{opp}}_\Theta\dot{w}^{-1} = \mathsf{P}_\Theta$. A Cartan subspace of S_Θ is

$$\mathfrak{a}_\Theta = \bigoplus_{\beta\in\Delta\smallsetminus\Theta} \mathbb{R}\, h_\beta\ ,$$

and the Lie algebra of S_Θ is

$$\mathfrak{s}_\Theta = \mathfrak{a}_\Theta \oplus \mathfrak{m}_\Theta \oplus \bigoplus_{\beta\in\Sigma\cap\mathrm{Span}(\Delta\smallsetminus\Theta)} \mathfrak{g}_\beta\ ,$$

where $\mathfrak{m}_\Theta$ is a compact Lie algebra. Hence $\Delta \smallsetminus \Theta$ is a set of simple positive roots for S_Θ and the Dynkin diagram of S_Θ is completely determined. We have the following :

Proposition 1.1. *Let $\mathfrak{b}_\Theta$ be the orthogonal complement of $\mathfrak{a}_\Theta$ in $\mathfrak{a}$ (with respect to the Killing form). Then $\mathfrak{b}_\Theta$ is the intersection of the spaces $\ker(\beta)$ for β varying in $\Delta \smallsetminus \Theta$. Moreover the elements of $\mathfrak{b}_\Theta$ commute with all elements of $\mathfrak{l}_\Theta$, and $\mathsf{B}_\Theta := \exp(\mathfrak{b}_\Theta)$ is a central factor in L_Θ.*

1.5. **Irreducible factors for the action of the Levi.** For every θ in Σ^+_Θ, we set

$$\Sigma_\theta := \left\{\beta \in \Sigma^+ \mid \beta - \theta \in \operatorname{Span}(\Delta \smallsetminus \Theta)\right\}, \tag{1}$$

$$\mathfrak{u}_\theta := \bigoplus_{\alpha\in\Sigma_\theta} \mathfrak{g}_\alpha\ ,\ \ \mathfrak{u}_{-\theta} := \bigoplus_{\alpha\in\Sigma_\theta} \mathfrak{g}_{-\alpha}. \tag{2}$$

We will often use that, for an element β of Σ^+, β belongs to Σ_θ if and only if $\beta-\theta$ is zero in restriction to $\mathfrak{b}_\Theta$. When θ belongs to Θ, Σ_θ is the set of roots of the form $\theta+\alpha$ where α is a linear combination of simple roots in $\Delta\smallsetminus\Theta$. Obviously Σ_θ is disjoint from Σ_η when η and θ are distinct elements of Θ, and for arbitrary η and θ in Σ^+_Θ, the sets Σ_θ and Σ_η are either disjoint of equal.

The following result was established by Kostant [25].

Theorem 1.2. *The subspaces $\mathfrak{u}_\beta$ (resp. $\mathfrak{u}_{-\beta}$), for β varying in Σ^+_Θ, are the irreducible factors of the action of L_Θ on $\mathfrak{u}_\Theta$ (resp. $\mathfrak{u}^{\mathrm{opp}}_\Theta$).*

The Weyl group $\mathsf{W}_{\mathsf{S}_\Theta}$ is generated by the s_α for α in $\Delta\smallsetminus\Theta$, we then have:

Proposition 1.3. *The Weyl group $\mathsf{W}_{\mathsf{S}_\Theta}$ of S_Θ satisfies*

$$\mathsf{W}_{\mathsf{S}_\Theta}(\Sigma_\theta) \subset \Sigma_\theta$$

for every θ in Θ.

Finally, for any θ in Θ, we introduce the following subalgebra. Let $\mathfrak{g}^H_\theta := x^\perp_{-\theta} \cap \mathfrak{g}_\theta$ —we denote by $x^\perp_{-\theta}$ the orthogonal for the Killing form—, this is a hyperplane in $\mathfrak{g}_\theta$ —hence the choice of the superscript H— not containing x_θ, and let

$$\mathfrak{v}_\theta := \mathfrak{g}^H_\theta \oplus \sum_{\beta\in\Sigma^+_\Theta\smallsetminus\{\theta\}} \mathfrak{g}_\beta\ .$$

Then $\mathfrak{u}_\Theta = \mathbb{R}x_\theta \oplus \mathfrak{v}_\theta$. We set $\mathsf{V}_\theta = \exp(\mathfrak{v}_\theta)$. We denote by H_θ the connected Lie group (isogenic to $\mathsf{SL}_2(\mathbb{R})$) whose Lie algebra is generated by the $\mathfrak{sl}_2$-triple $(x_\theta, x_{-\theta}, h_\theta)$.

Proposition 1.4. *For any θ in Θ, the vector space $\mathfrak{v}_\theta$ is an ideal in the Lie-algebra $\mathfrak{u}_\Theta$. In particular, we have*

$$\mathsf{U}_\Theta = \exp(\mathfrak{g}_\theta) \ltimes \mathsf{V}_\theta\ .$$

The conjugates, by elements of H_θ, of the group V_θ are contained in P_Θ.

Proof. Since θ is a simple root, $\Sigma^+_\Theta \subset \Sigma^+$, and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, we have $[\mathfrak{v}_\theta, \mathfrak{v}_\theta] \subset \mathfrak{v}_\theta$, and $[x_\theta, \mathfrak{v}_\theta] \subset \mathfrak{v}_\theta$. Thus, $\mathfrak{v}_\theta$ is an ideal.

Let $\dot{s}_\theta$ be an element of H_θ representing the non-trivial element of the Weyl group of H_θ (one can choose for example $\dot{s}_\theta = \exp(\frac{\pi}{2}(x_\theta - x_{-\theta})))$. The group H_θ is generated by $\dot{s}_\theta$ and $\exp(\langle x_\theta\rangle) \subset \exp(\mathfrak{g}_\theta)$. Since V_θ is invariant by conjugation by $\exp(\mathfrak{g}_\theta)$, it remains to prove that the conjugate of V_θ by $\dot{s}_\theta$ is contained in P_Θ.

Let $\mathsf{V} = \exp(\sum_{\alpha\in\Sigma^+\smallsetminus\{\theta\}} \mathfrak{g}_\alpha)$. By the inclusions $\mathsf{V}_\theta \subset \exp(\mathfrak{g}^H_\theta)\mathsf{V} \subset \mathsf{U}_\Delta \subset \mathsf{P}_\Delta \subset \mathsf{P}_\Theta$, it is enough to prove that V and $\exp(\mathfrak{g}^H_\theta)$ are invariant by conjugation by $\dot{s}_\theta$. For V, this follows from the well known fact that $\dot{s}_\theta$ induces a permutation of $\Sigma^+\smallsetminus\{\theta\}$; for $\exp(\mathfrak{g}^H_\theta)$, this follows from the fact that $\mathfrak{g}^H_\theta$ is the trivial H_θ-module. The proposition is proved. ◻

1.6. **Parabolic subgroups and flag manifolds.** We consider the partial flag variety associated to P_Θ. We choose a setup that allows us to identify the tangent spaces at points z in the flag variety with the Lie algebra $\mathfrak{u}_\Theta^{\mathrm{opp}}$. For this we consider a group G isomorphic to G_0 and we let $\mathcal{I}$ be the space of isomorphisms from G_0 to G. On $\mathcal{I}$ the group G_0 acts on the right by pre-conjugation and G acts on the left by post-conjugation (in fact the groups of automorphisms of G and of G_0 act respectively on the left and on the right). We also fix once and for all $\mathcal{G}$ a connected component of $\mathcal{I}$. As G and G_0 are connected, $\mathcal{G}$ is invariant by the actions of G and G_0. For example, one could choose G to be equal to G_0 and $\mathcal{G}$ to be the connected component of the identity in the group of automorphisms of G_0. In general, the space $\mathcal{G}$ identifies (not naturally) with the adjoint form of G_0.

The left and right actions of G and G_0 on $\mathcal{G}$ being locally free, we have, for every φ in $\mathcal{G}$, natural identifications $\mathsf{T}_\varphi\mathcal{G} \simeq \mathfrak{g}$ and $\iota_\varphi^{\mathcal{G}} \colon \mathsf{T}_\varphi\mathcal{G} \simeq \mathfrak{g}_0$; the composition of these identifications gives a map $\mathfrak{g}_0 \to \mathfrak{g}$ which is precisely the differential $\varphi_* := \mathsf{T}_e\varphi$ of φ at the identity.

We consider the flag variety $\mathcal{F}_\Theta := \mathcal{G}/\mathsf{P}_\Theta$, which can be identified with a connected component in the set of subalgebras $\mathfrak{u}$ of $\mathfrak{g}$ isomorphic to $\mathfrak{u}_\Theta$. The group G acts on the left on $\mathcal{F}_\Theta$. Let $\pi^{\mathcal{F}}$ be the projection from $\mathcal{G}$ to $\mathcal{F}_\Theta$. For every φ in $\mathcal{G}$ and $x = \pi^{\mathcal{F}}(\varphi)$, the differential $\mathsf{T}_\varphi\pi^{\mathcal{F}}$ is a map from $\mathsf{T}_\varphi\mathcal{G}$ to $\mathsf{T}_x\mathcal{F}$; composing this map with the identification $\mathfrak{g}_0 \simeq \mathsf{T}_\varphi\mathcal{G}$ gives a projection

$$\pi_\varphi^{\mathcal{F}} \colon \mathfrak{g}_0 \longrightarrow \mathsf{T}_x\mathcal{F}_\Theta \ ,$$

whose kernel is $\mathfrak{p}_\Theta$. This gives us, by restriction, an identification of $\mathfrak{u}_\Theta^{\mathrm{opp}}$ with $\mathsf{T}_x\mathcal{F}_\Theta$ whose inverse will be denoted by $\iota_\varphi^{\mathcal{F}} \colon \mathsf{T}_x\mathcal{F}_\Theta \to \mathfrak{u}_\Theta^{\mathrm{opp}}$.

Similarly, we introduce the *opposite flag variety* $\mathcal{F}_\Theta^{\mathrm{opp}} := \mathcal{G}/\mathsf{P}_\Theta^{\mathrm{opp}}$. As a G-space, it is isomorphic to $\mathcal{G}/\mathsf{P}_{\iota(\Theta)}$ and the G-isomorphism is unique. The projection $\mathcal{G} \to \mathcal{F}_\Theta^{\mathrm{opp}}$ will be denoted by $\pi^{\mathcal{F}^{\mathrm{opp}}}$, and for every φ in $\mathcal{G}$, letting $y = \pi^{\mathcal{F}^{\mathrm{opp}}}(\varphi)$, we have isomorphisms $\iota_\varphi^{\mathcal{F}^{\mathrm{opp}}} \colon \mathsf{T}_y\mathcal{F}_\Theta^{\mathrm{opp}} \to \mathfrak{u}_\Theta$ and $\pi_\varphi^{\mathcal{F}^{\mathrm{opp}}} \colon \mathfrak{u}_\Theta \to \mathsf{T}_y\mathcal{F}_\Theta^{\mathrm{opp}}$.

The stabilizer in G of a point z in $\mathcal{F}_\Theta$ (or in $\mathcal{F}_\Theta^{\mathrm{opp}}$) will be denoted by P_z, the unipotent radical of P_z will be denoted by U_z. Their Lie algebras are denoted $\mathfrak{p}_z$ and $\mathfrak{u}_z$ respectively.

We say that a point z in $\mathcal{F}_\Theta$ and a point w in $\mathcal{F}_\Theta^{\mathrm{opp}}$ are *transverse*, and write $z \pitchfork w$, if

$$\mathfrak{p}_z \oplus \mathfrak{u}_w = \mathfrak{g} \ .$$

This is equivalent to the fact $\mathsf{P}_z \cap \mathsf{P}_w$ is a Levi factor of both P_z and P_w. Let finally

$$\mathcal{L}_\Theta := \{(z,w) \in \mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}} \mid z \pitchfork w\} \ ,$$

and observe that $\mathcal{L}_\Theta$ is canonically isomorphic to $\mathcal{G}/\mathsf{L}_\Theta$. The natural map $\mathcal{G} \to \mathcal{L}_\Theta$ will be denoted by $\pi^{\mathcal{L}}$, it is the corestriction of the map $(\pi^{\mathcal{F}}, \pi^{\mathcal{F}^{\mathrm{opp}}})$. The differential of $\pi^{\mathcal{L}}$ induces an onto morphism between the vector bundles $\mathsf{T}\mathcal{G}$ and $\pi^{\mathcal{L}*}\mathsf{T}\mathcal{L}_\Theta$ that induces an isomorphism $\mathsf{T}\mathcal{G}/\mathfrak{l}_\Theta \simeq \pi^{\mathcal{L}*}\mathsf{T}\mathcal{L}_\Theta$. The differential of $\pi^{\mathcal{L}}$ at a point

$(z,w) = \pi^{\mathcal{L}}(\varphi)$, composed with the identification of $\mathfrak{g}_0$ with $\mathsf{T}_\varphi\mathcal{G}$ induces a map

$$\pi^{\mathcal{L}}_\varphi\colon \mathfrak{g}_0 \longrightarrow \mathsf{T}_{(z,w)}\mathcal{L}_\Theta\ ,$$

which gives rise to an identification $\iota^{\mathcal{L}}_\varphi\colon \mathsf{T}_{(z,w)}\mathcal{L}_\Theta = \mathsf{T}_z\mathcal{F}_\Theta \oplus \mathsf{T}_w\mathcal{F}^{\mathrm{opp}}_\Theta \to \mathfrak{u}^{\mathrm{opp}}_\Theta \oplus \mathfrak{u}_\Theta$, more precisely the map $\iota^{\mathcal{L}}_\varphi$ is equal to $(\iota^{\mathcal{F}}_\varphi, \iota^{\mathcal{F}^{\mathrm{opp}}}_\varphi)$ where $\iota^{\mathcal{F}}_\varphi$ and $\iota^{\mathcal{F}^{\mathrm{opp}}}_\varphi$ are given above.

We denote by $\mathsf{L}_{z,w} = \mathsf{P}_z \cap \mathsf{P}_w$ the stabilizer of a pair (z,w) of transverse points in $\mathcal{F}_\Theta \times \mathcal{F}^{\mathrm{opp}}_\Theta$.

It will very often be the case in this paper that we are in the situation that the opposite parabolic $\mathsf{P}^{\mathrm{opp}}_\Theta$ is conjugate to P_Θ. In such case we will use the natural identification $\mathcal{F}^{\mathrm{opp}}_\Theta \simeq \mathcal{F}_\Theta$ and use the notion of transversality and the maps $\iota^{\mathcal{F}^{\mathrm{opp}}}_\varphi$ with elements of $\mathcal{F}_\Theta$ as well.

1.7. **Loxodromic elements.** An element g in G is Θ-*loxodromic* if and only if g has an attracting fixed point in $\mathcal{F}_\Theta$. In this case, g has exactly one attracting fixed point z in $\mathcal{F}_\Theta$ and one repelling fixed point w in $\mathcal{F}^{\mathrm{opp}}_\Theta$, and those fixed points z and w are transverse.

An element g_0 is *hyperbolic* if we can find an isomorphism ψ from G_0 to G such that $g_0 = \psi(\exp(a))$ where a is in the closed Weyl chamber. Recall that a is uniquely determined.

The *Kostant–Jordan decomposition* of g is the unique commuting product $g = g_h g_k g_u$ where g_u is unipotent, g_k generates a subgroup whose closure is compact, and g_h is hyperbolic. The unique element a of the closed Weyl chamber such that $\psi(\exp(a)) = g_h$ is called the *Jordan projection* of g.

We observe that we have the following:

Proposition 1.5. *An element g is Θ-loxodromic if and only if its hyperbolic part h is Θ-loxodromic. In that case both g and h have the same repelling and attracting fixed points.*

An algebraic definition is given by:

Proposition 1.6. *Let x in $\mathcal{F}_\Theta$ and y in $\mathcal{F}^{\mathrm{opp}}_\Theta$ be transverse points.*

(1) *Let h be an hyperbolic element of G, which is Θ-loxodromic with attracting fixed point x in $\mathcal{F}_\Theta$ and repelling fixed point y in $\mathcal{F}^{\mathrm{opp}}_\Theta$. Let ψ be an isomorphism from G_0 to G such that $\psi(\mathsf{P}_\Theta) = \mathsf{P}_x$, $\psi(\mathsf{P}^{\mathrm{opp}}_\Theta) = \mathsf{P}_y$, $h = \psi(\exp(a))$ with a in $\mathfrak{a}$. Then we have*

$$\langle a \mid \theta \rangle > 0,$$

for all θ in Θ,

(2) *Conversely, assume that ψ is an isomorphism from G_0 to G satisfying $\psi(\mathsf{P}_\Theta) = \mathsf{P}_x$, $\psi(\mathsf{P}^{\mathrm{opp}}_\Theta) = \mathsf{P}_y$, let a be an element of the (closed) Weyl chamber $\mathfrak{a}^+$ such that for all θ in Θ, we have $\langle a \mid \theta \rangle > 0$ then $\psi(\exp(a))$ is Θ-loxodromic with attracting fixed point x and repelling fixed point y.*

Proof. This follows from the fact that the tangent space to $\mathcal{F}_\Theta$ at x identifies with $\mathfrak{u}^{\mathrm{opp}}_\Theta$ and the tangential action of h given by $\exp(\mathrm{Ad}(a))$. We refer to [19, Proposition 3.3]. ◻

Remark 1.7. Of course the condition $\psi(\mathsf{P}_\Theta) = \mathsf{P}_x$ is equivalent to $\pi^{\mathcal{F}}(\psi) = x$, and the conditions $\psi(\mathsf{P}_\Theta) = \mathsf{P}_x$, $\psi(\mathsf{P}^{\mathrm{opp}}_\Theta) = \mathsf{P}_y$ are equivalent to $\pi^{\mathcal{L}}(\psi) = (x, y)$.

1.8. **Characters.** If a linear form η on $\mathfrak{a}$ is given, the *η-character* of g is the exponential of the evaluation of η on the Jordan projection a of g, it will be denoted by

$$\chi_\eta(g) = \exp(\langle a \mid \eta\rangle) .$$

1.9. **Dominant forms.** We introduce the notion of Θ-*compatible dominant forms and weights* in the dual of the Cartan subspace.

Definition 1.8 (Θ-COMPATIBLE DOMINANT FORM)**.** An element η of $\mathfrak{a}^*$ is called

- *dominant* if $\langle \eta, \theta\rangle \geqslant 0$ for all θ in Δ;
- Θ-*compatible* if $\langle \eta, \beta\rangle = 0$ for all β in $\Delta \smallsetminus \Theta$.

Equivalently, η is a Θ-compatible dominant form if and only if the restriction of η to $\mathfrak{a}_\Theta$ is zero and $\langle h_\theta \mid \eta\rangle \geqslant 0$ for all θ in Θ, since $\langle h_\theta \mid \eta\rangle = 2\frac{\langle \eta,\theta\rangle}{\langle \theta,\theta\rangle}$ (recall that $\mathfrak{a}_\Theta$ is the intersection $\bigcap_{\beta\in\Delta\smallsetminus\Theta} \ker \beta$). When η is a weight we will speak of a Θ-*compatible dominant weight*.

Observe that if a non-zero dominant form is Θ-compatible, there exists θ in Θ such that $\langle \eta, \theta\rangle > 0$. Among those Θ-compatible dominant forms are the fundamental weights $\{\omega_\theta\}_{\theta\in\Theta}$; more generally a linear form η is Θ-compatible and dominant if and only if it belongs to the convex cone generated by $\{\omega_\theta\}_{\theta\in\Theta}$ (since $\eta = \sum_{\theta\in\Delta} \langle h_\theta \mid \eta\rangle\, \omega_\theta$).

2. Cross-ratios

In this section we associate to every Θ-compatible dominant weight η a cross-ratio b^η defined on the flag varieties $\mathcal{F}_\Theta = \mathcal{G}/\mathsf{P}_\Theta$ and $\mathcal{F}^{\mathrm{opp}}_\Theta = \mathcal{G}/\mathsf{P}^{\mathrm{opp}}_\Theta$. This family of cross-ratios satisfies the following multiplicative properties

$$\mathrm{b}^{\eta_1}\mathrm{b}^{\eta_2} = \mathrm{b}^{\eta_1+\eta_2} , \quad \mathrm{b}^{n\eta} = (\mathrm{b}^\eta)^n . \tag{3}$$

Up to restricting the domain of definition of these cross-ratios, we give an integral formula that allows us to give a construction of b^η for every Θ-compatible dominant form η. This is possible due to a symplectic reinterpretation of such cross-ratio, which generalizes results of [27].

2.1. **Cross-ratios and periods.** Let F and F' be sets, which could be either a flag manifold and its opposite, or the boundary at infinity of a hyperbolic group (and itself). Let U be a subset of $F \times F'$. For flag manifolds, U will be the set of pairs of transverse points; for the boundary at infinity, U will be the set of pairs of distinct points. We consider the subset O of $F^2 \times (F')^2$ defined by

$$O := \left\{(x, y, X, Y) \in F^2 \times (F')^2 \mid (x, Y) \in U,\ (y, X) \in U\right\} .$$

We recall from [27] that a *cross-ratio* is a non-constant function b defined on O, taking values in a field $\mathbb{F}$, and satisfying the two cocycle identities

$$\mathrm{b}(x, w, X, Y)\, \mathrm{b}(w, y, X, Y) = \mathrm{b}(x, y, X, Y) ,$$
$$\mathrm{b}(x, y, X, W)\, \mathrm{b}(x, y, W, Y) = \mathrm{b}(x, y, X, Y) .$$

Observe that, in these equations, if the left-hand side is defined, so is the right and that the cocycle identities imply the equalities $\mathrm{b}(x,x,X,Y)=1$ and $\mathrm{b}(x,y,X,X)=1$. Our cocycle identities are multiplicative and, when $\mathbb{F}$ is $\mathbb{R}$, the cross-ratio may also be negative. This is the definition from [27] (up to the ordering of the arguments). It differs from Hamenstädt [23] since we do not impose additional restrictive symmetries; it also differs from Otal [35] where the cocycle identities are additive and symmetries are imposed as well.

Example 2.1. For a field $\mathbb{F}$, the classical projective cross-ratio on $\mathbf{P}^1(\mathbb{F})$ is a cross-ratio in this sense. In projective coordinates it is defined by

$$[x,y,X,Y]=\frac{(X-x)(Y-y)}{(X-y)(Y-x)} .$$

The convention for the order of elements in the projective cross-ratio is characterized by the fact that $[\infty,0,1,z]=z$. The projective cross-ratio is $\mathsf{PGL}_2(\mathbb{F})$-invariant.

We assume from now on that F and F' are topological spaces and U is open. We also assume that, for every x and y in F, there is X in F' such that (x,X) and (y,X) are in U.

Let also γ act on F and on F' with exactly one attracting fixed point γ^+ and one repelling fixed point γ^- on F and such that its diagonal actions preserve U and b; then the period of γ with respect to b is

$$\mathrm{p}(\gamma):=\mathrm{b}(\gamma^+,\gamma^-,y,\gamma(y)) ,$$

for any y in F' such that (γ^+,y) and (γ^-,y) belong to U (this is independent of the choice of y thanks to the cocycle identities).

Example 2.2. When $\mathbb{F}$ is a valued field and if we take γ acting on $\mathbf{P}^1(\mathbb{F})$ defined by $\gamma(z)=\lambda z$ with the absolute value of λ being >1 so that 0 is the repelling fixed point of γ and ∞ is the attracting fixed point, the period of γ is given by

$$\mathrm{p}(\gamma)=\lambda .$$

2.2. **Projective cross-ratios.** The cross-ratio on the projective line generalizes to higher dimension. Let E be a vector space over a field $\mathbb{F}$. We apply the general setting above to $F=\mathbf{P}(E)$, $F'=\mathbf{P}(E^*)$ and U the set of pairs of transverse elements, so that

$$O=\{(x,y,X,Y)\mid X\pitchfork y \text{ and } Y\pitchfork x\}.$$

The *projective cross-ratio* is the $\mathbb{F}$-valued function on the open subset O given by

$$\mathrm{b}^E(x,y,X,Y)=\frac{\langle\bar{x}\mid\bar{X}\rangle\langle\bar{y}\mid\bar{Y}\rangle}{\langle\bar{y}\mid\bar{X}\rangle\langle\bar{x}\mid\bar{Y}\rangle} , \tag{4}$$

where $\bar{u}$ denotes a non-zero vector in the line u. This does not depend on the choice of $\bar{x},\bar{y},\bar{X}$ and $\bar{Y}$.

When E is finite dimensional and when $\mathbb{F}$ is a valued field, $\mathbf{P}(E)$ and $\mathbf{P}(E^*)$ have natural topologies. In this case, the period of an element g of $\mathsf{PGL}(E)$ for b^E is the ratio between the eigenvalue of greatest modulus and the eigenvalue of lowest modulus of one (any) representative of g in $\mathsf{GL}(E)$.

The projective cross-ratio behaves well under tensor product.

Lemma 2.3. *Let E_1 and E_2 be vector spaces over a field $\mathbb{F}$, and let $E = E_1 \otimes E_2$. The natural maps $E_1 \times E_2 \to E$ and $E_1^* \times E_2^* \to E^*$ induce maps $\mathbf{P}(E_1) \times \mathbf{P}(E_2) \to \mathbf{P}(E)$ and $\mathbf{P}(E_1^*) \times \mathbf{P}(E_2^*) \to \mathbf{P}(E^*)$. All these maps will be denoted $(a, b) \mapsto a \otimes b$. One then has the identities*

$$\mathrm{b}^E(x_1 \otimes x_2, y_1 \otimes y_2, X_1 \otimes X_2, X_1 \otimes X_2) = \mathrm{b}^{E_1}(x_1, y_1, X_1, Y_1)\mathrm{b}^{E_2}(x_2, y_2, X_2, Y_2)$$

for all (x_1, y_1, X_1, Y_1) and (x_2, y_2, X_2, Y_2) such that $X_1 \pitchfork y_1$, $Y_1 \pitchfork x_1$, $X_2 \pitchfork y_2$, and $Y_2 \pitchfork x_2$.

Proof. Using that $\langle v_1 \otimes v_2 \mid \varphi_1 \otimes \varphi_2 \rangle = \langle v_1 \mid \varphi_1 \rangle \langle v_2 \mid \varphi_2 \rangle$, this comes from a direct calculation. □

2.3. **The cross-ratio associated to a dominant weight.** Let G_0 be a semisimple Lie group, $\Theta \subset \Delta$ a subset of simple roots, we do not assume here that Θ is invariant under the involution opposition. We denote, as usual, by $\mathcal{F}_\Theta = \mathcal{G}/\mathsf{P}_\Theta$ the flag variety associated to P_Θ and by $\mathcal{F}_\Theta^{\mathrm{opp}} = \mathcal{G}/\mathsf{P}_\Theta^{\mathrm{opp}}$ the flag variety associated to $\mathsf{P}_\Theta^{\mathrm{opp}}$. The space $\mathcal{L}_\Theta = \mathcal{G}/\mathsf{L}_\Theta$ is the unique open G-orbit in $\mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}}$.

For a Θ-compatible dominant weight η, let τ be the associated representation $\mathsf{G} \to \mathsf{PGL}(E)$ on a real vector space E [19, Lemma 3.2] (thus τ is the irreducible proximal representation with highest weight η, well defined up to isomorphism). This means that, for every ψ in $\mathcal{G}$, there is a (unique up to scalar) vector v in E such that, for every X in $\mathfrak{a}$, $\tau_* \circ \psi_*(X)(v) = \langle X \mid \eta \rangle v$; this vector v is mapped to 0 by $\tau_* \circ \psi_*(\mathfrak{u}_\Theta)$ and is also an eigenvector of $\tau_* \circ \psi_*(\mathfrak{l}_\Theta)$. From this we deduce that there are unique equivariant maps $\Xi_\eta \colon \mathcal{F}_\Theta \to \mathbf{P}(E)$ and $\Xi_\eta^* \colon \mathcal{F}_\Theta^{\mathrm{opp}} \to \mathbf{P}(E^*)$. If we consider an element of $\mathcal{F}_\Theta$ as a nilpotent Lie algebra in $\mathfrak{g}$, its image under Ξ_η is the unique eigenline in E for this subalgebra.

The *cross-ratio associated to* η is (with $F = \mathcal{F}_\Theta$, $F' = \mathcal{F}_\Theta^{\mathrm{opp}}$, and $U = \mathcal{L}_\Theta$)

$$\mathrm{b}^\eta(u, v, w, z) \coloneqq \mathrm{b}^E\Big(\Xi_\eta(u), \Xi_\eta(v), \Xi_\eta^*(w), \Xi_\eta^*(z)\Big)\,, \tag{5}$$

for all $(u, v, w, z) \in O \subset \mathcal{F}_\Theta \times \mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}} \times \mathcal{F}_\Theta^{\mathrm{opp}}$ (i.e. such that (v, w) and (u, z) belong to $\mathcal{L}_\Theta$), where b^E is the $\mathbb{R}$-valued projective cross-ratio defined in Equation (4). It follows directly from the definition that the cross-ratio associated to η is a semi-algebraic function.

Assume now that h in G is such that both h and h^{-1} are Θ-loxodromic elements (equivalently, h is $(\Theta \cup \iota(\Theta))$-loxodromic) and denote their attracting fixed points in $\mathcal{F}_\Theta$ respectively h^+ and h^-. Hence h^- is the repelling fixed point of h. The *η-period* of h is

$$\mathrm{p}^\eta(h) = \mathrm{b}^\eta(h^+, h^-, x, h(x))\,,$$

for (any) x in $\mathcal{F}_\Theta^{\mathrm{opp}}$ transverse to both h^- and h^+ (i.e. the η-period is the period with respect to the cross-ratio b^η).

The periods are related to the η-characters (Section 1.8).

Proposition 2.4. *For any element h such that both h and h^{-1} are Θ-loxodromic, we have*

$$\mathrm{p}^{\eta}(h) = \chi_{\eta}(h) \cdot \chi_{\eta}(h^{-1}) .$$

Note that $\chi_{\eta}(h^{-1}) = \chi_{\iota(\eta)}(h)$ where ι is the opposition involution.

Proof. Let h^+ and h^- be the attracting fixed points for the respective action of h and h^{-1} on $\mathcal{F}_{\Theta}$ and let L_{h^-,h^+} be the subgroup of G stabilizing them, this subgroup is isomorphic to L_{Θ}. The cocycle identities and the G-invariance of b^{η} imply that $\Psi\colon g \mapsto \mathrm{b}^{\eta}(h^+, h^-, x, g(x))$ is a homomorphism defined on L_{h^-,h^+}. Thus $\Psi(g) = 1$ for g in the semisimple part of L_{h^-,h^+} and for g in the compact factor of the center of L_{h^-,h^+}. The result now follows from an explicit computation for the elements g of the form $\exp(a)$ with a in the Weyl chamber, using that the highest eigenvalue of $\exp(a)$ on E is $\exp(\langle a|\eta\rangle)$ (cf. Section 2.2). ◻

By construction the cross-ratio is multiplicative in η:

Lemma 2.5. *For every Θ-dominant weights η_1, η_2 and any integer n it holds*

$$\mathrm{b}^{\eta_1}\mathrm{b}^{\eta_2} = \mathrm{b}^{\eta_1+\eta_2} , \quad \mathrm{b}^{n\eta} = (\mathrm{b}^{\eta})^{\,n} .$$

Proof. This follows readily since, if $\tau_i\colon \mathsf{G} \to \mathsf{PGL}(E_i)$ are the representations associated to the weights η_i for $i = 1, 2$, then the representation τ associated to $\eta = \eta_1 + \eta_2$ can be realized as an irreducible factor of $E_1 \otimes E_2$ and the associated equivariant maps Ξ_{η}, Ξ^*_{η} are given by $\Xi_{\eta_1} \otimes \Xi_{\eta_2}$, respectively $\Xi^*_{\eta_1} \otimes \Xi^*_{\eta_2}$. The claim then follows from Lemma 2.3. The second claim follows from the first by induction on n. ◻

2.4. **A symplectic reinterpretation.** We now give a symplectic reinterpretation of the cross-ratio b^{η} in analogy with [27, Section 4.4]. We first consider the case of the projective cross-ratio. The product $E \times E^*$ of the real vector space E and its dual carries a canonical symplectic form; this is the natural symplectic form ω^E on the cotangent bundle $T^*E = E \times E^*$, one has also $\omega^E = -d\beta^E$ where β^E is the canonical 1-form (or Liouville 1-form); explicitely, for (v, φ) in $E \times E^*$ and $(\dot{v}, \dot{\varphi})$ in $\mathsf{T}_{(v,\varphi)}(E \times E^*) \simeq E \times E^*$, one has $\beta^E_{(v,\varphi)}(\dot{v}, \dot{\varphi}) = \langle \dot{v} \mid \varphi\rangle$. The real multiplicative group $\mathbb{R}^*$ acts symplectically on $E \times E^*$ by $\lambda(x, X) = (\lambda x, \lambda^{-1}X)$, with a moment map given by $\mu(x, X) = \langle x \mid X\rangle$. The symplectic reduction at 1 —that is the space $\mu^{-1}(1)/\mathbb{R}^*$— then identifies with

$$U = \{(x, X) \in \mathbf{P}(E) \times \mathbf{P}(E^*) \mid \langle \bar{x} \mid \bar{X}\rangle \neq 0\} ,$$

which hence carries a symplectic form that we call ω. More explicitly [27, Section 4.4.3], if we identify the tangent space to $\mathbf{P}(E) \times \mathbf{P}(E^*)$ at a pair (L, P) —where L is a line transverse to the hyperplane P— with $(L^* \otimes P) \oplus (P^* \otimes L)$ we have

$$\omega\left((f, g), (h, j)\right) = \mathrm{Trace}(f \circ j) - \mathrm{Trace}(h \circ g) . \tag{6}$$

The following is proved in [27, Proposition 4.7].

Proposition 2.6. *Let f be a continuous, piecewise C^1 map from $[0,1]^2$ to U such that, for every t in $[0,1]$, $f(0,t)=(x,*)$ and $f(1,t)=(y,*)$, and for every s in $[0,1]$, $f(s,0)=(*,X)$ and $f(s,1)=(*,Y)$. Then*

$$\mathrm{b}^E(x,y,X,Y)=\exp\left(\int_{[0,1]^2} f^*\omega\right).$$

A map f satisfying the conditions in Proposition 2.6 can be constructed if and only if there are vectors $\bar{x},\bar{y},\bar{X},\bar{Y}$ representing x,y,X,Y such that the pairings $\langle \bar{x}\mid\bar{X}\rangle,\langle \bar{y}\mid\bar{X}\rangle,\langle \bar{x}\mid\bar{Y}\rangle,\langle \bar{y}\mid\bar{Y}\rangle$ are all positive.

The goal of the section is to obtain a version of Proposition 2.6 that calculates the cross-ratios b^η defined on other flag varieties. We will introduce a "curvature" form Ω on $\mathcal{L}_\Theta=\mathcal{G}/\mathsf{L}_\Theta$ with values in $\mathfrak{b}_\Theta$ and show:

Proposition 2.7. *Let η be a Θ-compatible dominant weight. Let f be a continuous, piecewise C^1 map from $[0,1]^2$ to $\mathcal{L}_\Theta$ such that, for every t in $[0,1]$, $f(0,t)=(x,*)$ and $f(1,t)=(y,*)$, and for every s in $[0,1]$, $f(s,0)=(*,X)$ and $f(s,1)=(*,Y)$. Then*

$$\mathrm{b}^\eta(x,y,X,Y)=\exp\left(\int_{[0,1]^2} f^*\left(\langle\Omega\mid\eta\rangle\right)\right).$$

Remark 2.8. If, as above, $\tau\colon \mathsf{G}\to\mathsf{PGL}(E)$ is a representation on a real vector space E with highest weight η, $\langle\Omega\mid\eta\rangle$ is the curvature form associated to the action of L_Θ on the vector space E.

Proposition 2.7 enables us to extend the definition of b^η for every Θ-compatible dominant form η. For a general η, the cross-ratio b^η is defined on the subspace $O^\square$ of O consisting of all quadruples (x,y,X,Y) bounding a continuous, piecewise C^1 square as in the assumptions of Proposition 2.7 and $\mathrm{b}^\eta(x,y,X,Y)$ is defined by the integral formula in that proposition. Note that the conditions (x,y,X,Y) in $O^\square$ and (y,z,X,Y) in $O^\square$ imply (x,z,X,Y) in $O^\square$ (respectively (x,y,X,Y) in $O^\square$ and (x,y,Y,Z) in $O^\square$ imply (x,y,X,Z) in $O^\square$) and that the cocycles identities also hold. This family of cross-ratios is multiplicative in η (i.e. $\mathrm{b}^{t_1\eta_1+t_2\eta_2}=(\mathrm{b}^{\eta_1})^{t_1}(\mathrm{b}^{\eta_2})^{t_2}$) so that it is completely determined by the cross-ratios $\{\mathrm{b}^{\omega_\theta}\}$ associated to the fundamental weights.

2.4.1. *The curvature form.* Recall that, for every φ in $\mathcal{G}$, we have isomorphisms $\iota^{\mathcal{G}}_\varphi\colon \mathsf{T}_\varphi\mathcal{G}\to\mathfrak{g}_0$, $\iota^{\mathcal{L}}_\varphi\colon \mathsf{T}_w\mathcal{L}_\Theta\to\mathfrak{u}^{\mathrm{opp}}_\Theta\oplus\mathfrak{u}_\Theta$ (where $w=\pi^{\mathcal{L}}(\varphi)$).

We introduce the following $\mathfrak{b}_\Theta$-valued forms on $\mathcal{G}$: for φ in $\mathcal{G}$ and v in $\mathsf{T}_\varphi\mathcal{G}$

$$\beta^{\mathcal{G}}_\varphi(v)=p\Big(\iota^{\mathcal{G}}_\varphi(v)\Big),\quad \Omega^{\mathcal{G}}=-d\beta^{\mathcal{G}},$$

where $p\colon\mathfrak{g}_0\to\mathfrak{b}_\Theta$ is the orthogonal projection.

One has $\Omega^{\mathcal{G}}_\varphi(v,w)=p\Big([\iota^{\mathcal{G}}_\varphi(v),\iota^{\mathcal{G}}_\varphi(w)]\Big)$ for φ in $\mathcal{G}$ and v,w in $\mathsf{T}_\varphi\mathcal{G}$.

The form $\beta^{\mathcal{G}}$ is a section of the vector bundle $(\mathsf{T}\mathcal{G})^*\otimes\mathfrak{b}_\Theta$. As $\beta^{\mathcal{G}}$ is equivariant and as the action of L_Θ on $\mathfrak{b}_\Theta$ is trivial (Proposition 1.1), the form $\beta^{\mathcal{G}}$, seen as a section of $(\mathsf{T}\mathcal{G})^*\otimes\mathfrak{b}_\Theta$, descends to a section of the vector bundle $(\mathsf{T}\mathcal{G}/\mathfrak{l}_\Theta)^*\otimes\mathfrak{b}_\Theta\simeq(\pi^{\mathcal{L}*}\mathsf{T}\mathcal{L}_\Theta)^*\otimes\mathfrak{b}_\Theta$ over $\mathcal{G}$. This section is also equivariant and, again by triviality of the action of L_Θ on

$\mathfrak{b}_\Theta$, it descends to a section of the vector bundle $(\mathsf{T}\mathcal{L}_\Theta)^* \otimes \mathfrak{b}_\Theta$, that is, to a $\mathfrak{b}_\Theta$-valued 1-form β on $\mathcal{L}_\Theta = \mathcal{G}/\mathsf{L}_\Theta$. One has $\pi^{\mathcal{L}*}\beta = \beta^{\mathcal{G}}$. We then define $\Omega = -d\beta$.

From the construction, one has:

Proposition 2.9. *The forms $\Omega^{\mathcal{G}}$ and Ω are closed, one has $\Omega^{\mathcal{G}} = \pi^{\mathcal{L}*}\Omega$.*

We call Ω the *curvature form* on $\mathcal{G}/\mathsf{L}_\Theta$. Some readers will recognize a curvature form of some bundle as in [3].

We describe now a special case for the group $\mathsf{SL}_m(\mathbb{R})$. Using the standard numbering for the simple roots, let Θ_0 consist of the first simple root so that $\mathcal{G}/\mathsf{P}_{\Theta_0} = \mathbf{P}^{m-1}(\mathbb{R})$. The quotient $U = \mathcal{G}/\mathsf{L}_{\Theta_0}$ is the space of pairs of a line and a hyperplane transverse to each other. In this case $\mathfrak{b}_{\Theta_0} \simeq \mathbb{R}$.

Proposition 2.10 ([27, Proposition 4.7])**.** *In the case that $\mathcal{G}/\mathsf{P}_\Theta = \mathbf{P}^{m-1}(\mathbb{R})$, the curvature form on U coincides with the symplectic form as in Equation (6).*

2.4.2. *Linear representations and cross-ratios.* Let η be a Θ-compatible dominant weight and let $\tau\colon \mathsf{G} \to \mathsf{PGL}(E)$ be the associated irreducible representation. The equivariant maps $\Xi_\eta\colon \mathcal{F}_\Theta \to \mathbf{P}(E)$ and $\Xi^*_\eta\colon \mathcal{F}^{\mathrm{opp}}_\Theta \to \mathbf{P}(E^*)$ combine to an equivariant map $\Psi_\eta\colon \mathcal{L}_\Theta \to U$, where U is the space of pairs of transverse points in $\mathbf{P}(E) \times \mathbf{P}(E^*)$.

Proposition 2.11. *If ω is the symplectic form on the open subset U of $\mathbf{P}(E) \times \mathbf{P}(E^*)$ (cf. Proposition 2.10), then $\Psi^*_\eta(\omega) = \langle \Omega \mid \eta \rangle$, where Ω is the curvature form on $\mathcal{G}/\mathsf{L}_\Theta$.*

Proof. Assume for simplicity that $\tau\colon \mathsf{G} \to \mathsf{PGL}(E)$ lifts to $\mathsf{GL}(E)$, the general case follows through covering theory, by observing that all our computations are local.

Let β^E be the Liouville form on $E \times E^*$ and let μ the moment map for the $\mathbb{R}$-action.

Let us fix an equivariant map $\zeta\colon \mathcal{G} \to E \times E^*$ lifting the map $\Psi_\eta \circ \pi^{\mathcal{L}}$ and with values in $\mu^{-1}(1)$, i.e. in the space of pairs (v, ℓ) such that $\langle v \mid \ell \rangle = 1$. It is then enough to check that $\zeta^*\beta^E = \langle \beta^{\mathcal{G}} \mid \eta \rangle$.

Let φ be in $\mathcal{G}$. By construction $\zeta(\varphi) = (v, \ell)$ where v is an eigenvector with weight η for the action of the Cartan subspace $\mathfrak{a}$ under $\tau_* \circ \varphi_*$, and ℓ in E^* is an eigenvector with weight $\iota(\eta)$, in other words the kernel of ℓ is the sum of the weight spaces associated to the weights different from η. This means that

$$\begin{aligned}\beta^E_{(v,\ell)}(\zeta_* \circ \varphi_*(X)) &= \langle \zeta_* \circ \varphi_*(X) \mid \ell \rangle \\ &= \langle \tau_* \circ \varphi_*(X)(v) \mid \ell \rangle \\ &= \Big\langle \langle X \mid \eta \rangle\, v \Big| \ell \Big\rangle \\ &= \langle X \mid \eta \rangle\end{aligned}$$

for every X in $\mathfrak{a}$.

One thus has $\left(\zeta^*\beta^E\right)_\varphi(\varphi_*(X)) = \langle X \mid \eta \rangle$ for every X in $\mathfrak{a}$. Since η is Θ-compatible, we also get $\left(\zeta^*\beta^E\right)_\varphi(\varphi_*(X)) = \langle p(X) \mid \eta \rangle$ for every X in $\mathfrak{a}$, indeed $X - p(X)$ belongs to $\mathfrak{a}_\Theta$ and η is zero in restriction to $\mathfrak{a}_\Theta$. This last equality also holds

- for every X in $\mathfrak{g}_\alpha$ since $\zeta_* \circ \varphi_*(X)$ sends v to the kernel of ℓ and $p(X) = 0$;

- for every X in $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ since v is also an eigenvector for this compact Lie subalgebra and is thus cancelled by X and again $p(X) = 0$;
- hence for every X in $\mathfrak{g}_0$ since we have the decomposition $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$.

Another way to formulate this equality is $\left(\zeta^*\beta^E\right)_\varphi(v) = \langle p(\iota_\varphi^{\mathcal{G}}(v)) \mid \eta\rangle$. In view of the definition of $\beta^{\mathcal{G}}$ this proves the proposition. □

Proof of Proposition 2.7. The result then follows from Proposition 2.11 and the projective case proven in [27, Proposition 4.7], recalled here in Proposition 2.6. □

3. Photons

In this section
we use the subgroup H_θ of G_0 which is locally isomorphic to $\mathsf{PSL}_2(\mathbb{R})$ to associate to each root θ in Θ a class of curves in $\mathcal{F}_\Theta$ that we call *θ-photons*, such that

(1) each θ-photon Φ is an orbit for the action of a subgroup H_Φ which is the image of H_θ by an element $\psi\colon \mathsf{G}_0 \to \mathsf{G}$ of $\mathcal{G}$,
(2) Given any θ-photon Φ, there is a *photon projection* p_Φ from an open set in $\mathcal{F}_\Theta^{\mathrm{opp}}$ to Φ, and p_Φ is equivariant with respect to the action of H_Φ.
(3) This projection has some nice properties with respect to the cross-ratio b^η associated to a Θ-compatible dominant form. In particular it satisfies that if (x, y, u) is a triple of pairwise distinct points in Φ and z and w are such that $p_\Phi(z) = p_\Phi(w) = u$, then
$$\mathrm{b}^\eta(x, y, z, w) = 1\ .$$

A key step in the proof of the Collar Lemma is then to relate the cross-ratio b^η to the projective cross-ratio on the photon, this is done in Proposition 3.27.

Remark 3.1. To motivate the terminology *photon*, recall that the Einstein universe is a flag manifold for the group $\mathsf{SO}(2, n)$. It admits a conformal structure of signature $(1, n-1)$, for which lightlike geodesics called photons play an important role. In this case a θ-photon is precisely a lightlike geodesic or a photon in the classical sense.

3.1. Photon subgroups and photons.

Let us consider, for θ in Θ, the connected subgroup H_θ in G_0 whose Lie algebra is generated by the $\mathfrak{sl}_2$-triple $(x_\theta, x_{-\theta}, h_\theta)$ (cf. Section 1.3). Observe that $\dim(\mathsf{H}_\theta \cap \mathsf{P}_\Theta) = 2$.

Given an element ψ of $\mathcal{G}$, we then consider the group $\psi(\mathsf{H}_\theta)$. Recall that $\pi^{\mathcal{F}}$ denotes the projection from $\mathcal{G}$ to $\mathcal{F}_\Theta$ (Section 1.6). We introduce the following definition.

Definition 3.2. A *θ-photon through x* is a subset $\Phi = \psi(\mathsf{H}_\theta)\cdot x$ of $\mathcal{F}_\Theta$, for some ψ such that $\pi^{\mathcal{F}}(\psi) = x$.

Note that in the situation of the definition, $\mathsf{P}_x \cap \psi(\mathsf{H}_\theta) = \psi(\mathsf{P}_\Theta \cap \mathsf{H}_\theta)$ is a parabolic subgroup of $\psi(\mathsf{H}_\theta)$.

Remark 3.3. In the case of the Grassmaniann $\mathrm{Gr}_p(\mathbb{R}^n)$, photons appeared in the work of Van Limbeek and Zimmer [37] under the name "rank one lines". Photons have also been introduced by Galiay [18] in the Shilov boundary of a tube type Hermitian Lie group.

The family of θ-photons through x will be denoted by $\mathbf{\Phi}(x)$. We have:

Proposition 3.4. *Let x be a point in $\mathcal{F}_\Theta$. Then:*

1. *Let Φ be a photon through x. The subgroup $\psi(\mathsf{H}_\theta)$ depends only on the photon Φ and neither on the choice of x in Φ, nor on the isomorphism ψ such that $\Phi = \psi(\mathsf{H}_\theta)\cdot x$.*
2. *The unipotent radical U_x of P_x acts trivially on the family $\mathbf{\Phi}(x)$.*
3. *The group $\mathsf{L}_x = \mathsf{P}_x/\mathsf{U}_x$ acts transitively on the family $\mathbf{\Phi}(x)$ (equivalently L_θ acts transitively).*
4. *The center of L_x acts trivially on the family $\mathbf{\Phi}(x)$.*

Remark 3.5. In view of point (1) we will denote $\mathsf{H}_\Phi := \psi(\mathsf{H}_\theta)$; this point will be made more precise in Proposition 3.9.

Proof. We first prove (2). Let u be in U_x and let Φ be a photon through x. Choose $\psi\colon \mathsf{G}_0 \to \mathsf{G}$ such that $\pi^{\mathcal{F}}(\psi) = x$ and $\Phi = \psi(\mathsf{H}_\theta)\cdot x$. In particular $\mathsf{U}_x = \psi(\mathsf{U}_\Theta)$. The element $\psi^{-1}(u)$ can be written as the product sv with s in $\exp(\mathbb{R}x_\theta)$ and v in V_θ (Proposition 1.4). This implies that for every $y = \psi(s')\cdot x$ in Φ (where s' in H_θ), one has

$$u\cdot y = \psi(svs')\cdot x = \psi(ss's'^{-1}vs')\cdot x = \psi(ss')\cdot x,$$

since $s'^{-1}vs'$ belongs to P_Θ (Proposition 1.4). Hence $u\cdot\Phi = \Phi$ and the second item is proved. Similar considerations show the first and the fourth items.

Let Φ_1 and Φ_2 be photons through x. Let ψ_i $(i = 1, 2)$ in $\mathcal{G}$ be such that $\pi^{\mathcal{F}}(\psi_i) = x$ and $\Phi_i = \psi_i(\mathsf{H}_\theta)\cdot x$. Since $\pi^{\mathcal{F}}(\psi_2) = \pi^{\mathcal{F}}(\psi_1) = x$, there is p in P_x such that $\psi_2 = \mathrm{int}_p \circ\psi_1$ (where $\mathrm{int}_p\colon \mathsf{G}\to\mathsf{G} \mid g \mapsto pgp^{-1}$). One then has $\Phi_2 = p\cdot\Phi_1$. This implies the transitivity in the third item. ◻

We have:

Proposition 3.6. *A θ-photon is diffeomorphic to $\mathbf{P}^1(\mathbb{R})$. More precisely the action on Φ of the (connected) group H_Φ factors through the adjoint group associated to H_Φ. This adjoint group is isomorphic to $\mathsf{PSL}_2(\mathbb{R})$, and Φ is equivariantly diffeomorphic to $\mathbf{P}^1(\mathbb{R})$.*

Proof. The flag variety $\mathcal{F}_\Theta$ can be identified with a G-orbit in the space of Lie subalgebras in $\mathfrak{g}$ isomorphic to $\mathfrak{u}_\Theta$. In view of the next lemma, which is of independent interest, applied to $\mathsf{H} = \mathsf{H}_\theta$, its action on $V = \mathfrak{g}_0$, and $W = \mathfrak{u}_\Theta$, it is thus enough to note that the stabilizer of $\mathfrak{u}_\Theta$ in $\mathfrak{h}_\theta$ is a Borel subalgebra. ◻

Lemma 3.7. *Let H be a connected Lie group such that $\mathfrak{h} \simeq \mathfrak{sl}_2(\mathbb{R})$ and B the subgroup of H which is the normalizer of the Borel subalgebra $\mathfrak{b}$ consisting of upper triangular matrices.*

Let V be a finite dimensional real vector space, and $\tau\colon \mathsf{H}\to\mathsf{GL}(V)$ be a continuous morphism with tangent Lie algebra morphism $\tau_*\colon \mathfrak{h}\to\mathrm{End}(V)$. Assume that W is a linear subspace of V whose stabilizer in $\mathfrak{h}$ —via the morphism τ_*— is equal to $\mathfrak{b}$. Then the stabilizer of W in H is equal to B. Therefore the action of H on the orbit Ψ of W in *the corresponding Grassmannian factors through the adjoint group, this adjoint group is isomorphic to* $\mathsf{PSL}_2(\mathbb{R})$*, and* Ψ *is equivariantly diffeomorphic to* $\mathbf{P}^1(\mathbb{R})$*.*

Proof. Recall that for every positive integer n there is a unique (up to isomorphism) irreducible $\mathfrak{sl}_2(\mathbb{R})$-module of dimension n, that this module integrates into a continuous homomorphism defined on $\mathsf{SL}_2(\mathbb{R})$, and that every $\mathfrak{sl}_2(\mathbb{R})$-module decomposes as a sum of irreducible modules.

This implies that the representation τ_* integrates into a continuous homomorphism $\hat{\tau}\colon \mathsf{SL}_2(\mathbb{R}) \to \mathsf{GL}(V)$. The images in $\mathsf{GL}(V)$ of the homomorphisms τ and $\hat{\tau}$ then coincide. Thus it is enough to prove the conclusion for $\mathsf{H} = \mathsf{SL}_2(\mathbb{R})$.

Let d be the dimension of W. Let

$$E = \bigwedge\nolimits^{d} V\ ,$$

ρ the associated representation of $\mathsf{GL}(V)$ on E, and q the $\mathsf{GL}(V)$-equivariant (injective) map from the Grassmannian of d-planes in V to $\mathbf{P}(E)$. Then an element g of $\mathsf{GL}(V)$ stabilizes the subspace W if and only if of $\rho(g)$ stabilizes $q(W)$.

Hence we can assume that $d = 1$. In this case, we know from the representation theory of $\mathsf{SL}_2(\mathbb{R})$, that the $\mathfrak{sl}_2(\mathbb{R})$-module generated by the $\mathfrak{b}$-invariant line W is irreducible; this means that we can assume that V is irreducible. However for irreducible modules the conclusion is well known (cf. [17, Section 11.1]). ◻

As a direct corollary of Proposition 3.6 we have:

Corollary 3.8. *Let* Φ *be a photon and let* x *be in* Φ*. For any* ψ *in* $\mathcal{G}$ *such that* $\pi^{\mathcal{F}}(\psi) = x$ *and* $\Phi = \psi(\mathsf{H}_\theta)\cdot x$*, one has*

$$\Phi = \overline{\psi(\exp(\langle x_{-\theta}\rangle))\cdot x}\ .$$

Proposition 3.9. *Assume that two photons are tangent at a point* x*. Then they coincide.*

Proof. Given any ψ such that $\pi^{\mathcal{F}}(\psi) = x$, we have a projection

$$\pi^{\mathcal{F}}_{\psi} : \mathfrak{g}_0 \to \mathsf{T}_x\mathcal{F}_\Theta\ .$$

Observe that the restriction of $\pi^{\mathcal{F}}_{\psi}$ to any vector subspace intersecting $\mathfrak{p}_\Theta$ trivially is injective.

Let ψ and φ be in $(\pi^{\mathcal{F}})^{-1}(x)$ such that the photons $\Phi_\psi = \psi(\mathsf{H}_\theta)\cdot x$ and $\Phi_\varphi = \varphi(\mathsf{H}_\theta)\cdot x$ are tangent at x.

By item (3) of Proposition 3.4, we can assume that there is g in L_Θ such that $\psi = \varphi \circ \mathrm{int}_g$. By Corollary 3.8

$$\Phi_\psi = \overline{\psi\left(\exp\langle x_{-\theta}\rangle\right)\cdot x}\ ,\ \text{and}\ \Phi_\varphi = \overline{\varphi\left(\exp\langle x_{-\theta}\rangle\right)\cdot x}\ . \tag{7}$$

Since Φ_ψ is tangent at x to Φ_φ, by construction we have

$$\pi^{\mathcal{F}}_{\varphi}(x_{-\theta}) = \pi^{\mathcal{F}}_{\psi}(x_{-\theta})\ .$$

It follows that

$$\pi^{\mathcal{F}}_{\varphi}(x_{-\theta}) = \pi^{\mathcal{F}}_{\varphi}(\mathrm{Ad}(g)x_{-\theta})\ .$$

However, both $x_{-\theta}$ and $\mathrm{Ad}(g)x_{-\theta}$ lie in the L_Θ-invariant subspace $\mathfrak{u}_\Theta^{\mathrm{opp}}$, which intersects $\mathfrak{p}_\Theta$ trivially and thus $\pi_\psi^{\mathcal{F}}$ is injective in restriction to $\mathfrak{u}_\Theta^{\mathrm{opp}}$. It follows that

$$\mathrm{Ad}(g)x_{-\theta} = x_{-\theta}\ .$$

Hence

$$\psi\left(\exp\langle x_{-\theta}\rangle\right) = \varphi \circ \mathrm{int}_g\left(\exp\langle x_{-\theta}\rangle\right) = \varphi\left(\exp\left(\langle \mathrm{Ad}(g)x_{-\theta}\rangle\right)\right) = \varphi\left(\exp\langle x_{-\theta}\rangle\right)\ .$$

Thus by Equation (7), $\Phi_\psi = \Phi_\varphi$ and the two photons coincide. ◻

Examples 3.10. Let us illustrate what the photons are for some of the groups admitting a positive structure (cf. Section 4).

(1) For **the symplectic group** $\mathrm{Sp}(2n, \mathbb{R})$, the generalized flag variety is $\mathcal{F}_\Theta = \mathrm{Lag}(\mathbb{R}^{2n})$ the space of Lagrangians. This is also the Shilov boundary of this Hermitian group. In this case Θ consists of a single element θ.

Let x be in $\mathcal{F}_\Theta$ and fix a symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ such that x is the Lagrangian $\langle f_1, \ldots, f_n\rangle$. Then $\mathsf{T}_x\mathcal{F}_\Theta$ and $\mathfrak{u}_\Theta$ both identify with the space of symmetric $n \times n$ matrices, and, under this identification, $\mathfrak{g}_\theta$ corresponds to the matrices whose only non-zero entry is in position $(1, 1)$. An example of a photon Φ through x consists of the set of all Lagrangians L that contains the subspace $V := \langle f_2, \ldots f_n\rangle$ and are contained in $W := \langle e_1, f_1, \ldots, f_n\rangle$.

The isomorphism between the photon Φ and the projective line $\mathbf{P}(W/V)$ is given by $L \mapsto L/V$.

(2) For **the orthogonal group** $\mathrm{SO}(p + 1, p + k)$, with $p \geqslant 1, k \geqslant 2$, the flag variety $\mathcal{F}_\Theta = \mathcal{F}_{1,\ldots,p}$ is the space of partial isotropic flags consisting of p nested isotropic subspaces of dimension 1 up to p. The set Θ is $\{\alpha_1, \ldots, \alpha_p\}$, with the standard numbering of the simple roots, in the Dynkin diagram α_i is connected to α_{i+1} for $i < p$. We pick a basis $\{e_1, \ldots, e_{2p+k+1}\}$, such that the orthogonal form is given by

$$\sum_{i=1}^{p+1} x_i x_{2p+k+2-i} - \sum_{i=1}^{k-1} x_{p+1+i}^2\ ,$$

and choose x to be the flag whose j-th subspace is given by $x^{(j)} = \langle e_1, \ldots, e_j\rangle$. To ease further notation we also set $x^{(p+1)} := \langle e_1, \ldots, e_{p+1}\rangle$ and $x^{(0)} := \{0\}$. Then a photon associated to the root α_i for $i \leqslant p - 1$ is the set

$$\Phi_i = \{F \in \mathcal{F}_\Theta \,|\, F^j = x^{(j)} \text{ for all } j = 1, \ldots, p,\ j \neq i\}.$$

A photon associated to the root α_p is the set

$$\Phi_p = \{F \in \mathcal{F}_\Theta \,|\, F^j = x^{(j)} \text{ for all } j \neq p \text{ and } F^{(p)} \subset x^{(p+1)}\}.$$

In all cases, the isomorphism with $\mathbf{P}^1(\mathbb{R}) = \mathbf{P}(x^{(i+1)}/x^{(i-1)})$ is now given by $F \mapsto F^{(i)}/x^{(i-1)}$.

3.2. **The set of θ-light-like vectors.** Let Z_θ be the L_Θ-orbit of $x_{-\theta}$ in $\mathfrak{u}_{-\theta}$.

Lemma 3.11. *The projectivization* $\mathbf{P}(Z_\theta)$ *is a closed subset in* $\mathbf{P}(\mathfrak{u}_{-\theta})$.

Proof. The group L_Θ is the almost product of S_Θ, $\exp(\mathfrak{b}_\Theta)$ and a compact factor M_Θ. Since the action of L_Θ on $\mathfrak{u}_{-\theta}$ is irreducible, the subgroup $\exp(\mathfrak{b}_\Theta)$ acts by homotheties on $\mathfrak{u}_{-\theta}$. This implies that $\mathbf{P}(Z_\theta)$ is the $\mathsf{M}_\Theta \times \mathsf{S}_\Theta$-orbit of the element $t_{-\theta}$ in $\mathbf{P}(\mathfrak{u}_{-\theta})$ represented by $x_{-\theta}$. Since $\mathfrak{g}_{-\theta}$ is the highest weight space, the stabilizer of $t_{-\theta}$ in S_Θ is a parabolic subgroup of S_Θ and hence the S_Θ-orbit of $t_{-\theta}$ is compact. Since M_Θ is compact, the result follows. □

We first prove:

Proposition 3.12. *If* $\pi^{\mathcal{F}}(\varphi) = \pi^{\mathcal{F}}(\psi) = x$ *then* $\pi^{\mathcal{F}}_\varphi(Z_\theta) = \pi^{\mathcal{F}}_\psi(Z_\theta)$.

Proof. Consider the isomorphisms $\bar{\pi}^{\mathcal{F}}_\varphi, \bar{\pi}^{\mathcal{F}}_\psi$ from $\mathfrak{g}_0/\mathfrak{p}_\Theta$ to $\mathsf{T}_x\mathcal{F}_\Theta$ obtained by modding out the common kernel of $\pi^{\mathcal{F}}_\varphi$, $\pi^{\mathcal{F}}_\psi$. It is enough to prove that $\bar{\pi}^{\mathcal{F}}_\varphi(Z'_\theta) = \bar{\pi}^{\mathcal{F}}_\psi(Z'_\theta)$ where Z'_θ denote the canonical image of Z_θ in $\mathfrak{g}_0/\mathfrak{p}_\Theta$.

Since $\pi^{\mathcal{F}}(\varphi) = \pi^{\mathcal{F}}(\psi)$, it follows that $\varphi = \psi \circ \mathrm{int}_g$ for some g in P_Θ. Thus $\pi^{\mathcal{F}}_\varphi = \pi^{\mathcal{F}}_\psi \circ \mathrm{Ad}(g)$. Since U_Θ acts trivially on $(\mathfrak{u}_{-\theta} \oplus \mathfrak{p}_\Theta)/\mathfrak{p}_\Theta$, it also acts trivially on Z'_θ; hence it follows that $\pi^{\mathcal{F}}_\varphi(Z_\theta) = \pi^{\mathcal{F}}_\psi \circ \mathrm{Ad}(h)(Z_\theta)$ for the element h in L_Θ equal to g modulo U_Θ. Since Z_θ is L_Θ-invariant, the result follows. □

Proposition 3.12 allows us to set:

Definition 3.13. A *θ-light-like vector* in $\mathsf{T}_x\mathcal{F}_\Theta$ is a vector in

$$Z^\theta_x := \pi^{\mathcal{F}}_\varphi(Z_\theta) ,$$

for one (equivalently every) isomorphism φ in $\mathcal{G}$ such that $\pi^{\mathcal{F}}(\varphi) = x$.

We now have:

Proposition 3.14. *There exists a unique θ-photon through any θ-light like vector.*

Proof. Proposition 3.9 proves uniqueness. If u belongs to Z^θ_x, then $v = \iota^{\mathcal{F}}_\psi(u)$ belongs to Z_θ for any ψ such that $\pi^{\mathcal{F}}(\psi) = x$. By construction $v = \mathrm{Ad}(g) \cdot x_{-\theta}$ for g in L_Θ. Hence $u = \pi^{\mathcal{F}}_\varphi(x_{-\theta})$ for $\varphi = \psi \circ \mathrm{int}_g$. Then u is tangent to the photon $\varphi(\mathsf{H}_\theta) \cdot x$. □

The set $\mathbf{\Phi}(x)$ of photons through x identifies by the above discussion with $\mathsf{L}_x/\mathrm{stab}_{\mathsf{L}_x}(\Phi) \simeq \mathbf{P}(Z_\theta)$ for some Φ in $\mathbf{\Phi}(x)$. Thus $\mathbf{\Phi}(x)$ is a compact manifold (see Lemma 3.11).

3.3. **Photon projection.** Given a θ-photon Φ, we may now define the *photon projection p_Φ*. Let

$$O_\Phi := \{y \in \mathcal{F}^{\mathrm{opp}}_\Theta \mid \text{there exists } x \text{ in } \Phi \text{ such that } x \pitchfork y\} ,$$

and observe that O_Φ is an open set. We have:

Proposition 3.15. *Let y be in O_Φ. Then:*

(1) There exists a unique z in Φ such that z is not transverse to y.
(2) The group $\mathsf{H}_\Phi \cap \mathsf{U}_y$ is a 1-parameter unipotent subgroup.
(3) $p_\Phi(y)$ is the unique fixed point of $\mathsf{H}_\Phi \cap \mathsf{U}_y$ on Φ.

Proof. Since y belongs to O_Φ, there exists w in Φ such that $w \pitchfork y$. It follows that we can find ψ in $\mathcal{G}$ (i.e. ψ is an isomorphism from G_0 to G) such that $\psi(\mathsf{P}_\Theta) = \mathsf{P}_w$ and $\psi(\mathsf{U}_\Theta^{\mathrm{opp}}) = \mathsf{U}_y$ (i.e. such that $\pi^{\mathcal{L}}(\psi) = (w, y)$). We can (and will) further assume that the photon subgroup H_Φ is $\psi(\mathsf{H}_\theta)$ (cf. Proposition 3.4.(3)). It follows that $\mathsf{H}_\Phi \cap \mathsf{U}_y$ is the unipotent group $V_y = \psi(\exp(\mathbb{R}x_{-\theta}))$ (thus item (2) holds). Since by Proposition 3.6 a photon is identified with $\mathbf{P}^1(\mathbb{R})$ as an $\mathsf{SL}_2(\mathbb{R})$-space, it follows that the orbit of w under V_y is $\Phi \smallsetminus \{z\}$ for some z in Φ. Observe now that $V_y \cdot w$ is precisely $\Phi \cap O_y$, where

$$O_y := \mathsf{U}_y \cdot w = \{x \in \mathcal{F}_\Theta \mid x \pitchfork y\}\ .$$

This implies that z is not transverse to y and concludes item (1). Since furthermore the action of V_y on $\Phi \smallsetminus \{z\}$ is simply transitive, we get item (3). □

We now define:

Definition 3.16. Given a photon Φ, the *photon projection* is the map p_Φ from O_Φ to Φ, which associates to y, the point $p_\Phi(y)$ which is the unique point in Φ not transverse to y.

From the definition, it follows that the graph of p_Φ is algebraic, hence p_Φ is continuous.

Remark 3.17. In the case of the Shilov boundary of an tube type Hermitian Lie group, the photon projection is also defined in [18, Section 6.2.2].

We can rephrase point (1) of Proposition 3.15 by saying that transversality between an element of Φ and an element of O_Φ can be asserted using the photon projection:

Corollary 3.18. *Let y be in O_Φ and let x be in Φ. Then y is transverse to x if and only if $p_\Phi(y)$ is not equal to x.*

The following result is also a direct consequence of the definition:

Proposition 3.19. *The photon projection p_Φ is equivariant under H_Φ.*

As a corollary we give another characterization of the photon projection.

Corollary 3.20. *Let x be a point in a photon Φ, y a point in O_Φ, then $p_\Phi(y) = x$ if and only if we have the following inclusion:*

$$\mathsf{H}_\Phi \cap \mathsf{U}_x \ \subset\ \mathsf{U}_y\ .$$

Proof. Suppose that $p_\Phi(y) = x$. By Proposition 3.15.(3), the intersection $\mathsf{H}_\Phi \cap \mathsf{U}_y$ is contained in P_x; this intersection is therefore the unipotent subgroup of H_Φ fixing x. As this unipotent subgroup of H_Φ is contained in U_x we also get the inclusion $\mathsf{H}_\Phi \cap \mathsf{U}_x \subset \mathsf{U}_y$.

Conversely assume that $\mathsf{H}_\Phi \cap \mathsf{U}_x \subset \mathsf{U}_y$. Then $\mathsf{H}_\Phi \cap \mathsf{U}_x$ and $\mathsf{H}_\Phi \cap \mathsf{U}_y$ are two unipotent subgroups of H_Φ contained one in the other. This forces the equality

$\mathsf{H}_\Phi \cap \mathsf{U}_x = \mathsf{H}_\Phi \cap \mathsf{U}_y$ and thus the inclusion $\mathsf{H}_\Phi \cap \mathsf{U}_y \subset \mathsf{P}_x$. By Proposition 3.15.(3) we conclude that $p_\Phi(y) = x$. □

In the case when Θ is invariant by the opposition involution, points in a given fiber of p_Φ are not transverse:

Corollary 3.21. *Suppose that Θ is invariant by the opposition involution so that we identify $\mathcal{F}_\Theta^{\mathrm{opp}}$ with $\mathcal{F}_\Theta$ and transversality makes sense between elements of $\mathcal{F}_\Theta$. Let x be a point in a photon Φ. If y and z in O_Φ, are transverse, then $p_\Phi(y) \neq p_\Phi(z)$.*

Proof. If $p_\Phi(y) = p_\Phi(z)$, then $\mathsf{U}_y \cap \mathsf{U}_z$ is not reduced to zero by Corollary 3.20, hence y and z are not transverse. □

Examples 3.22. Let us illustrate the photon projections in the Examples 3.10 we discussed before, we use the notation (and the photons) introduced there.

(1) For **the symplectic group** $\mathrm{Sp}(2n, \mathbb{R})$ with $x = \langle f_1, \ldots, f_n\rangle$ and Φ the photon given by the orbit of H_θ (here the stabilizer of x in H_θ is the standard opposite Borel subgroup). Then
$$O_\Phi = \{L \in \mathrm{Lag}(\mathbb{R}^{2n}) \mid \dim(L \cap \langle e_1, f_1, \ldots, f_n\rangle) = 1\}\ ,$$
and the projection sends L in O_Φ to the Lagrangian $(L\cap\langle e_1, f_1, \ldots, f_n\rangle)\oplus\langle f_2, \ldots, f_n\rangle$.

(2) For **the orthogonal group** $\mathrm{SO}(p+1, p+k)$, and x the flag whose j-th subspace is given by $x^{(j)} = \langle e_1, \ldots, e_j\rangle$ $(j = 1, \ldots, p)$. For $1 \leqslant i \leqslant p$, let Φ_i be the photon through x associated to α_i in Θ, then we have
$$O_{\Phi_i} = \left\{F \in \mathcal{F}_{1,\ldots,p} \,\middle|\, F^{(j)\perp} \cap x^{(j)} = \{0\} \text{ for all } j \neq i, \text{ and } \dim(F^{(i)\perp} \cap x^{(i+1)}) = 1\right\},$$
(again with the notation $x^{(p+1)} = \langle e_1, \ldots, e_{p+1}\rangle$). The projection sends F to the flag whose j-th subspace is $x^{(j)}$ for $j \neq i$ and the i-th subspace is $x^{(i-1)} \oplus (F^{(i)\perp} \cap x^{(i+1)})$.

3.4. **Photon projection and photon cross-ratio.** Let η be a Θ-compatible dominant form. Let Φ be a θ-photon. In this section we will prove the following:

Proposition 3.23. *Let z and y be in O_Φ such that $p_\Phi(y) = p_\Phi(z)$. Then for all x and w in Φ which are pairwise transverse to z and y (i.e distinct from $p_\Phi(y) = p_\Phi(z)$), we have*
$$\mathrm{b}^\eta(x, w, z, y) = 1\ .$$

Let us first show the following:

Lemma 3.24 (Infinitesimal lemma)**.** *Let u be a tangent vector to Φ at a point z. Let $c\colon [-1, 1] \to \mathcal{F}_\Theta^{\mathrm{opp}}$, be a curve differentiable at 0, with $y = c(0)$ transverse to z, such that $p_\Phi(c(t))$ is constant in t. Let $v = \dot{c}(0)$. Then*
$$\Big\langle \Omega\left((u, 0), (0, v)\right) \Big| \eta \Big\rangle = 0\ .$$

Proof. We can assume that u is non-zero.

Let us write $c(t) = c_0(t) \cdot y$ with $c_0(t)$ in U_z (this is possible in a neighborhood of 0). Let $x = p_\Phi(c(t))$, we have that $x \neq y$. Since $\mathsf{H}_\Phi \cap \mathsf{U}_x$ acts simply transitively on

$\Phi \setminus \{x\}$, there is w in the Lie algebra of $\mathsf{H}_\Phi \cap \mathsf{U}_x$ such that $u = \frac{d}{ds}|_{s=0} \exp(sw) \cdot z$. By Corollary 3.20, we have, for all real s,

$$\exp(sw) \in \mathsf{H}_\Phi \cap \mathsf{U}_x \ \subset \mathsf{U}_{c(t)} = c_0(t)\mathsf{U}_y c_0(t)^{-1} \ .$$

Thus for all real number s and all t close enough to 0

$$c_0(t)^{-1} \exp(sw) c_0(t)^{-1} \in \mathsf{U}_y \ .$$

After taking the derivatives at $s = 0$ and $t = 0$, it follows that

$$[w, \dot{c}_0(0)] \in \mathfrak{u}_y \ .$$

Let φ in $\mathcal{G}$ such that $\pi^{\mathcal{L}}(\varphi) = (z, y)$, hence $\varphi_*^{-1}(\dot{c}_0(0))$ belongs to $\mathfrak{u}_\Theta^{\mathrm{opp}}$, $\varphi_*^{-1}(w)$ belongs to $\mathfrak{u}_\Theta$ and the previous equation means that $\varphi_*^{-1}([w, \dot{c}_0(0)])$ belongs to $\mathfrak{u}_\Theta$. By the construction of Ω, one has

$$\left\langle \Omega\left((u,0),(0,v)\right) \middle| \eta \right\rangle = \left\langle p(\varphi_*^{-1}([w, \dot{c}_0(0)])) \middle| \eta \right\rangle = 0$$

since $p(\varphi_*^{-1}[(w, \dot{c}_0(0)])) = 0$. This concludes the proof. □

We need another lemma:

Lemma 3.25 (Fiber is connected). *Let x be a point in Φ, then the set $p_\Phi^{-1}(x) \subset O_\Phi$ is a connected submanifold.*

Proof. Since p_Φ is smooth (as it is algebraic) and by H_Φ-equivariance (Proposition 3.19), we see that p_Φ is a submersion so that $W := p_\Phi^{-1}(x)$ is a submanifold.

Let z be in $\Phi \setminus \{x\}$. Recall that by definition of the photon projection, W is included in the set O_z of points transverse to z. Let u be the unipotent subgroup $\mathsf{H}_\Phi \cap \mathsf{U}_z$. We then have a continuous map

$$\xi\colon O_z \to \mathsf{u} \ ,$$

characterized uniquely by $p_\Phi(w) = \xi(w) \cdot x$. Let us consider the map from $\mathsf{u} \times W$ to O_z given by

$$\psi(u, w) = u \cdot w.$$

The map ψ is a diffeomorphism: its inverse is given by

$$w \mapsto (\xi(w), \xi(w)^{-1} w) \ ,$$

(this follows from the u-equivariance of p_Φ). Thus O_z is diffeomorphic to $\mathsf{u} \times W$, hence W is connected since O_z is. □

Proof of Proposition 3.23. Let $p = p_\Phi(y) = p_\Phi(z)$.

Let $W = p_\Phi^{-1}(p)$. By Lemma 3.25, we can find a continuous, piecewise C^1 curve $c_1(t)$ joining y to z. Any point in W is transverse to any point in the interval in Φ joining x to w and not containing p; let $c_0\colon [0,1] \to \mathcal{F}_\Theta$ be a C^1 parameterization of this interval.

Then by Proposition 2.7 applied to $f\colon [0,1]^2 \to \mathcal{L}_\Theta \mid (s,t) \mapsto (c_0(s), c_1(t))$ and the Infinitesimal Lemma 3.24, the equality

$$\mathrm{b}^\eta(x, w, z, y) = 1 \ ,$$

holds. □

Using the cocycle identity we obtain as a corollary that

(8) $$\mathrm{b}^{\eta}(x,w,z_0,y_0)=\mathrm{b}^{\eta}(x,w,z_0',y_0')\ ,$$

if $p_\Phi(z_0)=p_\Phi(z_0')$ and $p_\Phi(y_0)=p_\Phi(y_0')$

Therefore Proposition 3.23 allows us to define the photon cross-ratio associated to the Θ-compatible dominant form η:

Definition 3.26. Let x,w,z,y be points in Φ satisfying the transversality condition $x\neq y$ and $w\neq z$. The *photon cross-ratio* on Φ associated to η is

$$\mathrm{b}^{\eta}_{\Phi}(x,w,z,y):=\mathrm{b}^{\eta}(x,w,z_0,y_0)\ ,$$

where z_0, y_0 are any points such that $p_\Phi(z_0)=z$, $p_\Phi(y_0)=y$.

3.5. **Photon cross-ratio and projective cross-ratio.** The photon cross-ratio is a cross-ratio on a projective line invariant under the projective group, therefore on positive quadruples it is a power of the projective cross-ratio. More generally for all quadruples, we have the following

Proposition 3.27. *Let η be a Θ-compatible dominant weight and θ in Θ, then for any θ-photon Φ*

(9) $$\mathrm{b}^{\eta}_{\Phi}(x,y,z,w)=[x,y,z,w]^{\langle h_\theta|\eta\rangle}\ ,$$

where $[a,b,c,d]$ denotes the projective cross-ratio on $\Phi\cong\mathbf{P}^1(\mathbb{R})$. In particular if ω_θ is the fundamental weight associated to θ,

(10) $$\mathrm{b}^{\omega_\theta}_{\Phi}(x,y,z,w)=[x,y,z,w]\ .$$

Proof. Let ψ be in $\mathcal{G}$ such that $\mathsf{H}_\Phi=\psi(\mathsf{H}_\theta)$. We can as well assume that $\Phi=\mathsf{H}_\Phi\cdot f_0$ where f_0 is the attracting fixed point in $\mathcal{F}_\Theta$ for the action of $h=\Psi(\exp(a))$ for some (and equivalently any) a in the open Weyl chamber. Let also f_∞ be the repelling fixed point in $\mathcal{F}_\Theta^{\mathrm{opp}}$ for h. The H_Φ-orbit $\Phi^\vee=\mathsf{H}_\Phi\cdot f_\infty$ is also equivariantly isomorphic to the projective line $\mathbf{P}^1(\mathbb{R})\simeq\mathsf{H}_\theta/\mathsf{B}_\theta$ (where B_θ is the standard Borel subgroup in H_θ). Precisely, the isomorphisms are given by

$$\begin{array}{ccc}\mathsf{H}_\theta/\mathsf{B}_\theta & \longrightarrow & \Phi\\ g\cdot\mathsf{B}_\theta & \longmapsto & \Psi(g)\cdot f_0\end{array}\qquad\qquad\begin{array}{ccc}\mathsf{H}_\theta/\mathsf{B}_\theta & \longrightarrow & \Phi^\vee\\ g\cdot\mathsf{B}_\theta & \longmapsto & \Psi(g\dot{s}_\theta)\cdot f_\infty,\end{array}$$

where $\dot{s}_\theta$ is an element of H_θ representing the non-trivial element in the Weyl group of H_θ.

These maps allow to define an H_Φ-equivariant identification $z\mapsto z^\vee$ from Φ to $\Phi^\vee$. Moreover by equivariance, this identification has the following properties:

- The point $z^\vee$ is not transverse to z. Indeed from the Schubert's cells decomposition, $\Psi(\dot{s}_\theta)\cdot f_\infty$ is not transverse to f_0.
- for all w in Φ distinct from z, the elements w and $z^\vee$ are transverse, indeed f_0 and f_∞ are transverse.

This implies that $p_\Phi(z^\vee)=z$ and thus we can use this map $z\mapsto z^\vee$ to calculate the photon cross-ratio. For x, y, z, and t in Φ,

$$\mathrm{b}^{\eta}_{\Phi}(x,y,z,t)=\mathrm{b}^{\eta}(x,y,z^\vee,t^\vee)\ .$$

Let now E be the real vector space underlying an irreducible proximal representation $\tau\colon \mathsf{G} \to \mathsf{GL}(E)$ of highest weight η. We choose a basis $(e_i)_{i=1}^d$ of weight vectors such that e_1 generates the highest weight space with respect to the Cartan subspace $\psi_*(\mathfrak{a})$ of G. The equivariant maps $\Xi\colon \mathcal{F}_\Theta \to \mathbb{P}(E)$ and $\Xi^*\colon \mathcal{F}_\Theta^{\mathrm{opp}} \to \mathbb{P}(E^*)$ are then given by $\Xi(g\cdot f_0) = \tau(g)\cdot[e_1]$ and $\Xi^*(g\cdot f_\infty) = \tau^*(g)\cdot[e_1^*]$ where $\tau^*\colon \mathsf{G} \to \mathsf{GL}(E^*)$ is the contragredient representation $g \mapsto {}^T\tau(g)^{-1}$. Then

$$\mathrm{b}^\eta(a,b,c,d) = \mathrm{b}^E(\Xi(a),\Xi(b),\Xi^*(c),\Xi^*(d))\,,$$

where $\mathrm{b}^E([v],[w],[\varphi],[\psi]) = \frac{\langle v|\varphi\rangle\langle w|\psi\rangle}{\langle v|\psi\rangle\langle w|\varphi\rangle}$. We now prove Equality (9). By continuity it is sufficient to treat the case when $x \neq y$, and by H_θ-equivariance we may assume that $x = \psi(\dot{s}_\theta)f_0$; in other words, x is the repelling fixed point in Φ for the Weyl chamber of H_θ, and

$$y = f_0\,,\ z = \psi(\exp(\lambda x_\theta)\dot{s}_\theta)\cdot f_0\,, \text{ and }\ t = \psi(\exp(\mu x_\theta)\dot{s}_\theta)\cdot f_0\,.$$

Thus the projective cross-ratio $[x,y,z,t]$ is equal to $[0,\infty,\lambda,\mu] = \lambda/\mu$. Furthermore we have

$$\begin{aligned}
\mathrm{b}^\eta_\Phi(x,y,z,t) &= \mathrm{b}^\eta(x,y,z^\vee,t^\vee)\\
&= \mathrm{b}^\eta(\psi(\dot{s}_\theta)f_0, f_0, \psi(\exp(\lambda x_\theta))f_\infty, \psi(\exp(\mu x_\theta))f_\infty)\\
&= \mathrm{b}^E(\tau\circ\psi(\dot{s}_\theta)[e_1],[e_1],\tau^*\circ\psi(\exp(\lambda x_\theta))[e_1^*],\tau^*\circ\psi(\exp(\mu x_\theta))[e_1^*])\\
&= \frac{\langle \tau\circ\psi(\dot{s}_\theta)e_1 \mid \tau^*\circ\psi(\exp(\lambda x_\theta))e_1^*\rangle\,\langle e_1 \mid \tau^*\circ\psi(\exp(\mu x_\theta))e_1^*\rangle}{\langle \tau\circ\psi(\dot{s}_\theta)e_1 \mid \tau^*\circ\psi(\exp(\mu x_\theta))e_1^*\rangle\,\langle e_1 \mid \tau^*\circ\psi(\exp(\lambda x_\theta))e_1^*\rangle}\\
&= \frac{\langle \tau\circ\psi(\dot{s}_\theta)e_1 \mid \tau^*\circ\psi(\exp(\lambda x_\theta))e_1^*\rangle}{\langle \tau\circ\psi(\dot{s}_\theta)e_1 \mid \tau^*\circ\psi(\exp(\mu x_\theta))e_1^*\rangle}
\end{aligned}$$

since $\langle e_1 \mid \tau^*\circ\psi(\exp(\mu x_\theta))e_1^*\rangle = \langle \tau\circ\psi(\exp(\mu x_\theta))e_1 \mid e_1^*\rangle = \langle e_1 \mid e_1^*\rangle = 1$ and similarly $\langle e_1 \mid \tau^*\circ\psi(\exp(\lambda x_\theta))e_1^*\rangle = 1$. The proposition will be proven if we can show that there is a non-zero number c such that, for all λ in $\mathbb{R}$,

$$\langle \tau\circ\psi(\dot{s}_\theta)e_1 \mid \tau^*\circ\psi(\exp(\lambda x_\theta))e_1^*\rangle = c\lambda^{\langle\eta|h_\theta\rangle}\,. \tag{11}$$

For this, note first that

$$\langle \tau\circ\psi(\dot{s}_\theta)e_1 \mid \tau^*\circ\psi(\exp(\lambda x_\theta))e_1^*\rangle = \langle \tau\circ\psi(\exp(\lambda x_\theta))\tau\circ\psi(\dot{s}_\theta)e_1 \mid e_1^*\rangle\,.$$

Furthermore, denoting $\tau_*\colon \mathfrak{g} \to \mathrm{End}(E)$ the Lie algebra homomorphism associated to τ and $\psi_*\colon \mathfrak{g}_0 \to \mathfrak{g}$ the isomorphism associated to ψ, classical calculations in $\mathfrak{sl}_2$-modules give that $\tau\circ\psi(\dot{s}_\theta)e_1$ is a non-zero multiple of $(\tau_*\circ\psi_*(x_{-\theta}))^{\langle h_\theta|\eta\rangle}e_1$ and that

$$\begin{aligned}
\langle(\tau_*\circ\psi_*(x_\theta))^k(\tau_*\circ\psi_*(x_{-\theta}))^{\langle h_\theta|\eta\rangle}e_1 \mid e_1^*\rangle &= 0 \text{ if } k \neq \langle h_\theta \mid \eta\rangle \text{ and}\\
\langle(\tau_*\circ\psi_*(x_\theta))^{\langle h_\theta|\eta\rangle}(\tau_*\circ\psi_*(x_{-\theta}))^{\langle h_\theta|\eta\rangle}e_1 \mid e_1^*\rangle &\neq 0.
\end{aligned}$$

Using the equality $\tau\circ\psi(\exp(\lambda x_\theta)) = \sum_k \frac{1}{k!}\lambda^k(\tau_*\circ\psi_*(x_\theta))^k$ gives the existence of c such that Equation (11) holds, hence the wanted conclusion. □

Remark 3.28. Galiay obtained Proposition 3.27 for the photons in the Shilov boundary of tube type Hermitian Lie groups, see [18, Lemma 6.11].

4. Positivity

Now we will restrict to semisimple Lie groups G_0 that admit a *positive structure relative to a subset* Θ of Δ as defined in [21, Definition 3.1]. By definition, this means that for every θ in Θ there exists a convex acute open cone c_θ inside $\mathfrak{u}_\theta$, which is invariant by L_Θ°. (Note that in [21], the symbol c_θ stands for the *closed* invariant cone, but, since the closed cone does not play a big role in the present work, we simplify notation and a denote here by c_θ the open cone.)

As the action of L_Θ° on $\mathfrak{u}_\theta$ is irreducible, there exist exactly two such invariant cones (namely c_θ and $-c_\theta$). We distinguish between the two invariant cones by requesting that the element x_θ (of the $\mathfrak{sl}_2$-triple associated to θ, cf. Section 1.3) belongs to the closure of c_θ (cf. [21, Theorem 3.13]). Similarly the cone $c_{-\theta}$ in $\mathfrak{u}_{-\theta}$ is the invariant cone whose closure contains $x_{-\theta}$. Equivalently one can set $c_{-\theta} = -\sigma(c_\theta)$ where σ is the Cartan involution.

Remark 4.1. There are exactly four families of simple Lie groups admitting a positive structure with respect to some subset Θ of Δ (see [21, Theorem 1.1]). Up to isogeny, these correspond to the following cases:

(1) G_0 is a split real form, and $\Theta = \Delta$;
(2) G_0 is Hermitian of tube type and of real rank r and $\Theta = \{\alpha_r\}$, where α_r is the long simple restricted root;
(3) G_0 is $\mathrm{SO}(p+1, p+k)$, $p > 1$, $k > 1$ and $\Theta = \{\alpha_1, \dots, \alpha_p\}$, where $\alpha_1, \dots, \alpha_p$ are the long simple restricted roots;
(4) G_0 is the real form of F_4, E_6, E_7, or of E_8 whose system of restricted roots is of type F_4, and $\Theta = \{\alpha_1, \alpha_2\}$, where α_1, α_2 are the long simple restricted roots.

In general a semisimple Lie group admits a positive structure relative to Θ if it is the almost product of simple Lie groups G_i, $i = 1, \dots, n$, where each G_i admits a positive structure relative to Θ_i and $\Theta = \Theta_1 \cup \dots \cup \Theta_n$. Here the parabolic subgroup P_Θ is the almost direct product of the parabolic subgroups in the factors G_i and the flag manifold $\mathcal{F}_\Theta$ is the product of the flag manifolds corresponding to the different factors.

The fact that G_0 admits a positive structure relative to Θ implies in particular that

(1) the parabolic subgroup P_Θ is conjugate to its opposite,
(2) we have $\dim \mathfrak{g}_\alpha = 1$ for all α in Θ.

Furthermore the positive structure gives rise to an open and sharp semigroup N_Θ in U_Θ invariant under L_Θ°. The properties of N_Θ will be reflected in the properties of the diamonds that we introduce next. We refer to [21] for a precise description of N_Θ and its algebraic properties.

4.1. **Diamonds.** Let x and y be transverse points in $\mathcal{F}_\Theta$ (recall that $\mathcal{F}_\Theta^{\mathrm{opp}} \simeq \mathcal{F}_\Theta$), and consider an element ψ in $\mathcal{G}$ such that $\pi^{\mathcal{L}}(\psi) = (x, y)$. Note that ψ depends only on (x, y) up to precomposition by the conjugation by an element in L_Θ. Then by [20, Proposition 2.6] and [21, Theorem 8.1] we have:

Proposition 4.2. *Given (x, y) and ψ as above, the set $\psi(\mathsf{N}_\Theta) \cdot x$ is a connected component of the set*

$$\{z \in \mathcal{F}_\Theta \mid z \pitchfork x \text{ and } z \pitchfork y\} .$$

Such a connected component D is called a *diamond with extremities x and y*. A diamond with extremities x and y will be denoted $D(x, y)$; observe the slight abuse of notation since $D(x, y)$ does not only depend on x and y: there are $2^{\sharp\Theta}$ diamonds with the same extremities.

The map $\pi_\psi^{\mathcal{F}}$ (see Section 1.6) gives an identification of $\mathfrak{u}_\Theta^{\mathrm{opp}}$ with $\mathsf{T}_x\mathcal{F}$ and the map $\pi_\psi^{\mathcal{F}^{\mathrm{opp}}}$ gives an identification of $\mathfrak{u}_\Theta$ with $\mathsf{T}_y\mathcal{F}$. The tangent cone at x of the closure of a diamond $D = \psi(\mathsf{N}_\Theta) \cdot x$ is exactly the image by $\pi_\psi^{\mathcal{F}}$ of the closed cone $\sum_{\theta\in\Theta} \bar{c}_{-\theta}$ inside $\mathfrak{u}_\Theta^{\mathrm{opp}}$ (cf. [21, Section 8.5]):

- The tangent vectors belonging to this tangent cone will be called *non-negative* (with respect to D);
- The tangent vectors in the (relative) interior of the tangent cone, i.e. those belonging to $\pi_\psi^{\mathcal{F}}\left(\sum_{\theta\in\Theta} c_{-\theta}\right)$, are called *positive*.

Equivalently, a vector v in $\mathsf{T}_x\mathcal{F}_\Theta$ is positive (respectively non-negative) with respect to D if $\iota_\psi^{\mathcal{F}}(v)$ belongs to $\sum_{\theta\in\Theta} c_{-\theta}$ (respectively to $\sum_{\theta\in\Theta} \bar{c}_{-\theta}$).

Note that the shape of a diamond near its extremities should really be thought of as a "cusp"; indeed the diamond is open in $\mathcal{F}_\Theta$ whereas the dimension of its tangent cone at x is of positive codimension in $\mathsf{T}_x\mathcal{F}_\Theta$ as soon as the set Θ has at least 2 elements.

The tangent cone at y of D is the image by $\pi_\psi^{\mathcal{F}^{\mathrm{opp}}}$ of $\sum_{\theta\in\Theta} \bar{c}_\theta$.

The subset $\psi(\mathsf{N}_\Theta^{-1}) \cdot x$ is also a diamond with extremities x and y and is called the *diamond opposite to D* and will be denoted by $D^\vee$. Its tangent cone at x is the image by $\pi_\psi^{\mathcal{F}}$ of $\sum_{\theta\in\Theta} -\bar{c}_{-\theta}$ whereas its tangent cone at y is the image by $\pi_\psi^{\mathcal{F}^{\mathrm{opp}}}$ of $\sum_{\theta\in\Theta} -\bar{c}_\theta$. This opposite diamond $D^\vee$ depends only on D and not on the isomorphism ψ.

4.2. **Positive tuples.** When z belongs to a diamond D with extremities x and y, the triple (x, z, y) of $\mathcal{F}_\Theta^3$ will be called *positive*. Positive triples form a G-invariant and S_3-invariant subset of $\mathcal{F}_\Theta^3$. When z belongs to D and w to $D^\vee$, the quadruple (x, z, y, w) is called *positive*. Positive quadruples form a G-invariant subset of $\mathcal{F}_\Theta^4$ that is invariant by the cyclic permutation $(x, z, y, w) \mapsto (z, y, w, x)$ as well as the double transposition $(x, z, y, w) \mapsto (z, x, w, y)$.

Finally, for any k greater than 4, positive k-tuples are characterized in [20, Section 2.4] as those $(x_1, x_2, \ldots, x_k)$ in $\mathcal{F}_\Theta^k$ such that (x_i, x_j, x_ℓ, x_m) is a positive quadruple for all $1 \leqslant i < j < \ell < m \leqslant k$.

Let E be a set with a cyclic ordering. A map $f \colon E \to \mathcal{F}_\Theta$ will be called *positive* if, for every $k \geqslant 3$ and for every cyclically ordered k-tuple $(t_1, \ldots, t_k)$ of E^k, the k-tuple

$(f(t_1),\dots,f(t_k))$ of $\mathcal{F}_\Theta^k$ is positive. In view of the definition of positive k-tuples, it is enough to check this property with $k=3$ or 4 and, in the case when $\sharp E>3$, only with $k=4$.

Examples of positive maps are *positive circles* (as well as their restrictions to intervals). These arise as orbits in $\mathcal{F}_\Theta$ for certain 3-dimensional subgroups. We refer to [20, Section 2.5] or to [21, Section 7] for more details. We will need the following statement that can be easily obtained using positive circles.

Lemma 4.3. *Let (x,y) be in $\mathcal{L}$ and let D be a diamond with extremities x and y. There exists then a smooth positive arc $c\colon[0,1]\to\mathcal{F}_\Theta$ such that $c(0)=x$, $c(1)=y$, and $c(t)$ is in D for all t in $(0,1)$.*

Finally:

Definition 4.4. We say that a quadruple of points (X,Y,x,y) is *semi-positive* if X and Y are both transverse to x and y, and moreover (X,Y,x,y) is the limit of a sequence of positive quadruples.

Observe that if (X,Y,x,y) is semi-positive, then (Y,X,y,x) is semi-positive as well.

We prove now that the photon projection of a positive quadruple is a cyclically ordered quadruple (on the projective line) and that photon projections give rise to semi-positive quadruples. Note here that we can (and will) identify $\mathcal{F}_\Theta^{\mathrm{opp}}$ with $\mathcal{F}_\Theta$ so that the photon projection p_Φ is indeed defined on the open subset of $\mathcal{F}_\Theta$ of elements that are transverse to some point in Φ. With this in mind:

Proposition 4.5. *Assume that (X,Y,x,y) is a positive four-tuple in $\mathcal{F}_\Theta$. Let Φ be a photon through X, then*

1. *the configuration $(X,p_\Phi(Y),p_\Phi(x),p_\Phi(y))$ in the projective line Φ is positive,*
2. *the configurations $(p_\Phi(Y),Y,x,y)$ and $(Y,p_\Phi(Y),y,x)$ in $\mathcal{F}_\Theta$ are semi-positive.*

Proof. Since Y, x, and y are transverse to X in Φ, we indeed have that Y, x, and y belong to O_Φ so that we can consider the photon projections $p_\Phi(Y)$, $p_\Phi(x)$, and $p_\Phi(y)$.

Corollary 3.21 gives that $X,p_\Phi(Y),p_\Phi(x),p_\Phi(y)$ are pairwise distinct. Equally if C is a positive circle though X and Y, the restriction of p_Φ to C is an injective continuous map to the photon Φ and thus sends positive configurations in $\mathcal{F}_\Theta$ to positive configurations in Φ (with respect to the positive structure on the projective line Φ). Given any positive configuration (X,Y,x_0,y_0), we can find a deformation (X,Y,x_t,y_t) through positive configurations so that (X,Y,x_1,y_1) is on a positive circle [20, Lemma 3.7]. Hence for any positive (X,Y,x,y), the configuration $(X,p_\Phi(Y),p_\Phi(x),p_\Phi(y))$ is positive with respect to Φ. This proves the first item.

In particular, $p_\Phi(x)$ and $p_\Phi(y)$ both lie in the same connected component of $\Phi\smallsetminus\{X,p_\Phi(Y)\}$. Let I be the other component of $\Phi\smallsetminus\{X,p_\Phi(Y)\}$, we now observe that for all Z in I, Z is distinct from $p_\Phi(x)$, from $p_\Phi(y)$, and from $p_\Phi(Y)$, hence transverse (by the definition of p_Φ) to x, to y, and to Y. It follows by continuity and transversality that (Z,Y,x,y) is positive. Letting Z tend to $p_\Phi(Y)$, we get that $(p_\Phi(Y),Y,x,y)$ is semi-positive. Since double transpositions preserve positivity $(Y,p_\Phi(Y),y,x)$ is also semi-positive. ◻

As an important consequence of the previous proposition and of Proposition 3.27, we have that the photon cross-ratio of a positive quadruple is positive.

Proposition 4.6. *Let η be a Θ-compatible dominant non-zero weight. Given a photon Φ through a point x, as well as y, z and w in $\mathcal{F}_\Theta$ such that (x,y,z,w) is positive, then* $\mathrm{b}^\eta(p_\Phi(w),x,y,z) > 1$.

Proof. Indeed

$$\mathrm{b}^\eta(p_\Phi(w),x,y,z) = \mathrm{b}^\eta_\Phi(p_\Phi(w),x,p_\Phi(y),p_\Theta(z)) = [p_\Phi(w),x,p_\Phi(y),p_\Theta(z)]^{\langle h_\theta|\eta\rangle} > 1$$

since the quadruple $(p_\Phi(w),x,p_\Phi(y),p_\Theta(z))$ in $\Phi \simeq \mathbf{P}^1(\mathbb{R})$ is a positive configuration on the projective line so that its cross-ratio $[p_\Phi(w),x,p_\Phi(y),p_\Theta(z)]$ is > 1. ◻

4.3. **Positivity of bracket.** We prove here Theorem 4.7 which is an important step towards positivity of the cross-ratio.

Recall that we denote by $p\colon \mathfrak{g}_0 \to \mathfrak{b}_\Theta$ the orthogonal projection onto $\mathfrak{b}_\Theta$ (its kernel is $\mathfrak{a}_\Theta \oplus \mathfrak{z}_\mathfrak{k}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$). Our goal is to prove the following result:

Theorem 4.7 (Positivity of bracket). *Let η be a Θ-compatible dominant form. Let θ be an element of Θ. Let u and v be respectively elements of the open cones c_θ and $c_{-\theta}$, then*

$$\langle p([u,v]) \mid \eta\rangle \geqslant 0\ , \tag{12}$$

If furthermore, $\langle \eta, \theta\rangle > 0$, then

$$\langle p([u,v]) \mid \eta\rangle > 0\ . \tag{13}$$

We first begin by introducing and discussing boundary roots.

4.3.1. *Boundary roots.* In Section 1.5 we introduced the subsets Σ_θ, for any θ in Σ^+_Θ. Boundary roots are extremal elements of Σ_θ:

Definition 4.8. Let θ be in Θ. A *boundary root with respect to* θ is a root β in Σ_θ such that there exists u in $\mathfrak{a}$ for which

$$\beta(u) > \alpha(u)\ ,$$

for every α in $\Sigma_\theta \smallsetminus \{\beta\}$.

We denote by B_θ the set of boundary roots with respect to θ.

As, for all α,β in Σ_θ, the difference $\beta-\alpha$ is zero in restriction to $\mathfrak{b}_\Theta$ and as $\mathfrak{a} = \mathfrak{b}_\Theta \oplus \mathfrak{a}_\Theta$ (cf. Proposition 1.1), we can always assume that the element u in the definition belongs to $\mathfrak{a}_\Theta$, the Cartan subspace of S_Θ.

We first have:

Proposition 4.9. *Every root θ in Θ is a boundary root with respect to θ.*

Proof. Let v be in the opposite of the standard Weyl chamber. One has $\alpha(v) < 0$ for every positive root α. For all α in $\Sigma_\theta \smallsetminus \{\theta\}$, $\alpha - \theta$ is a sum of simple roots; hence $\alpha(v) - \theta(v) < 0$. Thus $\theta(v) > \alpha(v)$, which is what we wanted to prove. ◻

As Σ_θ is invariant by the Weyl group W_{S_Θ}, the set of boundary roots is invariant by the Weyl group W_{S_Θ}.

When $\Sigma_\theta = \{\theta\}$, the only boundary root is θ. Thus, the definition of boundary roots is meaningful when $\Sigma_\theta \neq \{\theta\}$, namely when $\Theta \neq \Delta$ and θ is a "special root" (i.e. connected to $\Delta \smallsetminus \Theta$ in the Dynkin diagram, cf. [21, Section 3.4]). In this case the factor S of S_Θ that acts non-trivially on $\mathfrak{u}_\theta$ is of type A_d for some d and, if $\varepsilon_0-\varepsilon_1, \dots, \varepsilon_{d-1}-\varepsilon_d$ are the simple roots in type A_d where the ε_i are weights summing to zero (i.e. the ε_i are the weights of the standard representation V of S), the weights of S in $\mathfrak{u}_\theta$ are $2\varepsilon_i$ ($i = 0, \dots, d$, and $\theta = 2\varepsilon_0$) and $\varepsilon_i + \varepsilon_j$ ($0 \leqslant i < j \leqslant d$) (those are the weights of S acting on $\mathrm{Sym}^2\mathsf{V}$; cf. [21, Section 3.5]). With this, we can now prove the following:

Proposition 4.10. *Let θ be in Θ. The set B_θ of boundary roots with respect to θ is the W_{S_Θ}-orbit of θ. In particular, we have* $\dim \mathfrak{g}_\beta = 1$ *for all β in B_θ.*

Proof. For the proof we can restrict without loss of generality to the case when $\mathfrak{g}_0$ is simple. The case when $\Theta = \Delta$ corresponds to the case when $\mathfrak{g}_0$ is split over $\mathbb{R}$ and one has $\Sigma_\theta = \{\theta\}$ and $\dim \mathfrak{g}_\theta = 1$ so that the results follow immediately.

Otherwise the subsets Θ and $\Delta \smallsetminus \Theta$ of the set of simple roots are both non-empty and connected and there is a unique root α_Θ in Θ that is connected to $\Delta \smallsetminus \Theta$. When $\theta \in \Theta \smallsetminus \{\alpha_\Theta\}$, we have, similarly to the split case, $\Sigma_\theta = \{\theta\}$ and the result is immediate.

When $\theta = \alpha_\Theta$, we will use the notation introduced before the proposition: the weights in $\mathfrak{u}_\theta$ are $2\varepsilon_i$ ($i = 0, \dots, d$) and $\varepsilon_i + \varepsilon_j$ ($0 \leqslant i < j \leqslant d$). The Weyl group acts here as the permutation group S_{d+1} and has therefore two orbits on the weights: the orbit of $2\varepsilon_0$ and the orbit of $\varepsilon_0 + \varepsilon_1$. To conclude we examine which of these orbits are contained in B_θ.

Choosing a vector u in the open Weyl chamber of S_Θ (so that $(\varepsilon_i - \varepsilon_{i+1})(u) > 0$ for all $i = 0, \dots, d-1$) shows that $2\varepsilon_0$ is a boundary root (cf. also Proposition 4.9).

The weight $\varepsilon_0 + \varepsilon_1$ does not correspond to a boundary root since, for an element u in $\mathfrak{a}$, the inequalities $(\varepsilon_0 + \varepsilon_1)(u) > 2\varepsilon_0(u)$ and $(\varepsilon_0 + \varepsilon_1)(u) > 2\varepsilon_1(u)$ cannot be simultaneously satisfied. □

Recall that, for every root β, we fixed an $\mathfrak{sl}_2$-triple $(x_\beta, x_{-\beta}, h_\beta)$ with $x_{\pm\beta}$ in $\mathfrak{g}_{\pm\beta}$, in view of Point (2) of the previous proposition, we can and will assume that the element x_β belongs to the closure of c_θ. With these choices, the following proposition holds:

Proposition 4.11. *Let θ be in Θ. Let β be a boundary root with respect to θ and let t_β be in* $\mathbf{P}(\mathfrak{u}_\theta)$ *the element represented by x_β.*

(1) *The group L_Θ° acts transitively on c_θ.*
(2) *The sum $\sum_{\beta \in B_\theta} x_\beta$ belongs to c_θ.*
(3) *The convex set $\mathbf{P}(c_\theta)$ is contained in*

$$O_\theta = \mathbf{P}\left\{ \sum_{\alpha \in \Sigma_\theta} u_\alpha \mid \forall \alpha \in \Sigma_\theta,\ u_\alpha \in \mathfrak{g}_\alpha \text{ and } \forall \beta \in B_\theta,\ u_\beta \in \mathbb{R}_{>0} x_\beta \right\}.$$

Proof. Point 1 is [21, Proposition 5.1]. Point 2 is [21, Theorem 5.12].

Using that the action of L_Θ° is transitive on c_θ, that the stabilizers in L_Θ° of points in c_θ contain a maximal compact subgroup (cf. [21, Proposition 5.1]) and using the Iwasawa decomposition in L_Θ°, the proof of the last item follows from the statement and the proof of [2, Proposition 4.7] for $\beta = \theta$; by equivariance under W_{S_Θ} the property holds for every boundary root β. ◻

The following proposition (notably item (2)) explains the terminology boundary root.

Proposition 4.12. *Let θ be in Θ. Let β be a boundary root with respect to θ and let t_β be in $\mathbf{P}(\mathfrak{u}_\theta)$ the element represented by the line $\mathfrak{g}_\beta$.*

1. *The point t_β is an attracting point for the action of an hyperbolic element in the Cartan subgroup A of S_Θ on $\mathbf{P}(\mathfrak{u}_\theta)$.*
2. *The point t_β belongs to the boundary of the set $\mathbf{P}(c_\theta)$.*

Proof. By equivariance under the Weyl group W_{S_Θ}, it is enough to prove the statements for $\beta = \theta$.

The tangent space at t_θ to $\mathbf{P}(\mathfrak{u}_\theta)$ identifies A-equivariantly with

$$\bigoplus_{\alpha \in \Sigma_\theta \smallsetminus \theta} \mathfrak{g}_\theta^* \otimes \mathfrak{g}_\alpha \ .$$

Let u in $\mathfrak{a}_\Theta$ be as in the definition of boundary root. The eigenvalue of $\mathsf{T}_{t_\theta} \exp(u)$ on the factor $g_\theta^* \otimes \mathfrak{g}_\alpha$ of the above decomposition is $\exp(\alpha(u) - \theta(u))$. These quantities being strictly smaller than 1, this implies the first item.

The basin of attraction of u on $\mathbf{P}(\mathfrak{u}_\theta)$ is open and dense and thus intersects $\mathbf{P}(c_\theta)$. This implies that t_θ belongs to the closure of $\mathbf{P}(c_\theta)$; by point (3) of Propostion 4.11, it does not belong to $\mathbf{P}(c_\theta)$, proving Point (2). ◻

4.3.2. *Proof of Theorem 4.7.* Let η be a Θ-compatible dominant form and consider the map

$$\begin{aligned} q\colon \mathfrak{u}_\theta \times \mathfrak{u}_{-\theta} &\longrightarrow \mathbb{R} \\ (u,v) &\longmapsto \langle p([u,v]) \mid \eta \rangle \ . \end{aligned} \tag{14}$$

Observe that q is $\mathrm{Ad}(\mathsf{L}_\Theta)$-invariant. Thanks to Proposition 4.11 it is thus enough to check the property for $u = \sum_{\beta \in B_\theta} x_\beta$ (where B_θ is the set of boundary roots) and any v in $c_{-\theta}$ that is

$$v = \sum_{\alpha \in \Sigma_\theta} v_\alpha \ ,$$

with v_α in $\mathfrak{g}_{-\alpha}$ for every α in Σ_θ, and $v_\beta = \mu_\beta x_{-\beta}$ with $\mu_\beta > 0$ for every β in B_θ. Using the decomposition $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{z}_\mathfrak{k}(\mathfrak{a}) \oplus \bigoplus \mathfrak{g}_\alpha$, it follows that the projection of $[u,v]$ on $\mathfrak{a}$ is equal to

$$\sum_{\beta \in B_\theta} \mu_\beta h_\beta \ .$$

Hence, as η is zero on $\mathfrak{a}_\Theta$ and since $p([u,v])$ differs from the above element by an element in $\mathfrak{a}_\Theta$, and one has

$$\langle p([u,v]) \mid \eta \rangle = \sum_{\beta \in B_\theta} \mu_\beta \langle h_\beta \mid \eta \rangle = 2 \sum_{\beta \in B_\theta} \mu_\beta \frac{\langle \eta, \beta \rangle}{\langle \beta, \beta \rangle} .$$

Thus from the definition of a dominant form, we have

$$\langle p([u,v]) \mid \eta \rangle \geqslant 2\mu_\theta \frac{\langle \eta, \theta \rangle}{\langle \theta, \theta \rangle} .$$

From this last inequality, the lower bounds in Equations (12) and (13) of Theorem 4.7 follow.

5. Positivity of the cross-ratio

We continue with the setup of the previous section: G_0 is a semisimple Lie group admitting a positive structure with respect to Θ. The main result is the following:

Theorem 5.1 (Positivity of the cross-ratio)**.** *Let η be a Θ-compatible dominant non-zero form. Let b^η be the cross-ratio associated to η (cf. Section 2 and more particularly Section 2.4). For any positive quadruple (x,y,z,w) in $\mathcal{F}_\Theta$ we have*

$$\mathrm{b}^\eta(x,y,z,w) > 1 .$$

The terminology "positivity of the cross-ratio" becomes justified after one takes the logarithm.

The proof of Theorem 5.1 relies on the integral formula for the cross-ratio given in Section 2.4.

We state a useful corollary to Theorem 5.1:

Corollary 5.2. *Let η be a Θ-compatible dominant form, ω_θ a fundamental weight and (x,y,z,x) a positive quadruple. Then*

$$\mathrm{b}^\eta(x,y,z,w) \geqslant (\mathrm{b}^{\omega_\theta}(x,y,z,w))^{\langle h_\theta | \eta \rangle} .$$

In particular, for all γ in G

$$\mathrm{p}^\eta(\gamma) \geqslant (\mathrm{p}^{\omega_\theta}(\gamma))^{\langle h_\theta | \eta \rangle} .$$

Proof. Indeed, we can write

$$\eta = \eta_0 + \langle h_\theta \mid \eta \rangle \, \omega_\theta ,$$

where η_0 is a Θ-compatible dominant form. It then follows by Assertion (3) (p. 11) that

$$\mathrm{b}^\eta = (\mathrm{b}^{\omega_\theta})^{\langle h_\theta | \eta \rangle} \, \mathrm{b}^{\eta_0} .$$

and the statement follows from Theorem 5.1. ∎

5.1. **Infinitesimal positivity.** Denote as always $\mathcal{L}_\Theta = \mathcal{G}/\mathsf{L}_\Theta$.

Let x and y be transverse points in $\mathcal{F}_\Theta$. Let D be a diamond with extremities x and y. Recall from Section 4.1 that a tangent vector at x is non-negative with respect to $D = \psi(\mathsf{N}_\Theta)\cdot x$ if it belongs to the image by $\pi^{\mathcal{L}}_\psi$ of the closed cone $\sum_{\theta\in\Theta}\bar{c}_{-\theta}$ inside $\mathfrak{u}^{\mathrm{opp}}_\Theta$. We have:

Proposition 5.3. *Let η be a Θ-compatible dominant form and let v be a non-negative tangent vector at x (with respect to D) and w be a non-negative tangent vector at y (with respect to the opposite diamond $D^\vee$). Then*

$$\left\langle \Omega_{(x,y)}((v,0),(0,w)) \middle| \eta \right\rangle \geqslant 0 \,.$$

If furthermore η is non-zero and v and w are positive tangent vectors, then

$$\left\langle \Omega_{(x,y)}((v,0),(0,w)) \middle| \eta \right\rangle > 0 \,.$$

Proof. Recall the decomposition

$$\mathfrak{g}_0 = \mathfrak{l}_\Theta \oplus \mathfrak{u}_\Theta \oplus \mathfrak{u}^{\mathrm{opp}}_\Theta \,,$$

and $\pi^{\mathcal{L}}$ the projection from $\mathcal{G}$ to $\mathcal{L}_\Theta$. Let ψ be in $\mathcal{G}$ such that $\pi^{\mathcal{L}}(\psi) = (x,y)$ and $D = \psi(\mathsf{N}_\Theta)\cdot x$. We have an identification $\iota^{\mathcal{L}}_\psi$ (see Section 1.6) of $\mathsf{T}_{(x,y)}\mathcal{L}_\Theta$ with

$$\mathfrak{u}^{\mathrm{opp}}_\Theta \oplus \mathfrak{u}_\Theta.$$

By definition

$$\Omega((v,0),(0,w)) = p\left([\iota^{\mathcal{L}}_\psi((v,0)), \iota^{\mathcal{L}}_\psi((0,w))]\right) \,,$$

where p is the orthogonal projection from $\mathfrak{g}_0$ to $\mathfrak{b}_\Theta$. Hence the proposition reduces to Theorem 4.7 using that $\iota^{\mathcal{L}}_\psi((v,0))$ is a vector in $\sum_{\theta\in\Theta}\bar{c}_{-\theta}$, that $\iota^{\mathcal{L}}_\psi((0,w))$ is a vector in $\sum_{\theta\in\Theta} -\bar{c}_\theta$ (Section 4.1), and that the Lie bracket is antisymmetric. □

5.2. **Proof of Theorem 5.1.** We are in a setting where $\iota(\Theta) = \Theta$ so that $\mathcal{F}^{\mathrm{opp}}_\Theta \simeq \mathcal{F}_\Theta$ and $\mathcal{L}_\Theta$ is the open G-orbit in $\mathcal{F}_\Theta \times \mathcal{F}_\Theta$.

We begin the proof of Theorem 5.1 by showing that the hypotheses of Proposition 2.7 are always verified for positive quadruples:

Proposition 5.4. *Let (x,y,z,w) be a positive quadruple, then there exist C^1 arcs $c_0\colon [0,1]\to \mathcal{F}_\Theta$ and $c_1\colon [0,1]\to\mathcal{F}_\Theta$ such that $c_0(0) = x$, $c_0(1) = y$, $c_1(0) = z$, and $c_1(1) = w$ and such that, for all s in $(0,1)$ and all t in $(0,1)$, the sextuple $(x, c_0(s), y, z, c_1(s), w)$ is positive.*

For every such arcs c_0 and c_1 the map $f\colon [0,1]^2 \to \mathcal{F}_\Theta\times\mathcal{F}_\Theta$ defined by $f(s,t) = (c_0(s), c_1(t))$ takes value in $\mathcal{L}_\Theta$ and one has, for every t in $[0,1]$, $f(0,t) = (x,*)$ and $f(1,t) = (y,*)$, and for every s in $[0,1]$, $f(s,0) = (*,z)$ and $f(s,1) = (*,w)$.

Proof. Let D be the diamond with extremities x and z containing y; by positivity of the quadruple (x,y,z,w) the opposite diamond $D^\vee$ contains w. There exists a unique diamond D_0 contained in D and with extremities x and y and there exists a unique diamond D_1 contained in $D^\vee$ and with extremities z and w - see Figure 1.

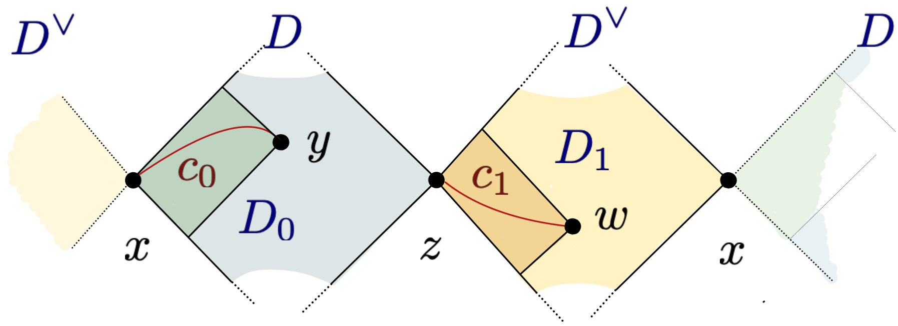

FIGURE 1. Configuration of the positive quadruple (x, y, z, w), the C^1 arcs c_0, c_1 and the diamonds $D, D^\vee, D_0, D_1$ in a $\mathbb{Z}$-covering of an annulus

We can now choose two arcs of positive circles: c_0 joining x to y and contained in D_0, c_1 joining z to w and contained in D_1 (Lemma 4.3). The inclusions of diamonds give that, for all s and t, the sextuple $(x, c_0(s), y, z, c_1(s), w)$ is positive.

Given such arcs c_0 and c_1, by positivity, for all s and t, $c_0(s)$ is transverse to $c_1(t)$. The map f given by

$$(s, t) \mapsto (c_0(s), c_1(t))$$

takes thus value in $\mathcal{L}_\Theta$ and has the wanted properties. □

We can now conclude the proof of Theorem 5.1 using c_0, c_1, and f as in the previous lemma. By Proposition 2.7,

$$\mathrm{b}^\eta(x, y, z, w) = \exp\left(\int_{[0,1]^2} f^*(\langle \Omega \mid \eta \rangle)\right) .$$

By definition if (u, v) belongs to $[0, 1]^2$,

$$f^*(\langle \Omega \mid \eta \rangle)_{(u,v))} = \left\langle \Omega\Big(\dot{c}_0(u), \dot{c}_1(v)\Big)\Big|\eta \right\rangle \cdot \mathrm{d}s \wedge \mathrm{d}t .$$

By Proposition 5.3, we have

$$\left\langle \Omega\Big(\dot{c}_0(u), \dot{c}_1(v)\Big)\Big|\eta \right\rangle > 0 .$$

The result now follows. □

6. THE PHOTON CROSS-RATIO BOUNDS THE θ-CHARACTER

In this section we relate the photon cross-ratios to characters of simple roots. In particular, we prove the following:

Theorem 6.1. *Let θ be an element of Θ, η a Θ-compatible dominant form such that $\langle \eta, \theta \rangle > 0$, and γ in G be a Θ-loxodromic element with attracting and repelling fixed points γ^+, γ^-, let x in $\mathcal{F}_\Theta$ be such that $(\gamma^+, \gamma^-, x, \gamma \cdot x)$ is a positive quadruple then*

$$\chi_\theta(\gamma)^{\langle h_\theta | \eta \rangle} \geqslant \min_{\Phi \in \boldsymbol{\Phi}(\gamma^-)} \mathrm{b}^\eta(p_\Phi(\gamma^+), \gamma^-, x, \gamma(x)) ,$$

where $\boldsymbol{\Phi}(\gamma^-)$ *is the family of* θ*-photons through* γ^-*.*

We first state and prove two preliminary results. Let a and b be two transverse points in $\mathcal{F}_\Theta$ and $\mathsf{L} \coloneqq \mathsf{L}_{a,b}$ be the stabilizer in G of the pair (a, b).

Proposition 6.2 (The compact case)**.** *Let* M *be a compact subgroup of* L*. Assume that* k *belongs to* M *and that* x *is transverse to both* a *and* b*, then for any* M*-invariant compact subset* M_0 *of* $\boldsymbol{\Phi}(a)$

$$\min_{\Phi \in M_0} \mathrm{b}^\eta(p_\Phi(b), a, x, k(x)) \leqslant 1 .$$

Proof. Let $\mathcal{S}$ be the M-orbit of x in $\mathcal{F}_\Theta$. All z in $\mathcal{S}$ are transverse to a and b, and hence to $p_\Phi(b)$ for all Φ in $\boldsymbol{\Phi}(a)$ by Corollaries 3.18 and 3.21. Thus the function

$$\Psi \colon (z, y, \Phi) \longmapsto \mathrm{b}^\eta(p_\Phi(b), a, z, y)$$

on $\mathcal{S} \times \mathcal{S} \times M_0$ is defined and continuous. We consider the function on $\mathcal{S}^2$

$$G(z, y) = \min_{\Phi \in M_0} \left| \mathrm{b}^\eta(p_\Phi(b), a, z, y) \right| ,$$

which is continuous by the continuity of Ψ and the compactness of M_0. As a consequence of the cocycle identity we have

$$G(z, y) \geqslant G(z, w) G(w, y) . \tag{15}$$

Since M_0 is M-invariant, for every g in M we have

$$G(g(z), g(y)) = G(z, y) .$$

By the compactness of $\mathcal{S}$ there is a constant A such that for all z and y in $\mathcal{S}$,

$$G(z, y) \leqslant A .$$

For any z and y in $\mathcal{S}$, let g in M be such that $y = g(z)$, we obtain by iterating the cocycle inequality (15) and using the M-invariance of G, that for all n

$$A \geqslant G(z, g^n(z)) \geqslant G(z, g(z))^n = G(z, y)^n .$$

This shows that $G(z, y) \leqslant 1$ for all y and z in the M-orbit of x. Hence $G(x, k(x))$ is at most 1 and this concludes the proof. □

In the next proposition, we use the Kostant–Jordan decomposition recalled in the beginning of Section 1.7.

Proposition 6.3 (A photon is preserved)**.** *Let* g *be a* Θ*-loxodromic element in* G *such that* a *and* b *are respectively the repelling and attracting fixed points of* g*. Let* $g = g_h g_u g_e$ *be the Kostant–Jordan decomposition of* g *(in* G*). Then there exists a* θ*-photon* Φ *in* $\boldsymbol{\Phi}(a)$ *invariant by* g_h *and by* g_u *and*

$$\chi_\theta(g)^{\langle h_\theta | \eta \rangle} = \mathrm{b}^\eta(p_\Phi(b), a, y, g_h g_u(y)) , \tag{16}$$

for all y *transverse to* a *and to* b*.*

Proof. Let ψ be an isomorphism of G_0 with G such that $\pi^{\mathcal{L}}(\psi) = (a,b)$ and $g_h = \psi(\exp(X))$ with X in the closed Weyl chamber $\mathfrak{a}^+$ — by the Kostant–Jordan decomposition as in Section 1.7. One also has $\psi(\mathsf{L}_\Theta) = \mathsf{L}_{a,b}$.

As g is Θ-loxodromic (see Section 1.7), the element X satisfies that $\langle X \mid \alpha\rangle > 0$ for all α in Θ and $\langle X \mid \alpha\rangle \geqslant 0$ for all α in $\Delta \smallsetminus \Theta$.

Let $E = \{v \in \mathfrak{u}_{-\theta} \mid \mathrm{ad}(X)v = -\langle X \mid \theta\rangle\, v\}$, and let Z_θ the L_Θ-orbit of $x_{-\theta}$ in $\mathfrak{u}_{-\theta}$. We know that the image by $\pi^{\mathcal{F}}_{\psi}$ of every vector v in Z_θ is tangent to a photon through a (Proposition 3.14). If furthermore this vector v is in E, the η-period of g_h on this photon satisfies the stated equality thanks to Proposition 3.27.

The proposition will be proved if we can find a vector in $E \cap Z_\theta$ that is also invariant by g_u (in which case the action of g_u on the corresponding photon will be trivial). Note that the space E is $\mathrm{Ad}(g_u)$-invariant since E is the intersection of $\mathfrak{u}_\theta$ with $\ker(\mathrm{ad}(X) - \langle X \mid \theta\rangle \,\mathrm{Id})$ and both these spaces are $\mathrm{Ad}(g_u)$-invariant. The projectivization $\mathbf{P}(Z_\theta \cap E)$ of Z_θ in $\mathbf{P}(E)$ is a closed $\mathrm{Ad}(g_u)$-invariant subset (cf. Lemma 3.11). Since $\mathrm{Ad}(g_u)$ is unipotent, every $\langle \mathrm{Ad}(g_u)\rangle$-orbit in $\mathbf{P}(E)$ accumulates to a point fixed by $\mathrm{Ad}(g_u)$. These last two remarks imply that $\mathbf{P}(Z_\theta \cap E)$ contains points fixed by $\mathrm{Ad}(g_u)$. This finishes the proof. ◻

Proof of Theorem 6.1. We can now prove the inequality of Theorem 6.1.

Let us write $\gamma = \gamma_0\gamma_e$ with $\gamma_0 = \gamma_h\gamma_u$, where γ_h, γ_u, and γ_e are pairwise commuting and respectively the hyperbolic, unipotent, and elliptic parts of γ. Let then M be the closure of the group generated by γ_e and M_0 be the compact set of photons Φ preserved by γ_0 in $\mathbf{\Phi}(\gamma^-)$ and satisfying Equation (16), namely such that

$$\chi_\theta(\gamma)^{\langle h_\theta \mid \eta\rangle} = \mathrm{b}^\eta(p_\Phi(b), a, y, \gamma_0\gamma_u(y))\ ,$$

for all y transverse to a and b. We observe that M_0 is invariant by M, and non-empty by Proposition 6.3 (applied with $a = \gamma^-$ and $b = \gamma^+$). Let finally Φ_0 be a photon in M_0 such that

$$\mathrm{b}^\eta(p_{\Phi_0}(\gamma^+), \gamma^-, x, \gamma_e(x)) = \min_{\Phi\in\mathsf{M}_0} \mathrm{b}^\eta(p_\Phi(\gamma^+), \gamma^-, x, \gamma_e(x)) \leqslant 1\ ,$$

where the inequality comes from Proposition 6.2. We then have by the cocycle identities

$$\begin{aligned}
&\mathrm{b}^\eta(p_{\Phi_0}(\gamma^+), \gamma^-, x, \gamma(x)) \\
&\qquad = \mathrm{b}^\eta(p_{\Phi_0}(\gamma^+), \gamma^-, x, \gamma_e(x))\, \mathrm{b}^\eta(p_{\Phi_0}(\gamma^+), \gamma^-, \gamma_e(x), \gamma_0\gamma_e(x)) \\
&\qquad \leqslant \mathrm{b}^\eta(p_{\Phi_0}(\gamma^+), \gamma^-, \gamma_0(\gamma_e(x)), \gamma_e(x)) \\
&\qquad = \chi_\theta(\gamma_0)^{\langle h_\theta \mid \eta\rangle} = \chi_\theta(\gamma)^{\langle h_\theta \mid \eta\rangle}\ .
\end{aligned}$$

It follows that

$$\min_{\Phi\in\mathbf{\Phi}(\gamma^-)} \mathrm{b}^\eta(p_\Phi(\gamma^+), \gamma^-, x, \gamma(x)) \leqslant \chi_\theta(\gamma)^{\langle h_\theta \mid \eta\rangle}\ ,$$

and the result follows. ◻

7. The collar inequality

Our goal in this section is to prove the main result of this paper, it generalizes Theorem C in the sense that general Θ-compatible dominant forms are allowed.

Theorem 7.1 (Collar Lemma in the group)**.** *Let* G *a semisimple Lie group admitting a Θ-positive structure. Let A and B be Θ-loxodromic elements of* G*. Denote by* (a^+, a^-) *and* (b^+, b^-) *the pair of attracting and repelling fixed points of A and B respectively in the flag variety* $\mathcal{F}_\Theta$*. Assume that the sextuple*

$$(a^+, b^-, a^-, b^+, B(a^+), A(b^+)) ,$$

is positive (see Figure 2). Let θ be an element of Θ, and η be a Θ-compatible dominant form with $\langle h_\theta \mid \eta \rangle > 0$*. Then*

$$\left(\frac{1}{\mathrm{p}^\eta(B)}\right)^{1/\langle h_\theta|\eta\rangle} + \frac{1}{\chi_\theta(A)} < 1 . \tag{17}$$

Observe that when $\langle h_\theta \mid \eta \rangle = 0$, the above inequality is still true but of little use.

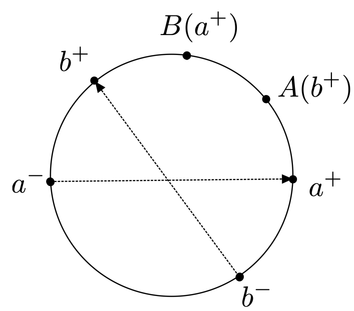

Figure 2. Positive sextuple

Proof. From Corollary 5.2 it is enough to prove the inequality whenever η is a fundamental weight ω_θ of θ.

Let Φ be a θ-photon through a^-. From Proposition 3.27 and the classical relation for the projective cross-ratio, we have

$$\mathrm{b}^{\omega_\theta}(a^-, p_\Phi(b^+), a^+, A(b^+)) + \mathrm{b}^{\omega_\theta}(a^-, p_\Phi(a^+), b^+, A(b^+)) = 1 .$$

We will now obtain a minoration of the first term in the left-hand side of this equation, we will then apply Theorem 6.1 in order to obtain the wanted majoration. In these computations, we will use freely that the cross-ratio is greater than 1 for positive quadruples (Theorem 5.1).

First step: We first bound from below the first term of the previous equality.

$$\mathrm{b}^{\omega_\theta}(a^-, p_\Phi(b^+), a^+, A(b^+)) > \mathrm{p}^{\omega_\theta}(B)^{-1} .$$

Let $L := \mathrm{b}^{\omega_\theta}(a^-, p_\Phi(b^+), a^+, A(b^+))$. By the cocyle relation we have

$$L = \mathrm{b}^{\omega_\theta}(a^-, b^+, a^+, A(b^+)) \cdot \mathrm{b}^{\omega_\theta}(b^+, p_\Phi(b^+), a^+, A(b^+)) .$$

The quadruple $(a^-, b^+, A(b^+), a^+)$ is positive, hence by Proposition 4.5, the quadruple $(b^+, p_\Phi(b^+), a^+, A(b^+))$ is also a semi-positive quadruple, thus Theorem 5.1 gives

$$\mathrm{b}^{\omega_\theta}(b^+, p_\Phi(b^+), a^+, A(b^+)) \geqslant 1 .$$

Then

$$\begin{aligned} L &\geqslant \mathrm{b}^{\omega_\theta}(a^-, b^+, a^+, A(b^+)) \\ &= \mathrm{b}^{\omega_\theta}(a^-, b^+, a^+, B(a^+)) \cdot \mathrm{b}^{\omega_\theta}(a^-, b^+, B(a^+), A(b^+)) \\ &= \mathrm{b}^{\omega_\theta}(a^-, b^-, a^+, B(a^+)) \cdot \mathrm{b}^{\omega_\theta}(b^-, b^+, a^+, B(a^+)) \cdot \mathrm{b}^{\omega_\theta}(a^-, b^+, B(a^+), A(b^+)) , \end{aligned}$$

where we used the cocycle identities twice. Since $(a^-, b^+, B(a^+), A(b^+))$ and $(a^-, b^-, a^+, B(a^+))$ are positive quadruples (the latter follows as $(b^-, a^-, B(a^+), a^+)$ is positive), their cross-ratios are greater than 1 and we get

$$L > \mathrm{b}^{\omega_\theta}(b^-, b^+, a^+, B(a^+)) = \mathrm{p}^{\omega_\theta}(B)^{-1} ,$$

which is what we wanted to prove.

Second step: We obtain from the first step that, for every photon Φ through a^-,

$$\mathrm{p}^{\omega_\theta}(B)^{-1} + \mathrm{b}^{\omega_\theta}(a^-, p_\Phi(a^+), b^+, A(b^+)) < 1 .$$

Letting Φ vary in $\mathbf{\Phi}(a^-)$, we get

$$\mathrm{p}^{\omega_\theta}(B)^{-1} + \max_{\Phi \in \mathbf{\Phi}(a^-)} (\mathrm{b}^{\omega_\theta}(a^-, p_\Phi(a^+), b^+, A(b^+))) < 1 .$$

Observe now that $(a^-, a^+, A(b^+), b^+)$ is a positive quadruple, thus Theorem 6.1 gives in particular that

$$\begin{aligned} \chi_\theta(A)^{-1} &\leqslant \Big(\min_{\Phi \in \mathbf{\Phi}(a^-)} \mathrm{b}^{\omega_\theta}(a^-, p_\Phi(a^+), A(b^+), b^+) \Big)^{-1} \\ &= \max_{\Phi \in \mathbf{\Phi}(a^-)} \mathrm{b}^{\omega_\theta}(a^-, p_\Phi(a^+), b^+, A(b^+)) . \end{aligned}$$

Thus combining the two last inequalities, we get

$$\mathrm{p}^{\omega_\theta}(B)^{-1} + (\chi_\theta(A))^{-1} < 1 .$$

This is the inequality that we wanted to prove. □

8. Positive representations of finite type and infinite type surfaces

In this section, we give the definition of positive representations in a setting that allows surfaces that are not closed, or not even of finite topological type, i.e. we do not assume that the fundamental group is finitely generated.

Let Σ be a —possibly non-compact— connected oriented surface whose fundamental group Γ contains a free group. Among loops not homotopic to zero, we distinguish between *peripheral loops* and *non-peripheral loops* in Σ: peripheral loops are curves in Σ which are freely homotopic to a multiple of a boundary component or a cusp, otherwise a loop is non-peripheral. We use the same terminology for conjugacy classes of elements of $\pi_1(\Sigma)$, seen as free homotopy classes of loops.

We denote by Λ the classes of non-peripheral elements of $\pi_1(\Sigma)$ up to positive powers; i.e. γ and γ' represent the same element in Λ if and only if there are positive integers n and n' such that $\gamma^n = \gamma'^{n'}$. The class in Λ of a non-peripheral element γ will be denoted by γ^+. The set Λ should be thought of as the set of attracting fixed points of non-peripheral elements of $\pi_1(\Sigma)$ in the boundary at infinity of the group. The conjugation induces a natural action of $\pi_1(\Sigma)$ on Λ. We will introduce a cyclic order on Λ. Since $\pi_1(\Sigma)$ might not be finitely generated, we cannot directly use the boundary at infinity. Instead, we use the following trick.

Proposition 8.1 (Reduction to finite type). *Given finitely many elements $\gamma_1, \dots, \gamma_p$ in $\pi_1(\Sigma)$, there exists an incompressible connected surface S of finite type, in Σ, whose fundamental group contains all the γ_i. If furthermore, none of the γ_i are peripheral, we can choose S such that all the γ_i remain non-peripheral in S.*

In the situation of the proposition, we say that S *encloses* $(\gamma_1, \dots, \gamma_p)$. Similarly, given finitely many elements $t_1, \dots, t_n$ in Λ, we say that an incompressible connected surface S of finite type *encloses* them if there are $\gamma_1, \dots, \gamma_n$ in $\pi_1(\Sigma)$ such that, for all i, $\gamma_i^+ = t_i$ and S encloses $(\gamma_1, \dots, \gamma_n)$.

If S encloses a curve γ, it also encloses every γ' representing the same element in Λ.

Given a n-tuple $(\gamma_1^+, \dots, \gamma_n^+)$ in Λ and a surface S of finite type enclosing the tuple, we say that $(\gamma_1^+, \dots, \gamma_n^+)$ is *S-cyclically oriented* if the tuple $(\gamma_{1,S}^+, \dots, \gamma_{n,S}^+)$ is cyclically oriented in $\partial_\infty\pi_1(S)$, where $\gamma_{i,S}^+$ is the attracting fixed point of γ_i in $\partial_\infty\pi_1(S)$.

Remarks 8.2.

- Note that if $\gamma_i^{n_i} = (\gamma_i')^{n_i'}$, then $\gamma_{i,S}^+ = \gamma_{i,S}'^+$ so that the definition makes sense.
- Of course, it is enough here to define cyclically oriented triples and the definition of cyclically oriented n-tuples follows by compatibility.
- When Σ is already of finite type, Λ is identified with a subset of $\partial_\infty\pi_1(\Sigma)$.

Proposition 8.3. *If $(\gamma_1^+, \dots, \gamma_n^+)$ is S_0-cyclically oriented for a subsurface S_0 of finite type enclosing them, it is S-cyclically oriented for any subsurface S of finite type enclosing them.*

Proof. Let S_0, S_1 be two incompressible finite type connected subsurfaces enclosing $(\gamma_1, \dots, \gamma_n)$. We find an incompressible finite type connected subsurfaces S containing both S_0 and S_1. Then there are embeddings $\iota_i \colon \partial_\infty\pi_1(S_i) \to \partial_\infty\pi_1(S)$, $i = 0, 1$, such that a tuple in $\partial_\infty\pi_1(S_i)$ is cyclically oriented if and only if its image under ι_i is. This proves the claim as $i_0(\gamma_{i,S_0}^+) = i_1(\gamma_{i,S_1}^+)$. □

As a conclusion, there is a well defined cyclic ordering on Λ. We use this to define the notion of Θ-*positive* representations.

In the next two definitions we assume that G_0 has a Θ-positive structure and we let G, and $\mathcal{F}_\Theta = \mathcal{G}/\mathsf{P}_\Theta$ be as in Section 1.

Definition 8.4. Let C be a set with a cyclic order. A map $\xi \colon C \to \mathcal{F}_\Theta$ is called *positive* if every cyclically ordered tuple in C is mapped to a positive tuple in $\mathcal{F}_\Theta$ by ξ (cf. also Section 4.1).

Definition 8.5. A representation $\rho\colon \pi_1(\Sigma) \to \mathsf{G}$ is said to be Θ-*positive* if there exists a ρ-equivariant positive map from Λ to $\mathcal{F}_\Theta$.

We then have:

Proposition 8.6. *Let ρ be a Θ-positive representation of $\pi_1(\Sigma)$ in G, if γ is a non-peripheral element, then $\rho(\gamma)$ is Θ-loxodromic, and ξ maps attracting fixed points to attracting fixed points.*

Proof. We can reduce using Propositions 8.1 to the case when Σ is a finite type surface. Then the result follows from [20, Proposition 3.18]. ◻

When Σ is closed, every non-trivial element in $\pi_1(\Sigma)$ is non-peripheral. In fact, in that case, Definition 8.5 agrees with the definition from [20] (cf. Proposition 5.7 in that reference).

9. Collar inequality for representations

In this section we collect the material of Sections 7 and 8 in order to produce Collar Lemmas for Θ-positive representations.

Theorem 7.1 applies to Θ-positive representations:

Corollary 9.1 (Collar Inequality)**.** *Let η be a Θ-compatible dominant form and let θ be in Θ. Assume that $\langle \theta, \eta \rangle > 0$. Let Σ be a connected oriented (not necessarily of finite type) surface whose fundamental group contains a free group. Then given a positive representation ρ of $\pi_1(\Sigma)$, two loops γ_0 and γ_1 geometrically intersecting, we have*

$$\left(\frac{1}{\mathrm{p}^\eta\left(\rho(\gamma_1)\right)}\right)^{1/\langle h_\theta | \eta\rangle} + \frac{1}{\chi_\theta\left(\rho(\gamma_0)\right)} < 1\,.$$

Proof. Let S be a finite type surface enclosing γ_0 and γ_1.

Let x be a point of intersection of γ_0 and γ_1, we choose (and denote them the same way) representatives γ_0 and γ_1 in $\pi_1(S,x)$. Let us denote by $\gamma_i^\pm$ the attracting/repelling fixed points of γ_i in $\partial_\infty \pi_1(S)$. The intersection hypothesis implies that, up to exchanging γ_1 and γ_1^{-1}, the sextuple

$$(\gamma_1^-,\ \gamma_0^-,\ \gamma_1^+, \gamma_1(\gamma_0^+),\ \gamma_0(\gamma_1^+),\ \gamma_0^+)\,,$$

is a positive configuration in $\partial_\infty \pi_1(S)$ (see for instance [30, Lemma 2.2]).

We denote by a^+, a^-, b^+ and b^- the images of respectively $\gamma_0^+, \gamma_0^-, \gamma_1^+$ and γ_1^- under the limit map. We also write

$$A := \rho(\gamma_0)\,,\quad B := \rho(\gamma_1)\,.$$

By Proposition 8.6, a^+ and a^- are the attracting and repelling fixed points of A, and b^+ and b^- are those of B. By positivity, it follows that

$$(b^-, a^-, b^+, B(a^+), A(b^+), a^+)$$

is also a positive configuration (see figure 2). Then the theorem follows from Theorem 7.1. ◻

Choosing the Θ-compatible dominant form η in Corollary 9.1 to be equal to a fundamental weight ω_θ we immediately get the following corollary which is Corollary D of the introduction.

Corollary 9.2. *Let ρ be a Θ-positive representation of a surface group $\pi_1(\Sigma)$. Let γ_0 and γ_1 be two geometrically intersecting loops and θ in Θ, then*

$$\frac{1}{\mathrm{p}^{\omega_\theta}(\rho(\gamma_0))} + \frac{1}{\chi_\theta(\rho(\gamma_1))} < 1 .$$

The inequality of this last corollary can be reformulated as

$$(\mathrm{p}^{\omega_\theta}(\rho(\gamma_0)) - 1)\,(\chi_\theta(\rho(\gamma_1)) - 1) > 1.$$

Remark 9.3. The previous corollary —or Corollary D— is actually equivalent to Corollary 9.1, by Corollary 5.2.

9.1. **Comparison with other Collar Lemmas.** Collar Lemmas originate from the work of Keen in hyperbolic geometry. For the holonomy $\rho\colon \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ of a hyperbolic structure, denoting by $\ell(\rho(\gamma))$ the length of the geodesic representative of $\rho(\gamma)$ in the hyperbolic surface, building on the results she proved in [24] the following sharp inequality was deduced (see [33, Section 6], [14, Corollary 4.1.2]): for γ_0 and γ_1 geometrically intersecting

$$\sinh\left(\frac{1}{2}\ell(\rho(\gamma_0))\right)\sinh\left(\frac{1}{2}\ell(\rho(\gamma_1))\right) > 1. \tag{18}$$

Moving to the higher rank setting there are several possible generalizations of this result, as many possible quantities can be understood as length of an element with respect to a representation. One possible direction is to replace the length with a suitable Finsler translation length; results in this direction are discussed in Section 9.1.2. Our collar lemma, as well as its predecessors discussed in Section 9.1.1 is a non-symmetric generalization, as it compares the character of a root with respect to that of a weight.

This asymmetry between roots and weights is key: on the one hand only by controlling the root we can deduce closedness in the space of representations, on the other hand it is proven in [4, Theorem 7.1] that for Hitchin representations in $\mathsf{PSL}_3(\mathbb{R})$ no collar lemma comparing the roots of two elements that intersect geometrically can exist, and in this respect our result, as well as the results discussed in Section 9.1.1, are optimal.

9.1.1. *Collar Lemmas comparing roots and weights for Θ-positive representations.* Many instances of Collar Lemmas comparing roots and weights for special classes of Θ-positive representations already appeared in the literature. None of these results are sharp, as the proofs always involve a crude minoration.

In the case of Hitchin representations into $\mathsf{PSL}_n(\mathbb{R})$ Lee and Zhang prove the following inequalities, for k in $\{1, \ldots, n-1\}$ [30, Proposition 2.12(1)]

$$(\mathrm{p}^{\omega_1}(\rho(\gamma_0)) - 1)\left(\chi_{\alpha_k}(\rho(\gamma_1)) - 1\right) > 1$$

here we denote by $\{\alpha_1, \ldots, \alpha_{n-1}\}$ the simple roots of $\mathsf{PSL}_n(\mathbb{R})$ in the standard numeration (i.e. α_i is connected to $\alpha_{i\pm1}$ in the Dynkin diagram) and denote $\omega_i := \omega_{\alpha_i}$.

In the case of maximal representations into $\mathsf{Sp}_{2n}(\mathbb{R})$ Burger and Pozzetti obtain [13, Theorem 3.3(2)]

$$\mathrm{p}^{\omega_1}(\rho(\gamma_0))^n\,(\chi_{\alpha_n}(\rho(\gamma_1))-1)>1\,.$$

In the case of Θ-positive representations into $\mathsf{SO}(p,q)$ for $p \leqslant q$ Beyrer and Pozzetti proved Corollary 9.2 [6, Theorem B]: for every $1 \leqslant k \leqslant p-1$

$$(\mathrm{p}^{\omega_k}(\rho(\gamma_0))-1)\left(\chi_{\alpha_k}(\rho(\gamma_1))-1\right)>1\,.$$

While all these results, as well as ours, rely on the positivity of the sextuple $(a^+, b^-, a^-, b^+, B(a^+), A(b^+))$ the strategy of proofs is different in the three cases. Our proof follows the approach outlined in [6] with the important new contribution of the introduction of photons which allows to treat all roots simultaneously regardless of the dimension of the associated root space. As such our proof is uniform for all Θ-positive representations, independent of the zoology of the group involved.

9.1.2. *Collar Lemmas through domination.* For a Hitchin representation into $\mathsf{PSL}_3(\mathbb{R})$, or a maximal representation in $\mathsf{SO}_0(2,n)$, Tholozan [36, Corollary 4] and Collier–Tholozan–Toulisse [15, Corollary 6] used domination to deduce a Collar Lemma: for a Hitchin representation ρ in $\mathsf{PSL}_3(\mathbb{R})$ Tholozan finds a Fuchsian representation whose spectrum dominates $\ell_1(A) := \log \mathrm{p}^{\omega_1}(A)$ and he deduces from the hyperbolic Collar Lemma

$$\sinh\left(\frac{1}{4}\ell_1(\rho(\gamma_0))\right)\sinh\left(\frac{1}{4}\ell_1(\rho(\gamma_1))\right)>1.$$

For a maximal representation ρ in $\mathsf{SO}_0(2,n)$ Collier, Tholozan, and Toulisse find a Fuchsian representation whose length spectrum dominates $\ell_1 = \log \mathrm{p}^{\omega_1}$, and deduce similarly

$$\sinh\left(\frac{1}{2}\ell_1(\rho(\gamma_0))\right)\sinh\left(\frac{1}{2}\ell_1(\rho(\gamma_1))\right)>1\,.$$

These Collar Lemmas are sharp, but since they don't control the root character, they do not guarantee that the limit of a converging sequence contains loxodromic elements in its image.

9.1.3. *Other Collar Lemmas.* Beyrer and Pozzetti show in [4, Theorem 1.1] that Collar Lemmas are not specific to Θ-positive representations and define other classes of representations in $\mathsf{PSL}_d(\mathbb{R})$ that satisfy the inequality

$$(\mathrm{p}^{\omega_k}(\rho(\gamma_0))-1)\left(\chi_{\alpha_k}(\rho(\gamma_1))-1\right)>1\,.$$

They exhibit in particular the class of $(k+2)$-positive representations for which this Collar Lemma holds (see [5, Corollary 6.20]). In particular $(k+2)$-positive representations form open subsets of the representation variety, but never connected components, outside of the Hitchin component.

This highlights that, even though we use the Collar Lemma in Section 9.2 to prove the closedness of the space of positive representations, this is not a mere consequence of the Collar Lemma, but really of the combination of the structure of limits of positive representations established in [20, Proposition 6.4] —see also [5, Theorem B]— and the Collar Lemma.

9.2. **Closedness of the space of positive representations.** As a consequence of the Collar Lemma together with results of [20] we obtain:

Corollary 9.4. *The space of positive representations is closed in the space of representations.*

Proof. Let $\{\rho_m\}_{m\in\mathbb{N}}$ be a sequence of positive representations converging to a representation ρ_∞. Let γ_0 and γ_1 be loops intersecting at least once. Let θ be an element of Θ. Since for every γ in $\pi_1(S)$, the sequence $\{\mathrm{p}^{\omega_\theta}(\rho_m(\gamma))\}_{m\in\mathbb{N}}$ is bounded by a constant $K(\gamma)$, it follows from the collar inequality (Corollary 9.2) that for all m,

$$\frac{1}{\chi_\theta(\rho_m(\gamma_1))} \leqslant 1 - \left(\frac{1}{K(\gamma_0)}\right) .$$

As a consequence there is a positive ε, such that for all m and all θ in Θ, we have

$$\chi_\theta\left(\rho_m(\gamma_1)\right) \geqslant 1 + \varepsilon .$$

Since the Jordan projection (and hence χ_θ) is continuous, it follows that

$$\chi_\theta\left(\rho_\infty(\gamma_1)\right) \geqslant 1 + \varepsilon .$$

Recall that h is Θ-loxodromic if and only if $\chi_\theta(h) > 1$ for all θ in Θ (Proposition 1.6). In particular $\rho_\infty(\gamma_1)$ is loxodromic. Let $\{x_m\}_{m\in\mathbb{N}}$ and $\{y_m\}_{m\in\mathbb{N}}$ be the repelling and attracting fixed points of $\rho_m(\gamma_1)$, then $\{x_m\}_{m\in\mathbb{N}}$ and $\{y_m\}_{m\in\mathbb{N}}$ converge to, respectively, the attracting and the repelling fixed points x_∞ and y_∞ of $\rho_\infty(\gamma_1)$ which are transverse. By [20, Proposition 6.4], ρ_∞ is positive. This concludes the proof. □

Appendix A. Extension to real closed fields

In this appendix, we explain how to extend the results obtained previously to all real closed fields by using the quantifier elimination Theorem of Tarski and Seidenberg. The importance of real closed field in the theory of surface group representations is outlined in the work of Brumfiel [9] and more recently Burger, Iozzi, Parreau and Pozzetti [11]. We will not address here challenges on the structure of the character variety of positive representations itself but only focus on the extension of our main results to positive representations defined over real closed fields.

A.1. **Real closed fields.** A *real closed field* is a totally ordered field so that every positive element is a square and every odd degree polynomial has a root. Obviously $\mathbb{R}$ is a real closed field.

The *Tarski–Seidenberg quantifier elimination theorem* loosely says that any semi-algebraic statement holding over $\mathbb{R}$ holds for any real closed field $\mathbb{F}$. Recall that a *semi-algebraic set* over an ordered field $\mathbb{K}$ is a subset of $\mathbb{K}^n$ which can be defined by finitely many algebraic equalities and inequalities with coefficients in $\mathbb{K}$. We will use two important consequences of Tarski–Seidenberg: the *Projection Theorem*, stating that the image a semi-algebraic set by a polynomial map is also semi-algebraic [7, Proposition 2.2.7], and the *transfer principle* stating that a semi-algebraic set defined over $\mathbb{R}$ is empty if and only if for any real closed extension $\mathbb{F}$ of $\mathbb{R}$ the subset of $\mathbb{F}^n$ defined by the same equalites and inequalities is empty [7, Proposition 5.3.5].

Our goals are now the following

- Define positive representations with values in a semisimple algebraic group defined over a real closed field $\mathbb{F}$,
- Show that Theorems A and C can be rephrased in terms of semi-algebraic subset, and thus hold over arbitrary real closed fields.

A.2. **Positive representations over real closed fields.** Let G be a semisimple real algebraic group equipped with a positive structure relative to Θ and $\mathcal{F}_\Theta$ the generalized flag manifold associated to P_Θ. Our first result is:

Proposition A.1. *The set of positive n-tuples is a semi-algebraic subset of $\mathcal{F}_\Theta^n$.*

Proof. The parametrization theorem of Guichard–Wienhard [21, Theorem 10.1] parametrizes a positive diamond as the image by a polynomial map —indeed the exponential map defined on a unipotent subalgebra is actually polynomial— of a product of cones in $\mathfrak{u}_\theta$ that are semi-algebraic and in fact defined by finitely many explicit inequalities [21, Section 5]. Thus by the Projection Theorem a positive diamond is a semi-algebraic set. □

Any semi-algebraic set defined over $\mathbb{K}$ admits a natural $\mathbb{F}$-extension for any real closed field $\mathbb{F}$ containing $\mathbb{K}$, which amounts to considering the set defined by the same polynomial equalities and inequalites in $\mathbb{F}^n$. Given a real closed field $\mathbb{F}$ we denote by $\mathcal{F}_\Theta(\mathbb{F})$ the $\mathbb{F}$-extension of the flag manifold $\mathcal{F}_\Theta$ on which $\mathsf{G}(\mathbb{F})$ acts and say that a triple in $\mathcal{F}_\Theta(\mathbb{F})^3$ is positive if it belongs to the $\mathbb{F}$-extension of the set of positive triples in $\mathcal{F}_\Theta^3$ (cf. [11, Example 6.17 (c)]).

We can now extend verbatim the definition of positive representations into algebraic groups with coefficients in real closed fields since it only involves the notion of positive triples and quadruples.

A.3. **The main results for real closed fields.** We now state Theorem A and Theorem C for real closed fields and prove the corresponding statements.

Theorem A.2 (Positivity of the cross-ratio). *Let G be a semisimple algebraic group admitting a positive structure relative to Θ, $\mathcal{F}_\Theta$ be the generalized flag manifold associated to Θ, λ a Θ-compatible dominant weight, b^λ the associated cross-ratio. Then for any real closed field $\mathbb{F}$ and every positive quadruple (x, y, X, Y) in $\mathcal{F}_\Theta(\mathbb{F})^4$ it holds*

$$\mathrm{b}^\lambda(x, y, X, Y) > 1 \ .$$

Proof. It follows from Equations (4) and (5) in Section 5 that the cross-ratio b^λ defines an algebraic function from the semi-algebraic subset O of $\mathcal{F}_\Theta^4$. As a result the set

$$\{(x, y, X, Y) \in \mathcal{F}_\Theta^4 | \, (x, y, X, Y) \text{ is positive}, \mathrm{b}^\lambda(x, y, X, Y) \leqslant 1\}$$

is a semi-algebraic subset defined over $\mathbb{R}$. Since by Theorem A.2 such set is empty over $\mathbb{R}$, it is empty over every real closed field $\mathbb{F}$ extending $\mathbb{R}$, which proves the desired statement. □

We now turn to Theorem C. Given a real closed field $\mathbb{F}$ and an element A in $\mathsf{G}(\mathbb{F})$ we say that a point a^+ in $\mathcal{F}_\Theta(\mathbb{F})$ is an attracting fixed point for A (respectively a^- is a repelling fixed point) if it is a fixed point for the A action on $\mathcal{F}_\Theta(\mathbb{F})$ and the action of $\mathrm{Ad}(g)$ on the Lie algebra of the unipotent radical of the stabilizer of a^+ has all eigenvalues of modulus in $\mathbb{F}$ strictly larger than one (respectively strictly smaller than one). Then the subset of $\mathsf{G}(\mathbb{F}) \times \mathcal{F}_\Theta(\mathbb{F})$ consisting of pairs (A, a) such that a is an attracting fixed point for A (respectively repelling) is semi-algebraic, and corresponds to the $\mathbb{F}$-extension of the subset of $\mathsf{G} \times \mathcal{F}_\Theta$ consisting of pairs satisfying the same property [11, Section 4.4].

Theorem A.3. *Let $\mathbb{F}$ be real closed. For every pair of Θ-loxodromic elements A and B in $\mathsf{G}(\mathbb{F})$ with attracting and repelling fixed points (a^+, a^-) and (b^+, b^-) such that the sextuple $(a^+, b^-, a^-, b^+, B(a^+), A(b^+))$ in $\mathcal{F}_\Theta(\mathbb{F})^6$ is positive and any θ in Θ it holds*

$$\frac{1}{\mathrm{p}^{\omega_\theta}(A)} + \frac{1}{\chi_\theta(B)} < 1 .$$

Proof. We need, again to show that the set P of pairs (A, B) in G^2 admitting attracting and repelling fixed points such that the sextuple

$$(a^+, b^-, a^-, b^+, B(a^+), A(b^+))$$

is positive and for which the Equation

$$\frac{1}{\mathrm{p}^{\omega_\theta}(A)} + \frac{1}{\chi_\theta(B)} \geqslant 1 . \tag{19}$$

holds is a semi-algebraic set.

On the one hand, let P_1 be the set of pairs (A, B) in G^2 admitting attracting and repelling fixed points such that the sextuple $(a^+, b^-, a^-, b^+, B(a^+), A(b^+))$ is positive, then P_1 is a semi-algebraic set. Indeed this follows from the discussion above concerning attracting fixed points, from the action of G on $\mathcal{F}_\Theta$ being algebraic, and the set of positive 6-tuples being semi-algebraic.

On the other hand, the set P_2 of pairs (A, B) in G^2 such that Equation (19) holds is a semi-algebraic set, since both the period $\mathrm{p}^{\omega_\theta}$ and the θ-character χ_θ are semi-algebraic functions. Indeed the periods are defined by the expressions $\mathrm{b}^{\omega_\theta}(b^+, b^-, a^+, B(a^+))$, which are semi-algebraic since the cross-ratio $\mathrm{b}^{\omega_\theta}$ and the function associating to B its attracting and repelling fixed points is semi-algebraic; that the character χ_θ is semi-algebraic follows from [11, Proposition 4.7].

Now the set $P = P_1 \cap P_2$ is semi-algebraic. Theorem C guarantees that is empty over $\mathbb{R}$, and it thus follows from the transfer principle that it is empty over any real closed field $\mathbb{F}$ extending $\mathbb{R}$, which concludes the proof. ◻

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