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ON MANIFOLDS LOCALLY MODELLED ON NON-RIEMANNIAN HOMOGENEOUS SPACES

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1. Introduction and Statement of Main Results

In this paper we continue the investigation of compact manifolds locally modelled on a homogeneous space of a finite dimensional Lie group. We recall that a manifold M is locally modelled on a homogeneous space $J\backslash H$ if there is an atlas on M consisting of local diffeomorphisms with open sets in $J\backslash H$ and where the transition functions are given by restrictions to open sets of translations by elements of H acting on $J\backslash H$. A basic example is given by a cocompact lattice in $J\backslash H$, namely a discrete subgroup $\Gamma \subset H$ such that Γ acts freely and properly discontinuously on $J\backslash H$ and such that $M = J\backslash H/\Gamma$ is compact. In this case, M is then locally modelled on $J\backslash H$. It is a basic open question to understand those homogeneous spaces $J\backslash H$ for which there is a compact form (i.e. space locally modelled on $J\backslash H$) or a cocompact lattice.

While there have been numerous approaches to this question (see, e.g. the references and brief discussion in [Z1]), one of these has involved the presence of a non-trivial centralizer $G \subset Z_H(J)$ for J. In the case in which H is simple and J actually has a non-trivial -diagonalizable 1-parameter subgroup in its center, then the main result of [BL] is that there are no compact forms for $J\backslash H$. The technique involves using the center to construct a suitable symplectic form. In [Z1], the case in which J may have trivial -split center but yet have a significant centralizer in H was studied. The main result of [Z1] is the assertion that if there is a cocompact lattice for $J\backslash H$ then under suitable hypotheses J must be compact. (In the event J is compact it is known that there are always cocompact lattices, at least if H is semisimple.) These hypotheses are

i. H and J are unimodular real algebraic groups;
ii. The centralizer $Z_H(J)$ contains a group G such that:
   (a) G is not contained in a proper normal subgroup of H;
   (b) G is a semisimple Lie group each of whose simple factors has -rank at least 2; and,

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(c) Every homomorphism $\tilde{G} \to J$ is trivial.

As a consequence, one obtained in [Z1] the conclusion that $\mathrm{SL}(m, \ )\backslash\mathrm{SL}(n, \ )$ (for the standard embedding) does not admit a cocompact lattice if $n > 5$ and $m < n/2$. However, the arguments in [Z1] do not give the conclusion that there is no compact form in these cases. It also leaves open the question of existence of cocompact lattices and compact forms if $m \geq n/2$. The techniques of [Z1] involve superrigidity for cocycles and Ratner's theorem on invariant measures on homogeneous spaces.

The aim of this paper is three-fold.

First, we dramatically simplify the proof of the main result of [Z1]. Our approach here enables us to use Moore's ergodicity theorem ([Z3]) in place of Ratner's theorem which is required in the proof given in [Z1]. (However, we remark that the proof in [Z1] using Ratner's theorem, although it is not explicitly stated there, reduces the general question of the existence of cocompact lattices, without assumption (c) above, to a purely algebraic question about Lie groups, i.e. one with nothing to do a priori with discrete subgroups. This seems to be a potentially useful approach to this significantly more general situation.)

Second, using these arguments, we extend the results of [Z1] to the case in which assumption (c) is replaced by:

(c′) : (i) Every non-trivial homomorphism $\tilde{G} \to J$ has compact centralizer in J;

(ii) There is a non-trivial -split 1-parameter group in $Z_H(JG)$ that is not contained in a normal subgroup of H.

In particular, we show that $\mathrm{SL}(m)\backslash\mathrm{SL}(2m)$ does not admit a cocompact lattice, nor does $J\backslash\mathrm{SL}(2m)$ where $J \subset \mathrm{SL}(m)$ is either $\mathrm{SO}(p,q)$, $p+q=m$, or $\mathrm{Sp}(2r, \ )$ with $2r = m$.

Third, we generalize these results to the case of compact forms. Thus our main result can be stated as follows:

THEOREM 1.1. *Let H be a real algebraic group and $J \subset H$ an algebraic subgroup. Suppose:*

i. H and J are unimodular;

ii. There is a compact manifold M locally modelled on $J\backslash H$;

iii. The centralizer $Z_H(J)$ of J in H contains a group G with the following properties:

(a) G is a semisimple Lie group each of whose simple factors has -rank at least 2;

(b) Every non-trivial homomorphism $\tilde{G} \to J$ has a compact centralizer in J;

(c) *If there is a non-trivial homomorphism $\tilde{G} \to J$, then there is a non-trivial 1-parameter -split subgroup $B \subset Z_H(JG)$ such that neither B nor G is contained in a proper normal subgroup of H.*

Then J is compact.

In particular, we have the following generalization of [Z1, Corollary 1.3].

COROLLARY 1.2. *If $n \geq 5$, $p \leq n/2$, then there is no compact manifold locally modelled on $SL(p, \ )\backslash SL(n, \ )$. The same is true for $J\backslash SL(n, \ )$ where J is any unimodular non-compact subgroup of $SL(p, \ )$ $(p \leq n/2)$.*

2. Cocompact Lattices

In this section we give an alternate proof for the results of [Z1] utilizing the conclusions of superrigidity for cocycles in a simpler way. Further considerations regarding uniqueness of the section defined by superrigidity allow us to generalize the homogeneous space under consideration and to deduce Theorem 1.1 for cocompact lattices.

We will use the following consequence of Moore's theorem ([Z3]).

LEMMA 2.1. *Let H be a Lie group and $\Gamma \subset H$ a discrete subgroup. Let $B \subset H$ be a non-trivial 1-parameter subgroup such that $Ad_H(B)$ is -split (i.e. -diagonalizable) and B is not contained in a proper normal subgroup. Suppose there is a set $Y \subset H/\Gamma$ of finite positive measure (with respect to the natural H-invariant volume on H/Γ) that is B-invariant. Then Γ is a lattice in H and $Y = H/\Gamma$.*

Proof: $\chi_Y \in L^2(H/\Gamma)$ is B-invariant. If H is semisimple, then Moore's theorem implies χ_Y (and hence Y) is H-invariant, from which the lemma follows. In general, the proof of Moore's theorem shows that χ_Y is invariant under all 1-parameter subgroups corresponding to non-0 root spaces of B. However, together with B, these generate a normal subgroup of H. By assumption, this must be equal to H, so again Y is H-invariant, proving the lemma. □

We now give a proof of the main result of [Z1].

THEOREM 2.2. *Assume the hypotheses of Theorem 1.1 with additional assumptions that $M = J\backslash H/\Gamma$ for a cocompact lattice Γ in $J\backslash H$, and that every local homomorphism $G \to J$ is trivial. Then J is compact and Γ is a lattice in H.*

Proof: If Γ is a cocompact lattice in $J\backslash H$, then $H/\Gamma \to J\backslash H/\Gamma$ is a principal J bundle on which the centralizer $Z_H(J)$ acts by principal bundle automorphisms preserving a volume density on the compact manifold $J\backslash H/\Gamma$

([Z1]). Let G be as in Theorem 1.1, with the additional assumption that every local homomorphism $G \to J$ is trivial (i.e. we have the situation in [Z1]). By superrigidity for cocycles, the cocycle $\alpha : J\backslash H/\Gamma \times G \to J$ defined by the action of G on H/Γ is then equivalent to a cocycle into a compact subgroup $K \subset J$. This means there is a measurable G-invariant section s of $K\backslash H/\Gamma \to J\backslash H/\Gamma$. Letting μ be the finite G-invariant measure on $J\backslash H/\Gamma$ defined by the G-invariant volume density, $s_*(\mu)$ will be a finite G-invariant measure on $K\backslash H/\Gamma$. Since K is compact, we can lift this via the bundle map $H/\Gamma \to K\backslash H/\Gamma$ to a finite G-invariant measure ν on H/Γ that projects to a smooth measure on $K\backslash H/\Gamma$. For any $j \in J$, $j_*\nu$ will also be G-invariant. We can write

$$\nu = \int^{\oplus} \nu_t d\mu(t)$$

where $t \in J\backslash H/\Gamma$ and ν_t is supported on a compact set (in fact a K-orbit) in the fiber over t in H/Γ. Thus, if $X \subset J$ is a compact set of positive Haar measure,

$$\int_{j \in X} j_*(\nu) dj$$

will be a finite G-invariant measure that is simply the restriction of the natural smooth measure on H/Γ to a set of finite positive measure. We now apply Lemma 2.1 to deduce the theorem. □

We now discuss some uniqueness results concerning superrigidity that will allow us to deduce Theorem 1.1 for cocompact lattices in general (i.e. without the assumption that all local homomorphisms $G \to J$ are trivial). We shall work first in the general framework of principal bundles, as we will need these results for the proof in the case of general compact forms as well.

Let $P \to M$ be a principal J bundle where M is a compact manifold and J is a connected real algebraic group. Let G be a connected semisimple Lie group such that every simple factor is of -rank at least 2. By passing to the universal cover when necessary, we may assume G is simply connected. Let L be an algebraic group which can be written as $L = BG$ where B is abelian and B centralizes G. We assume L acts on P by principal J-bundle automorphisms preserving a finite measure μ on M. Let $\pi : G \to J$ be a homomorphism (which necessarily factors via the algebraic universal cover of $Ad(g)$), and K a fixed maximal compact subgroup of $Z_J(\pi(G))$. We say (following the terminology of [4]) that a section s of $P \to M$ is π-simple if

$$s(gm) = gs(m)\pi(g)^{-1}c(g,m)$$

where $c(g,m) \in K$. The conclusion of superrigidity for cocycles, see [Z3,4], is that for G as above there always exists a π-simple section for some π

and some K, as long as the G-action is ergodic and the algebraic hull for the G-action is connected. For clarity of exposition we shall assume throughout sections 2 and 3 that the algebraic hull is indeed connected. In section 4 we indicate the necessary modifications for treating the general non-connected case. In the case when the G-action on M is not ergodic, we can decompose the action into ergodic components and apply superrigidity for cocycles to each ergodic component. If there is a $\bar{\pi}$-simple section where $\bar{\pi}$ is conjugate to π, then there is also a π-simple section. Since there are only finitely many conjugacy classes of homomorphisms $G \to J$ we deduce, without the assumption of ergodicity, the following.

LEMMA 2.3. *There is a G-invariant subset $X \subset M$ of positive measure, a homomorphism $G \to J$, and a π-simple section of $P|_X \to X$.*

We now observe:

LEMMA 2.4. *We can take the set $X \subset M$ in Lemma 2.3 to be L-invariant.*

Proof: Let s be the section of $P|_X \to X$ given by Lemma 2.3. For any $b \in B$, the section $s^b(m) = b^{-1}s(bm)$ is also π-simple for the bundle $P|_{b^{-1}X} \to b^{-1}X$. In this way, we can extend s to a π-simple section on $BX = LX$. □

LEMMA 2.5. *Suppose s and t are both π-simple sections for an ergodic G-space with finite invariant measure. Then there is an element $h \in Z_J(\pi(G))$ and functions $k_1, k_2 : M \to K$ such that $s(m) = t(m)k_1(m)hk_2(m)$ a.e.*

Proof: Let $s(m) = t(m)j(m)$ where $j : M \to J$. Write

$$s(gm) = gs(m)\pi^{-1}(g)c(g,m)$$

and

$$t(gm) = gt(m)\pi(g)^{-1}d(g,m)$$

where $c(g,m), d(g,m) \in K \subset Z_J(\pi(G))$ and K is compact. Then it follows that

$$j(gm) = d(g,m)^{-1}\pi(g)j(m)\pi(g)^{-1}c(g,m) \ . \tag{$*$}$$

Since K centralizes $\pi(G)$, G acts by conjugation on $K\backslash J/K$, and if we let $p : J \to K\backslash J/K$ be the natural map, then $(*)$ implies that $p \circ j : M \to K\backslash J/K$ is a G-map. Since J and G are algebraic and K is compact, G acts tamely (i.e. with locally closed orbits) on $K\backslash J/K$ and has algebraic stabilizers. Since the G action on M is ergodic with finite invariant measure, say μ, so is the action with the push forward measure $(p \circ j)_*\mu$. It follows that $p \circ j$ is essentially constant and that the essential image point is a G-fixed point. (This follows from the formulation of the Borel density theorem which asserts for an algebraic action of a semisimple group with no compact factors on a variety that finite invariant measures are supported on fixed points.) Let KhK be such a point. Then $\pi(g)h\pi(g)^{-1} \in KhK$ for all $g \in G$. Thus G acts on KhK and it is easy to see that for this action G preserves the product Haar measure. Thus, by Borel density again, G fixes h, i.e. $h \in Z_J(\pi(G))$. Since $j(m) \in KhK$ for a.e. m, the lemma follows. □

LEMMA 2.6. *Even without the assumption of ergodicity, for any two π-simple sections s, t we have*

$$s(m) = t(m)j(m)$$

where $j(m) \in Z_J(\pi(G))$.

Proof: Apply Lemma 2.5 to each ergodic component. □

LEMMA 2.7. *Consider the situation of Lemmas 2.3, 2.4. Assume $Z_J(\pi(G))$ is compact. Let $\beta : B \times X \to J$ be the cocycle corresponding to the action of B on $P|_X$. Then β is equivalent to a cocycle into $Z_J(\pi(G))$.*

Proof: Let s be a π-simple section for the action of G. Since B commutes with G, s^b is also π-simple for the G-action, where $s^b(x) = b^{-1}s(bx)$. By Lemma 2.6, $s^b(x) = s(x)j(b,m)$, i.e. $s(bx) = bs(x)j(b,m)$. Since $j(b,m) \in Z_J(\pi(G))$, the lemma follows. □

We can now prove Theorem 1.1 for cocompact lattices.

THEOREM 2.8. *Theorem 1.1 holds if $M = J\backslash H/\Gamma$ where Γ is a cocompact lattice for $J\backslash H$.*

Proof: We apply the above considerations to the action of BG on the principal J-bundle $H/\Gamma \to J\backslash H/\Gamma$. (We note that B also centralizes J and hence acts by bundle automorphisms.) Choose π and X as in Lemmas 2.3, 2.4. If π is trivial, we apply the argument of Theorem 2.2. If not, then by Lemma 2.7, $\beta|_{B\times X}$ is equivalent to a cocycle into a compact subgroup of J. Exactly as in the proof of Theorem 2.2 (but with the action of B rather than G), we deduce that there is a set of finite positive measure in H/Γ that is B-invariant. Applying Moore's theorem as in the proof of Theorem 2.2 completes the argument.

3. Proof of Theorem 1.1

We now give the additional arguments necessary to prove Theorem 1.1 in the general case of a compact manifold M locally modelled on $J\backslash H$. We begin with some general facts about locally homogeneous manifolds.

Let M be as in Theorem 1.1. Let $\Gamma = \pi_1(M)$. Then a classical construction of the developing map (see [G], e.g.) shows that there is a homomorphism $\Gamma \to H$ (the holonomy homomorphism) and a Γ-equivariant local diffeomorphism (not necessarily either injective or surjective) $\tilde{M} \to J\backslash H$ (with Γ acting on the right). We view $H \to J\backslash H$ as a principal J-bundle (with left J-action) and pull this back to $\tilde{M}$ to obtain a principal J-bundle $P^* \to \tilde{M}$ on which Γ acts by principal J-bundle automorphisms. Let $P = P^*/\Gamma$, so that $P \to M$ is a principal J-bundle (with J acting on the left). The map $P^* \to H$ is a $J \times \Gamma$-equivariant local diffeomorphism.

If N is a manifold, we let $\mathrm{Vect}(N)$ denote the Lie algebra of smooth vector fields on N. We identify the Lie algebra $\mathfrak{h} \hookrightarrow \mathrm{Vect}(H)$ as the right-invariant vector fields. In particular, Γ leaves $\mathfrak{h}$ pointwise fixed. Since $P^* \to H$ is a local diffeomorphism, the Lie algebra $\mathfrak{h}$ lifts canonically to a Lie algebra embedding $\mathfrak{h} \to \mathrm{Vect}(P^*)$. We denote the image by $\mathfrak{h}_{P^*}$ and corresponding elements by X_{P^*}. The Lie algebra $\mathfrak{j}_{P^*} \subset \mathfrak{h}_{P^*}$ is simply the Lie algebra defined by the left action of J on P^*. In particular, for $X \in \mathfrak{j}$, X_{P^*} is complete (which is not a priori necessarily true for a general $X \in \mathfrak{h}$). Because all $X \in \mathfrak{h}_{P^*}$ are Γ-invariant, there is a natural induced Lie algebra of vector fields on P which we denote by $\mathfrak{h}_P$. Clearly all $X_P \in \mathfrak{h}_P$ are non-vanishing. For $X \in \mathfrak{j}$, X_P is complete, the Lie algebra $\mathfrak{j}_P$ corresponding to the left action of J on P. If $X \in \mathfrak{z}_{\mathfrak{h}}(\mathfrak{j})$, then X_P is J-invariant and thus defines a vector field X_M on M. I.e., $X \mapsto X_M$ is an embedding of Lie algebras $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{j}) \hookrightarrow \mathrm{Vect}(M)$. Since M is compact, we obtain a corresponding action of $\widetilde{Z_H(J)}^\circ$ on M, and hence on P as well.

We can now use exactly the arguments applying superrigidity for cocycles in section 2 to deduce that there is a 1-parameter -split subgroup, say $\alpha(t)$ (which, as in section 2, is either B or a subgroup of G), for which there is a subset $Y \subset P$ of finite positive measure (with respect to the natural smooth measure on P) that is $\alpha(t)$-invariant. To show that J is compact it suffices to show that Y is essentially (i.e. modulo null sets) J-invariant. Namely, J acts properly on P, and if there is a subset of finite positive measure that is invariant for a measure preserving proper action, then J is compact. (Alternatively there is a J-invariant set of finite positive measure on each fiber of P, and hence a J-invariant subset of J of finite positive Haar measure. This implies J is compact.)

For the remainder of the proof, we shall identify $X \in \mathfrak{h}$ with $X_P \in \mathrm{Vect}(P)$. Let $f = \chi_Y$ be the characteristic function of Y. Then for $X \in \mathfrak{h}$, we can consider the distributional derivative $D_X(f)$. To see that Y is J-invariant, it suffices to see that for all $X \in \mathfrak{j}$, $D_X(f) = 0$. We claim in fact that $D_X(f) = 0$ for all $X \in \mathfrak{h}$. It suffices to show this for X in a set that generates $\mathfrak{h}$ as a Lie algebra, and since the root spaces of α with non-trivial root have this generating property (as in section 2 by virtue of iii(c) in the hypothesis of Theorem 1.1), it suffices to show $D_X(f) = 0$ for any such root vector X of α.

LEMMA 3.1. *If X is a root vector for α with non-trivial root, then for a.e. $y \in Y$, the integral curve $u_y(t)$, for X through y is defined for all t.*

Proof: For $y \in Y$, let $r(y) = \sup\{t \in \quad \mid u_y(t)$ is defined on $[0, t)\}$. We claim that for a.e. y, $r(y) = +\infty$. If not, then there is a set $Z \subset Y$ of positive measure and $R > 0$ such that $r(y) < R$ for all $y \in Z$. We can then choose $Z_1 \subset Z$ of positive measure and $r > 0$ such that $r(y) > r$ for $y \in Z_1$. Since $\alpha(t)$ preserves a finite measure on Y, we can apply Poincaré recurrence and deduce that for a.e. $y \in Z_1$, there is a sequence $t_n \to +\infty$ such that $\alpha(t_n)y \in Z_1$. If $\mathrm{Ad}(\alpha(t))X = e^{\lambda t}X$ where $\lambda > 0$, then we have $r(\alpha(t_n)y) > re^{\lambda t_n} \to +\infty$. This contradicts $r(\alpha(t_n)y) < R$. Thus $r(y) = +\infty$ for a.e. $y \in Y$. If $\lambda < 0$, we use the same argument with $t_n \to -\infty$. A similar argument shows $u_y(t)$ is defined for all $t < 0$, completing the proof of the lemma. □

We can summarize the situation as follows.

LEMMA 3.2. *Let $Y^* = \{u_y(t)y \mid y \in Y, t \in \quad, u_y$ the integral curve through y for X as above$\}$. Then $Y^* \subset P$ is a subset of (possibly infinite) positive measure. There is an action of the group $\{\alpha(t)u(s) \mid t, s \in \quad\}$ on this set which is measure preserving. The group $\{\alpha(t)u(s)\}$ is isomorphic to the $ax + b$ group, $\ {}^{+}_{\times}\quad$. There is a set of finite positive measure $Y \subset Y^*$ that is $\{\alpha(t)\}$-invariant.*

We now complete the proof of Theorem 1.1.

To see that $D_X(f) = 0$, we want to show that Y is $\{u(s)\}$-invariant. Consider the unitary representation of $\{\alpha(t)u(s)\}$ on $L^2(Y^*)$. The vector $f \in L^2(Y^*)$ (since Y has finite measure) and is $\{\alpha(t)\}$-invariant. By the unitary representation theory of the $ax + b$ group, this implies that f is invariant under all $\{\alpha(t)u(s)\}$, completing the proof. (See [Z3, Corollary 2.3.7] for precisely this result on the $ax + b$ group.)

Remark 3.3: Rather than call upon the unitary representation theory of the $ax + b$ group, one could proceed by adapting the standard uniform contraction-expansion argument to deduce invariance of f along the foliation given by the non-vanishing vector field X. However, the result on unitary representations we used is a convenient and easily usable way of incorporating this argument when one is in a group theoretic context. An examination of the proof of this representation theoretic result ([Z3, Corollary 2.3.7]) shows that it is in fact just a Fourier transform version of the "more direct" contraction-expansion argument.

4. Non-connected Algebraic Hulls

We have two commuting groups G and$B$ acting on a principal J-bundle $\pi : P \to M$. Let E be the space of ergodic components for the G-action on M. For each $e \in E$, let M_e be the corresponding component, let $R_e \subset J$ be the (real) algebraic hull of the action of G on the principal J-bundle $\pi^{-1}(M_e) \to M_e$. For each $e \in E$, R_e is a reductive group with compact center. (The algebraic hulls are "real algebraic", i.e. -point of -groups.) If R_e is connected for a set of $e \in E$ of positive measure then the argument in section 2 applies. If not then we proceed as follows.

LEMMA 4.1. *Suppose B and G commute and we have an ergodic BG action on a space X and let $X \to E$ be the ergodic decomposition into G-ergodic components. Suppose $\alpha : BG \times J \to J$ is a cocycle. Let $R_e \subset J$ be the algebraic hull of $\alpha|_{G \times X_e}$, for each $e \in E$. Then*
1) Almost all R_e, $e \in E$, are equal. Denote it by R.
2) The algebraic hull of α is contained in $N_J(R)$.

Proof: Let j, r_e be the Lie algebras of J, R_e, respectively. The map $\varphi : E \to Gr(j)/Ad(J)$ given by $\varphi(e) = r_e$ is measurable. It is B-invariant since B centralizes G. (I.e. the algebraic hulls, R_e and R_{be} are the "same", namely are the same conjugacy class.) Since B is ergodic on E (since GB is ergodic on X), and $Ad(J)$ acts tamely on $Gr(j)$, φ is constant a.e. Thus, we can, by changing α only up to coboundary, assume $\alpha|_{G \times X_e}$ takes values in a group R_e so that $R_e^0 = R^0$ are all identical.

Now consider the set $\mathcal{S}$ of algebraic subgroups $S \subset J$, s.t. $S^0 = R^0$. Then any element of J that conjugates two such subgroups is in $N(R^0)$. Further, $N(R^0)$ acts tamely on $\mathcal{S}$ (since this action can be embedded in the action of $N(R^0)$ on finite subsets of J/R^0 which is algebraic.) It follows in a manner similar to the previous paragraph, that all R^e are conjugate, and hence by a cohomology change, we can assume all R_e are equal, say $R_e = R$. This proves (1).

To show (2): Fix $b \in B$ and $e \in E$. For $x \in X_e$, let $\varphi(x) = \alpha(b, x)$. Then since b centralizes G, we have from the cocycle identity:

$$\varphi(gx)^{-1}\alpha(g, bx)\varphi(x) = \alpha(g, x) \ . \tag{$*$}$$

This implies that the map $\overline{\varphi} : X_e \to R\backslash J/R$ given by projection of $\varphi : X \to J$ satisfies $\overline{\varphi}(gx) = \overline{\varphi}(x)$ for $x \in X_e$. Thus $\overline{\varphi}$ is constant a.e. Hence, we can write

(i) $\varphi(x) = \theta_1(x) j_0 \theta_2(x)$ where $\theta_i : X_e \to R$ and $j_0 \in J$.
We also then have
(ii) $\varphi(gx) = \theta_1(gx) j_0 \theta_2(gx)$
and substituting (i) and (ii) into $(*)$ and simplifying, we have

$$\theta_1(gx)^{-1}\alpha(g,bx)\theta_1(x) = j_0\theta_2(gx)\alpha(g,x)\theta_2(x)^{-1}j_0^{-1}\ .$$

In particular $j_0\beta(g,x)j_0^{-1} \in R$, where β is the cocycle (equivalent to α) given by $\beta(g,x) = \theta_2(gx)\alpha(t,x)\theta_2(x)^{-1}$. Since the algebraic hull of β equals R and β takes values in R, we deduce in particular that the Zariski closure of the group generated by $\{\beta(g,x)\}$ is in fact R. Thus $j_0 \in N_J(R)$ and from equation (1), we deduce $\alpha(b,x) \in N_J(R)$. This holds for all $(b,x) \in B \times X$, and since $\alpha(G \times X) \subset R$, assertion (2) is proved.

Now return to our geometric situation in which G is simple of higher -rank. Fix an ergodic component $X \subset M$ for the $B \cdot G$ action. Let R be as in the lemma, i.e. the algebraic hull of the G action on a.e. G-ergodic component of X. Then, by the lemma, the algebraic hull for the action of $B \cdot G$ on $\pi^{-1}(X) \to X$ is contained in $N_J(R)$. By superrigidity, we know that R^0 is isomorphic to $\overline{\pi}(G)$ (in which $\pi(G)$ is of finite index) times a compact subgroups for some homomorphism π.

Notice that:

Lemma 4.2. *If $Z_j(\pi(G))$ is compact, then $N_J(R)/R^0$ is a compact group.*

We let $\alpha : B \cdot G \times X \to J$ be the cocycle corresponding to the action on $\pi^{-1}(X)$, with $\mathrm{Im}(\alpha) \subset N_J(R)$. Let $\overline{X} = X \times_\alpha N_J(R)/R^0$. Thus, $B \cdot G$ acts preserving the product measure. (However the action of BG on $\overline{X}$ is not necessarily ergodic.) Let $\overline{\alpha}$ be the lift of α to $\overline{X}$. Thus $\overline{\alpha}$ is the cocycle of the J-bundle

$$\begin{array}{c}\overline{P}\\ \downarrow\\ \overline{X}\end{array}$$

where $\overline{P}$ is the pull-back to $\overline{X}$ of $\pi^{-1}(X) \to X$. Standard arguments (see [Z3, Proposition 9.2.6]) show that for each ergodic component of the action of G on $\overline{X}$, the algebraic hull is R^0. We can thus obtain a π-simple section on each ergodic component view superrigidity. Arguing as in earlier sections, this implies that there is either a B or G finite invariant measure on $\overline{P}$ that projects to a set of positive measure on X. Thus, the same is true for $P \to M$.

Let ν be this finite measure on P. This can be described as $\int_{m\in M_0}^{\oplus} \nu_m$ where $M_0 \subset M$ is of positive measure and is B or G-invariant. Fixing a positive definite inner product on j (=Lie algebra of J) defines a $B \cdot G$

invariant Riemannian metric along the fibers of P. Fix $\varepsilon, r > 0$. Let $A_m^{\varepsilon,r} \subset \pi^{-1}(m)$ be the union of all balls of radius ε in $\pi^{-1}(m)$ whose ν_m-measure is at least r. Then for suitable ε, r, $\bigcup_m A_m^{\varepsilon,r}$ will be a B or G-invariant set of positive finite Lebesgue measure. The argument now proceeds as before.

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