POSITIVITY AND REPRESENTATIONS OF SURFACE GROUPS

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Abstract. In [20, 22] Guichard and Wienhard introduced the notion of Θ-positivity, a generalization of Lusztig's total positivity to real Lie groups that are not necessarily split.

Based on this notion, we introduce in this paper Θ-*positive representations* of surface groups. We prove that Θ-positive representations are Θ-Anosov. This implies that Θ-positive representations are discrete and faithful and that the set of Θ-positive representations is open in the representation variety. We show that the set of Θ-positive representations is closed within the set of representations that do not virtually factor through a parabolic subgroup. From this we deduce that for any simple Lie group G admitting a Θ-positive structure there exist components consisting of Θ-positive representations. More precisely we prove that the components parametrized using Higgs bundles methods in [6] consist of Θ-positive representations.

1. Introduction

An important feature of Teichmüller space, seen as a connected component of the space of representations of the fundamental group of a closed connected orientable surface S of genus at least 2 in $\mathsf{PSL}_2(\mathbb{R})$, is that it consists entirely of representations which are discrete and faithful. These representations are moreover quasi-isometries from $\pi_1(S)$

to $\mathsf{PSL}_2(\mathbb{R})$. This situation does not extend to the case of *any* semisimple group, notably for simply connected complex ones, where the representation variety is irreducible as an algebraic variety [32].

However, this phenomenon was shown to happen for *some* groups of higher rank. Two families of (unions of) connected components of the variety of representations of the fundamental group of a closed connected orientable surface S of genus at least 2, which consist entirely of discrete and faithful representations, have been singled out:

- Hitchin components, defined when G is a real split group [25, 15, 12, 27, 19],
- spaces of maximal representations, which are defined when G is Hermitian [14, 10, 8].

When G is $\mathsf{PSL}_2(\mathbb{R})$, the Hitchin component and the space of maximal representations both agree with Teichmüller space.

Both studies are closely related to the theory of Anosov representations as introduced in [27, 21]. Being Anosov is a notion defined for any reductive Lie group with respect to a choice of parabolic subgroup. Every Anosov representations is in particular faithful, discrete and a quasi-isometric embedding [28, 21, 11].

Representations in the Hitchin components as well as maximal representations can be characterized in terms of equivariant curves from the boundary at infinity of $\pi_1(S)$ into an appropriate flag variety, which preserves some positivity [12, 27, 19, 10]. For Hitchin components this positivity is based on Lustztig's total positivity [30], for maximal representations it is based on the maximality of the Maslov index and related to Lie semigroups in G.

In [22, 20], Guichard and Wienhard introduced the notion of Θ-*positivity*. This notion extends Lusztig's total positivity to generalized flag manifolds associated to a parabolic defined by a set Θ of simple roots.

They classified all possible simple Lie groups that admit a Θ-positive structure. These include real split Lie groups, for which Θ-positivity is Lusztig's total positivity, Hermitian Lie groups of tube type, where Θ-positivity is related to Lie-semigroups, but they also include two other families of Lie groups, namely the family of classical groups $\mathsf{SO}(p,q)$ —with $p \neq q$— and an exceptional family consisting of the real rank 4 form of F_4, E_6, E_7, and E_8 respectively. They conjectured that Θ-positivity provides the right underlying algebraic structure for the existence of components made solely of

discrete faithful representations. The goal of the present article is to fulfil that program.

As a corollary of our main result, we prove

Theorem A. *If* G *admits a* Θ*-positive structure, then there exist connected components of the representation variety of the surface group in* G *that consists solely of discrete and faithful representations.*

A Θ-positive structure on G implies in particular the existence of a positive semigroup in the unipotent radical of the parabolic group P_Θ, which then leads to the notion of *positive triples* and *positive quadruples* in the flag variety $\mathbf{F}_\Theta \simeq \mathsf{G}/\mathsf{P}_\Theta$. In the basic example of $\mathsf{G} = \mathsf{PSL}_2(\mathbb{R})$ and $\mathbf{F}_\Theta = \mathbf{P}^1(\mathbb{R})$, a triple is positive if it consists of pairwise distinct points and a quadruple is positive if it is cyclically ordered.

Let us give a geometric picture of the positivity in the flag variety $\mathbf{F}_\Theta$. For this let a and b be two points in $\mathbf{F}_\Theta$ which are transverse to each other. Then Θ-positivity provides the existence of preferred connected components of the set of all points in $\mathbf{F}_\Theta$ that are transverse to both a and b. We call these preferred components the *diamonds* (with extremities a and b). They are several, at least two, disjoint diamonds with given extremities. The semigroup property alluded to before translates into a nesting property of diamonds: if c is a point in a diamond $V(a,b)$ with extremities a and b, then there is exactly one diamond $V(c,b)$ (with extremities c and b) included in $V(a,b)$. These nesting properties of diamonds play an important role in our arguments.

If a and b are transverse, and c belongs to a diamond with extremities a and b, we say the triple (a,b,c) is *positive*. Similarly if a and b are transverse, and c and d belongs to the two opposite diamonds with extremities a and b, then the quadruple (a,c,b,d) is called *positive*. We show in section 3 that being positive is invariant under all permutations for a triple, and invariant under the dihedral group for a quadruple.

We define a map ξ from of a cyclically ordered set A to $\mathbf{F}_\Theta$ to be *positive* if ξ maps triples of pairwise distinct points to positive triples and cyclically ordered quadruples to positive quadruples.

This allows us to define the notion of a Θ-*positive representation*: A representation $\rho\colon \pi_1(S) \to \mathsf{G}$ is Θ-*positive* if there exists a non-empty subset A of $\partial_\infty \pi_1(S)$, invariant by $\pi_1(S)$, and a ρ-equivariant positive boundary map from A to $\mathbf{F}_\Theta$.

We prove

Theorem B. *Let* G *be a simple Lie group that admits a* Θ*-positive structure, and let* $\rho\colon \pi_1(S) \to \mathsf{G}$ *be a* Θ*-positive representation. Then* ρ *is a* Θ*-Anosov representation.*

As a direct consequence we obtain that a Θ-positive representation is faithful with discrete image, its orbit map into the symmetric space is a quasi-isometric embedding and the boundary map extends uniquely to a Hölder map [28, 21, 11, 7].

Theorem B provides a general proof of the Anosov property for all Hitchin representations and all maximal representations. This is especially relevant for the case of the Hitchin component of $\mathsf{SO}(p,p)$ and of F_4, E_6, E_7, and E_8 and the case of maximal representations into the exceptional Hermitian Lie group of tube type, which cannot be tightly embedded into $\mathsf{Sp}_{2n}(\mathbb{R})$ [9, 23, 24]. The Anosov property was established for the other Hitchin components in [27], and for all maximal representations which tightly embed into $\mathsf{Sp}_{2n}(\mathbb{R})$ in [8]. Fock and Goncharov established a key property of Anosov representations for every Hitchin representation, namely the existence of a continuous, transverse (and positive) boundary map [12, Theorem 7.2]; from this, the Anosov property can be established using for example [21, Theorem 4.11].

Using the openness of the set of Θ-Anosov representation, a further consequence of Theorem B is the following

Corollary C. *The set of* Θ*-positive representations* $\mathrm{Hom}_{\Theta-\mathrm{pos}}(\pi_1(S),\mathsf{G})$ *is an open subset of the set of all homomorphism* $\mathrm{Hom}(\pi_1(S),\mathsf{G})$.

We now consider the set $\mathrm{Hom}^*(\pi_1(S),\mathsf{G})$ of homomorphisms ρ of $\pi_1(S)$ in G that do not factor through a parabolic subgroup of G, not even when restricted to a finite index subgroup of $\pi_1(S)$. We show in Proposition 6.1 that $\mathrm{Hom}_{\Theta-\mathrm{pos}}(\Gamma,\mathsf{G})$ is a subset of $\mathrm{Hom}^*(\pi_1(S),\mathsf{G})$. We show

Theorem D. *The set of* Θ*-positive representations* $\mathrm{Hom}_{\Theta-\mathrm{pos}}(\pi_1(S),\mathsf{G})$ *is a union of connected components of* $\mathrm{Hom}^*(\pi_1(S),\mathsf{G})$.

Note that special Θ-positive representations arise from positive embeddings of $\mathsf{SL}_2(\mathbb{R})$ into G. These positive embeddings of $\mathsf{SL}_2(\mathbb{R})$ can be characterized and classified explicitly in terms of the embedding of the nilpotent generator of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. They have the property that the embedding induces a positive map from $\mathbf{P}^1(\mathbb{R})$ into $\mathbf{F}_\Theta$. We call the image of such a map a *positive circle*. These circles play an important role in some of our arguments. Precomposing a positive embedding $\mathsf{SL}_2(\mathbb{R}) \to \mathsf{G}$ with a discrete embedding of $\pi_1(S)$

into $SL_2(\mathbb{R})$, we obtain a Θ-positive representation. We call such representations Θ-*positive Fuchsian representations*.

Recently, Bradlow, Collier, García-Prada, Gothen, and Oliveira [6] developed the theory of magical $\mathfrak{sl}_2$-triples, which is very closely related to the theory of Θ-positivity. In fact a real simple Lie group is associated to a magical $\mathfrak{sl}_2$-triple if and only if it admits a Θ-positive structure. Using methods from the theory of Higgs bundles, they then show that for any simple Lie group G admitting a Θ-positive structure there exists a non-empty open and closed subset (*i.e.* a union of connected components) $\mathcal{P}_e(S, G)$ in $\mathrm{Rep}^+(\pi_1(S) = \mathrm{Hom}^+(\pi_1(S), G)/G$, where $\mathrm{Hom}^+(\pi_1(S), G)$ denotes the set of reductive representations, such that $\mathcal{P}_e(S, G)$ contains Θ-positive representations. They furthermore show that these $\mathcal{P}_e(S, G)$ are contained in $\mathrm{Rep}^*(\pi_1(S), G) = \mathrm{Hom}^*(\pi_1(S), G)/G$ [6, Theorem A, Theorem E]. Using this, we deduce from Theorem D

Theorem E. *Let G be a simple Lie group that admits a Θ-positive structure. Then the non-empty open and closed subsets $\mathcal{P}_e(S, G)$ in $\mathrm{Rep}^+(\pi_1(S), G)$ consist entirely of Θ-positive representations.*

In particular, the union of connected components $\mathcal{P}_e(S, G)$ of $\mathrm{Rep}^+(\pi_1(S), G))$ consists entirely of Θ-Anosov representations, and hence of discrete and faithful representations, and we deduce Theorem A as a corollary.

Observe that the same phenomenon already occurred for Hitchin components: Hitchin introduced the Hitchin component [25] for real split groups using Higgs bundle methods. From his interpretation one gets that all representations in Hitchin components are in $\mathrm{Hom}^*(\pi_1(S), G)$ [27, Lemma 10.1]. This last observation is then used in the proof by Labourie [27] as well as Fock and Goncharov [12] that these components consist entirely of discrete and faithful representations.

Note however that in our case Theorem E does not readily imply that the set $\mathcal{P}_e(S, G)$ agrees with the set of Θ-positive representations, even though we expect this to be true [36, 6]. Theorem E also does not imply that the set of Θ-positive representations is open and closed in $\mathrm{Hom}^+(\pi_1(S), G)/G$, a property we conjectured in [36, Conjecture 19].

In the case when G is locally isomorphic to $SO(p, q)$, $p \leqslant q$, Beyrer and Pozzetti [1] recently proved the *closedness* of the set of Θ-positive Anosov representations in $\mathrm{Hom}(\pi_1(S), G)$, thus by Theorem B also the closedness of the set of Θ-positive representations. They derive this as a consequence of a family of collar lemmas and fine properties of the boundary map they establish. A similar approach should also

work for the exceptional family of Lie groups admitting a Θ-positive structure.

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Outline of the paper: In Section 2, we recall the necessary algebraic material from [22, 20] and introduce the main definitions: diamonds, positive configurations, positive circles and positive maps. In Section 3, we prove three propositions concerning combinatorial properties of configurations, proper inclusion of diamonds and extension of positive maps. In Section 4, we introduce the diamond metric on diamonds and establish its properties. With these preparations we prove Theorem B and Corollary C in Section 5, Theorem D in Section 6, and Theorem E in Section 7.

<div align="center">CONTENTS</div>

2. Definitions

2.1. **Lie algebra notations.** Let G be a simple group.

The *roots* of G are the nonzero weights of the adjoint action of a Cartan subspace $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ of G. They form a root system $\Sigma \subset \mathfrak{a}^*$ (nonreduced in some cases) and the choice of a vector space ordering on $\mathfrak{a}^*$ gives rise to the set Σ^+ of positive roots, and the set Δ of simple roots. The α-weight space will be denoted by $\mathfrak{g}_\alpha \subset \mathfrak{g}$.

The parabolic subgroups of G are the subgroups conjugated to one of the standard parabolic subgroups P_Θ (for Θ varying in the subsets of Δ); namely P_Θ is the normalizer in G of the Lie algebra $\mathfrak{u}_\Theta := \bigoplus_{\alpha \in \Sigma^+ \setminus \text{span}(\Delta \setminus \Theta)} \mathfrak{g}_\alpha$. A parabolic subgroup is its own normalizer so that the space $\mathbf{F}_\Theta$ of parabolic subgroups conjugated to P_Θ is isomorphic to G/P_Θ. The unipotent radical of P_Θ is the subgroup $U_\Theta = \exp(\mathfrak{u}_\Theta)$.

Two parabolic subgroups P and P' are called *transverse* or *opposite* if their intersection $P \cap P'$ is a reductive subgroup (*i.e.* its unipotent radical is trivial); this is equivalent to having $\text{UniRad}(P) \cap P' = \{1\}$. In that case, there exists $\Theta \subset \Delta$ such that the pair (P, P') is conjugated to $(P_\Theta, P_\Theta^{\text{opp}})$ where P_Θ^{opp} is the normalizer of $\bigoplus_{\alpha \in \Sigma^+ \setminus \text{span}(\Delta \setminus \Theta)} \mathfrak{g}_{-\alpha}$. The intersection $L_\Theta := P_\Theta \cap P_\Theta^{\text{opp}}$ is a Levi factor of P_Θ (and of P_Θ^{opp}).

We will always work with a parabolic subgroup $P \simeq P_\Theta$ such that P_Θ is conjugated to its opposite; in this situation it makes sense to look at transverse elements in $\mathbf{F}_\Theta \simeq G/P_\Theta$. In particular we will use the following notation, for x in $\mathbf{F}_\Theta$,

$$P_x := \text{Stab}(x),$$
$$U_x := \text{UniRad}(P_x)$$
$$\Omega_x := \{y \in \mathbf{F}_\Theta \mid y \text{ is transverse to } x\},$$
$$S_x := \mathbf{F}_\Theta \setminus \Omega_x .$$

We will sometimes use that, if a and b are transverse points, then $L_{a,b} := P_a \cap P_b$ is a Levi factor of P_a and P_b. Recall that Ω_x is an open transitive orbit of U_x and that S_x is a proper algebraic subvariety of $\mathbf{F}_\Theta$.

Given a point a in $\mathbf{F}_\Theta$, a *unipotent pinning*, or *U-pinning* at a, is an identification s of U_Θ with U_a that exponentiates an isomorphism from $\mathfrak{u}_\Theta$ to $\mathfrak{u}_a$ induced by restriction of an automorphism of the Lie

algebra $\mathfrak{g}$. Observe that there are finitely many U-pinnings up to the action of L_x and that the associated Weyl group acts on those U-pinnings.

2.2. **Cones and semigroup.**

Definition 2.1. [20, Theorem 9.2] A *positive structure* with respect to $\mathbf{F}_\Theta$ (or a Θ-*positive structure*) is a semigroup N of $U = U_\Theta$ such that, denoting x and y the points of $\mathbf{F}_\Theta$ corresponding to P_Θ and P_Θ^{opp} respectively, $N \cdot y$ is a connected component of $\Omega_x \cap \Omega_y$.

In this case, N is invariant by conjugation by the connected component $L^\circ := L_\Theta^\circ$ of L_Θ and is a sharp semigroup: for any $h, k \in \overline{N}$, if $hk = 1$, then $h = k = 1$ (*i.e.* the only invertible element in $\overline{N}$ is the identity element).

We shall see that given a and b transverse in $\mathbf{F}_\Theta$ and an identification of U with U_a which sends N to a subgroup N_a of U_a, then $N_a \cdot b$ is a connected component of $\Omega_a \cap \Omega_b$.

In [20, 22], Guichard and Wienhard have classified all Θ-positive structures. In particular, they have shown that this new notion of positivity encompasses Lusztig's total positivity for real split groups, and positivity in the context of Shilov boundaries of Hermitian symmetric subspaces. Moreover they have shown that new classes of groups and parabolics appear; also, up to natural symmetries, the semigroup N in the definition is unique.

We first present some conclusion of their construction that we are going to use in this paper, then concentrate on the notions of *diamonds* and *positive configurations* that play a crucial role in this paper.

2.2.1. *The parameterization of the positive semigroup.* The papers [22, Theorem 4.5] and [20, Theorem 1.3] give a precise description of the possible parameterizations of the semigroup N. We recall here the material necessary for our purpose.

Fact 2.2. *There exist $N \geqslant 1$, and* C *a* L°-*invariant cone in* $(\mathfrak{u}_\Theta)^N$ *such that the map*

$$(\mathfrak{u}_\Theta)^N \longrightarrow U$$
$$(x_1, \ldots, x_N) \longmapsto \exp(x_1) \cdots \exp(x_N)$$

induces by restriction a L°-*equivariant diffeomorphism*

$$\Psi \colon C \longrightarrow N .$$

Furthermore the stabilizer in L° *of any point h in* C *is a compact subgroup of* L°.

The closure $\overline{\mathsf{C}}$ is also L°-invariant and the definition 2.1 implies that the cone $\overline{\mathsf{C}}$ is salient, *i.e.* the intersection of $\overline{\mathsf{C}}$ and $-\overline{\mathsf{C}}$ is reduced to $\{0\}$.

Remark 2.3. A little more precisely, we have that N is the length of the longest element in a finite Coxeter group associated with Θ and, for every $i = 1, \ldots, N$, there is a L°-invariant cone C_i contained in an L-irreducible factor of $\mathfrak{u}_\Theta$ (and C_i is open in that factor) and such that $\mathsf{C} = C_1 \times C_2 \times \cdots \times C_N$.

2.3. Diamonds.

Let a and b be two transverse points in $\mathbf{F}_\Theta$.

Definition 2.4. A *diamond* with *extremities a and b*, associated to a U-pinning s_a at a, is the subset

$$s_a(\mathsf{N}) \cdot b \ .$$

The terminology *diamond* was coined in [29] in the context of $\mathsf{G} = \mathsf{SO}(2, n)$. To give an idea, in that context $\mathbf{F}_\Theta$ is covered by charts which are identified with the Minkowski space $\mathbb{R}^{1,n-1}$. Then, in a suitable chart, a diamond is the intersection of the future time cone F^+ of a, with the past time cone F^- of b.

It follows that in that case there are precisely two diamonds with given extremities. However for the split case $\mathsf{SL}(3, \mathbb{R})$ Tengren Zhang has noticed that they are 4 diamonds with given extremities.

By definition, we observe that a diamond makes sense for a real closed field and is a semi-algebraic set.

We list some first properties of diamonds that are proved in [20, Section 10].

Proposition 2.5.
    (1) *A diamond with extremities a and b is a connected component of $\Omega_a \cap \Omega_b$.*
    (2) *Given a diamond $s_a(\mathsf{N}) \cdot b$, there exists a U-pinning s_b so that*

$$s_a(\mathsf{N}) \cdot b = s_b(\mathsf{N}) \cdot a \ .$$

    (3) *Given any diamond $V(a, b)$ then a belongs to the closure of $V(a, b)$*
    (4) *Let O be any neighbourhood of a and $V(a, b)$ a diamond. Then O contains a neighbourhood U of a so that $U \cap V(a, b)$ is connected.*

We also remark that

Proposition 2.6. *Given a diamond V there is a unique diamond V^* satisfying that if $V = s_b(\mathsf{N}) \cdot a$ then $V^* = s_b(\mathsf{N}^{-1}) \cdot a$. The diamond V^* is the* opposite diamond *to V. A diamond and its opposite are disjoint. Finally any point in V is transverse from any point in V^*.*

Proof. We just have to remark that the definition of the opposite diamond does not depend on the choice if extremities. More precisely if

$$V = s_b(\mathsf{N}) \cdot a = s_a(\mathsf{N}) \cdot b \,,$$

then

$$s_b(\mathsf{N}^{-1}) \cdot a = s_a(\mathsf{N}^{-1}) \cdot b \,.$$

by [20, Section 10] .   □

As a consequence of the proposition, if c is an element in a diamond with extremities a and b, will denote by

(1) $V_c(a,b)$ the unique diamond containing c with extremities a and b;
(2) $V^c(a,b) := V_c^*(a,b)$ the diamond opposite to the diamond containing c.

Proof of Proposition 2.5. The first item is a consequence of [20, Theorems 1.3 and 1.4]. The second item is a consequence of [20, Proposition 10.1].

The third item comes from [22, Remark 4.9] and from [20, Section 10.6]. In particular, if $x \in V$, then $x = s_b(n) \cdot a$ with $n \in \mathsf{N}$, while if $y \in V^*$, then $y = s_b(m^{-1}) \cdot a$ with $m \in \mathsf{N}$. Thus

$$x = s_b(nm) \cdot y \,.$$

Since N is a semigroup, this means that x belongs to a diamond with extremities y and b. By the first point x is transverse to y.

The fourth item follows from the fact that the identity belongs to the closure of N, while the fifth is a consequence of the parametrisation.   □

As an immediate consequence of the semigroup property we obtain the following result that we shall use freely:

Lemma 2.7 (NESTING PROPERTY). *Let c be a point in a diamond with extremities (a,b).*
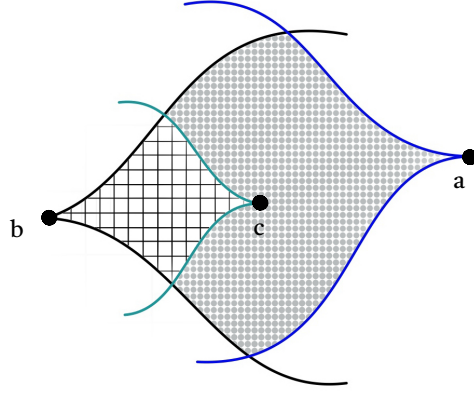
(1) *Then there exist a unique diamond $V(a,c)$ with extremities (a,c) respectively so that*

$$V(a,c) \subset V_c(a,b) \,.$$

(2) *Moreover, if $V(c,b)$ is the unique diamond with extremities (c,b) included in $V(a,b)$ then*

$$V(a,c) \cap V(c,b) \;=\; \emptyset \,.$$

(3) *Finally a belongs to the opposite diamond $V^*(c,b)$.*

FIGURE 1. The nesting of $V(c, b)$ in $V(a, b)$

Proof. Let us first construct diamonds $V^0(c, b)$ and $V^0(a, c)$ included in $V_c(a, b)$. Let us write $V_c(a, b) = \mathsf{N}_b \cdot a = \mathsf{N}_a \cdot b$ and consider the diamonds

$$V(c, b) = \mathsf{N}_b \cdot c \ , \ V(a, c) = \mathsf{N}_a \cdot c \ .$$

By construction $c = n_b \cdot a = n_a \cdot b$ with $n_b \in \mathsf{N}_b$ and $n_a \in \mathsf{N}_a$. By the semigroup property

$$\mathsf{N}_b \cdot n_b \subset \mathsf{N}_b \ , \ \mathsf{N}_a \cdot n_a \subset \mathsf{N}_a \ ,$$

which leads to the inclusions

$$V^0(a, c) \subset V_c(a, b) \ , \ V^0(c, b) \subset V_c(a, b) \ .$$

We now prove these specific diamonds are disjoint. By the construction and the inclusion above both $V(a, c)$ and $V(b, c)$ are connected components of $V_c(a, b) \setminus S_c$. It follows that either they are equal or disjoint. By the sharpness property of N, the identity element does not belong to the closure of $\mathsf{N} \cdot n_a$. Let thus O be an open set in U_a containing the identity and with trivial intersection with $\mathsf{N} \cdot n_a$. Then $O \cdot b$ is a neighborhood of b that does not intersect $\mathsf{N}_a \cdot c = \mathsf{N}_a n_a \cdot b$. Thus b does not belong to the closure of $V(a, c)$. Thus from the last item of Proposition 2.5, $V(a, c)$ is different from $V(c, b)$ and by the above discussion they are disjoint:

$$V^0(a, c) \cap V^0(b, c) = \emptyset \ .$$

Let us finally prove uniqueness. Let O be neighbourhood of a disjoint from $S_c \cup S_b$. Let U be the associated neighborhood to O by item (5) of Proposition 2.5. Let $V(a, c)$ be any diamond with extremities (a, c)

included in $V(a,b)$, then $V(a,c) \cap U$ is a union of connected components of $U \setminus S_a$. Since furthemore

$$V(a,c) \cap U \subset V(a,b) \cap U,$$

and the latter set is connected by item (5) of Proposition 2.5, hence a connected component of $U \setminus S_a$, it follows that

$$V(a,c) \cap U = V(a,b) \cap U.$$

Thus

$$V(a,c) \cap U = V^0(a,c) \cap U = V(a,b) \cap U \neq \emptyset.$$

Since two diamonds with the same extremities are either disjoint of equal it follows that $V(a,c) = V^0(a,c)$. This concludes the proof of the first two items.

For the last one, observe that $a = n_b^{-1}c \in N_b^{-1}c = V^*(c,b)$. $\square$

2.4. **Positive configurations.** The following definition plays a central role in this article:

Definition 2.8 (POSITIVE CONFIGURATION). Let us equip $\{1, \ldots, p\}$ with the usual cyclic order, with $p \geqslant 3$. We say the configuration $(a_1, \ldots, a_p)$ is *positive*, if there exists diamonds $V_{i,j}$ with extremities (a_i, a_j) for all $i \neq j$ so that

   (1) $V_{i,j} = V_{j,i}^*$,
   (2) a_i belongs to $V_{j,k}$, if (j,i,k) is oriented,
   (3) we have $V_{i,j} \subset V_{k,m}$, if (k,i,j,m) is oriented.

Proposition 3.1 will give easier criteria to understand positive triples and quadruples and show that the definition is equivalent to the definition given in the introduction.

Observe that the choice of $V_{i,j}$ is forced once the cyclic order is chosen.

By construction, every subconfiguration of a positive configuration is positive. On the real projective line, a configuration is positive exactly if it is cyclically oriented.

Moreover

Proposition 2.9. *Positivity of a configuration is invariant under cyclic and order reversing permutations. In particular*

   (1) *to be positive for a triple is invariant under all permutations,*
   (2) *to be positive for a quadruple is invariant under the dihedral group.*

Proof. The definition is invariant under cyclic transformation. If σ_0 is the reverse ordering, we choose the new diamonds $V^\circ_{\sigma_0(i),\sigma_0(j)} = V^*_{i,j}$. $\square$
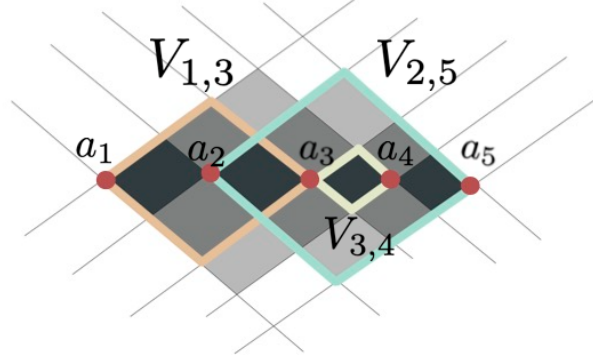
FIGURE 2. A positive 5-configuration and some diamonds

Finally, since the Definition 2.8 involves each time to check at most 4 indices, to check that a configuration is positive, it is enough to check that subquadruples are positive. In the next section we will give more properties of positive configurations.

2.5. **Positive circles and** $PSL_2(\mathbb{R})$. Let H be a subgroup in G locally isomorphic to $PSL_2(\mathbb{R})$. An H-*circle* in $\mathbf{F}_\Theta$ is a closed H-orbit, it can be parameterized by a *circle map* which is a $PSL_2(\mathbb{R})$-equivariant from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}_\Theta$. The group H is *proximal* if it contains a proximal element in $\mathbf{F}_\Theta$.

Proposition 2.10 (POSITIVE CIRCLE). *Given a positive structure, there exists $\mathcal{H}$, a G-orbit of pairs (H, C) so that H is a subgroup of G locally isomorphic to $PSL_2(\mathbb{R})$, C is a an H-circle, satisfying the following properties*

  (1) H *has a compact centralizer,*
  (2) H *is proximal and C is the set of attractive fixed points of hyperbolic elements of H,*
  (3) *given transverse points a and b in $\mathbf{F}_\Theta$, there exists (H, C) in $\mathcal{H}$ passing through a and b, such that*
     • *if c is a point in C different from a and b, then*

$$V_c(a, b) \cap C \text{ and } V_c^*(a, b) \cap C,$$

are the two connected components of $C \setminus \{a, b\}$.
- *If furthermore d belongs to the connected component of $C \setminus \{c, b\}$ not containing a, then*

$$V_d(b, c) \subset V_d(a, b) .$$

Proof. It is enough to construct one such group isomorphic to $\mathsf{PSL}_2(\mathbb{R})$ and to use the G-action that is transitive on the pairs of transverse points in $\mathbf{F}_\Theta$. For this, one just picks the Lie subgroup associated with a $\mathfrak{sl}_2$-triple given by the Jacobson–Morozov theorem applied to the nilpotent element $n = \log(h)$ for h a fixed element in the semigroup N.

For the third item, let $\mathsf{N}_2 = s_b(\exp(\mathbb{R}_{>0}n))$, so that N_2 is included in N. Then $C \setminus \{a, b\}$ has two connected components: which are respectively $\mathsf{N}_2 \cdot a$ and $\mathsf{N}_2^{-1} \cdot a$ which are included in diamonds opposite to each other.

Moreover, for the last statement, let us write $c = n \cdot a$ with n in N_2. Observe that

$$V_{n \cdot d}(c, b) = V_d(c, b) .$$

Then

$$V_d(c, b) = V_{n \cdot d}(c, b) = n V_d(a, b) = n \mathsf{N}_2 \cdot a \subset \mathsf{N}_2 a = V_d(a, b) ,$$

where we used in the inclusion the semigroup property. This concludes the proof.

$\square$

Remark 2.11. *A more detailed construction of this positive $\mathsf{PSL}_2(\mathbb{R})$ can be found in [20, Section 11].*

We fix once and for all such a class $\mathcal{H}$ and call for (H, C) in $\mathcal{H}$, H a *positive* $\mathsf{PSL}_2(\mathbb{R})$ and C a *positive circle*. As an important example of positive configuration, we have

Proposition 2.12. *Any cyclically ordered configuration of points in a positive circle is positive.*

Proof. It is enough to prove the results for triples and quadruples.

Let first (a_0, a_1, a_2) be a triple on a circle C. By the last property of circles, a_{i+1} belongs to a diamond with extremities (a_i, a_{i+2}). Let us define (where indices are taken modulo 3)

$$V_{i,i+2} := V_{a_{i+1}}(a_i, a_{i+2}) , V_{i,i+1} := V_{i+1,i}^* .$$

Then property (2) of Definition 2.8 is obviously satisfied and the triple is positive. Let now consider (a_0, a_1, a_2, a_3) a quadruple on a

circle C, such that a_{i+1} and a_{i+3} belongs to different components of $C \setminus \{a_i, a_{i+2}\}$. Observe that from the last property of circles we have

$$V_{a_{i+2}}(a_i, a_{i+3}) = V_{a_{i+1}}(a_i, a_{i+3}) .$$

We now define

$$W_{i,i+2} := V_{a_{i+1}}(a_i, a_{i+2}) ,$$
$$W_{i,i+3} := V_{a_{i+2}}(a_i, a_{i+3}) = V_{a_{i+1}}(a_i, a_{i+3}) ,$$
$$W_{i,i+1} := W^*_{i+1,i} .$$

It then follows from item (3) of Proposition 2.10 that $V_{a_{i+3}}(a_i, a_{i+2}) = V^*_{a_{i+1}}(a_i, a_{i+2})$, and thus that $W^*_{i,i+2} = W_{i+2,i}$.

Finally recall that $W_{i,i+1} = W^*_{i+1,i}$. Thus from item (3) of Proposition 2.10, $W_{i+1,i+2} \cap C$ is the component of $C \setminus \{a_{i+1}, a_{i+2}\}$ not containing a_i and a_{i+3}. Let d in $W_{i+1,i+2} \cap C$, then

$$W_{i+1,i+2} = V_d(a_{i+1}, a_{i+2}) \subset V_d(a_i, a_{i+3}) = W_{i,i+3} ,$$

where, for the inclusion, we applied twice the last part of the item (3) of Proposition 2.10.

This concludes the proof. □

2.6. **Positive maps.**

Definition 2.13 (Positive map). Let S be a cyclically ordered set containing at least three points. Then a map f from S to $\mathbf{F}_\Theta$ is *positive* is the image of every ordered quadruple is a positive quadruple, and the image of every ordered triple is a positive triple.[1]

Observe then that the image of every cyclically ordered configuration by a positive map is a positive configuration: indeed to be a positive configuration only depends on triple and quadruples.

Observe that by Proposition 2.12, the circle map of a positive circle is positive.

3. Properties of positivity

We prove in this section, three main propositions concerning positivity:

- The first one, Proposition 3.1, gives various combinatorial properties of positive triples, quadruples and configurations;
- The second one, Proposition 3.8, gives information about the limit of diamonds included in a given diamond;

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[1]When S has more than three points, the second requirement follows from the first

- The last one, Proposition 3.13, shows that positive maps share the property of monotone maps: they coincide on a dense subset with a left-continuous positive map.

3.1. **Combinatorics of positivity.** The following items of the next proposition are fundamental properties of positive triples and quadruples.

- The first one gives an easy criterion for positivity of triples, while the second and third for quadruples. In particular, this shows that the definition of positivity given in the introduction is equivalent to Definition 2.8.
- The fourth one gives a recursive way to build positive configuration and, formally, the previous property can be deduced from it.
- The fifth and sixth gives "exclusion" properties that are not used in this paper.

We are going to prove this proposition and its corollary in the context of a group defined over $\mathbb{R}$, although by Tarski Theorem, the statements will be true over every real closed field.

Proposition 3.1 (COMBINATORIAL PROPERTIES).   (1) *Assume (a, b) are transverse and c belongs to a diamond with extremities a and b, then (a, b, c) is positive.*

(2) *Assuming (a, c, b) is positive and d belongs to $V^a(c, b)$, then (a, c, d, b) is positive.*

(3) *Assuming (a, x_0, b) and (a, y_0, b) are positive then (a, x_0, b, y_0) is positive if and only if $V_{x_0}(a, b) = V^*_{y_0}(a, b)$.*

(4) *Assume that $(x_0, x_1, \ldots, x_p)$ is a positive configuration and that $y \in V^{x_2}(x_0, x_1)$ then*

$$(x_0, y, x_1, \ldots, x_p) \, ,$$

is a positive configuration.

(5) *[EXCLUSION FOR TRIPLES] Assume (a, b, c, d) is positive, then (a, c, b, d) is not positive.*

(6) *[EXCLUSION FOR QUADRUPLES] Let x_0, x_1, and x_2 be three points so that (a, x_i, b) is positive, then the three quadruples (a, x_0, b, x_1), (a, x_1, b, x_2), and (a, x_2, b, x_0) cannot be all positive.*

The proof of this proposition and of the next Corollary 3.2 will be given in Section 3.1.3. It is important to remark that all these properties are true for configurations on $\mathbf{P}^1(\mathbb{R})$.

Although the last three properties are not used in the sequel they are important in the study of positivity. Finally we have,

Corollary 3.2 (NECKLACE PROPERTY)**.** *Let* (a, b, c) *be a positive triple. Let* α, β *and* γ *be elements of* $V_a(b, c)$,$V_b(a, c)$ *and* $V_c(a, b)$ *respectively. Then* (α, β, γ) *is a positive triple.*

3.1.1. *Triples and quadruples.*

Lemma 3.3. *A triple* (a_0, a_1, a_2) *is positive if and only if* a_0, a_1, a_2 *all belongs to diamonds with extremities* (a_1, a_2), (a_2, a_0) *and* (a_0, a_1) *respectively.*

Proof. We just need to prove the "if" part. Let then

$$V_{i,i+1} := V^{a_{i+2}}(a_i, a_{i+1}) \,, \ V_{i,i+2} := V_{a_{i+1}}(a_i, a_{i+2}).$$

Observe that

$$V_{i,i+1} = V^{a_{i+2}}(a_i, a_{i+1}) = V^*_{a_{i+2}}(a_i, a_{i+1}) = V^*_{i+1,i} \,.$$

Then Lemma 2.7 provides all the necessary inclusions needed to prove the triple is positive. □

The following lemma gives a way to go from positive triple to positive quadruple.

Lemma 3.4. *Let* (a_0, a_1, a_2, a_3) *be a quadruple. Assume that all subtriples are positive. Then the quadruple* (a_0, a_1, a_2, a_3) *is positive, if and only if*

$$a_i \in V^*_{a_{i+2}}(a_{i+1}, a_{i+3}) \,, \tag{1}$$

$$a_{i+2} \in V_{a_{i+1}}(a_i, a_{i+3}) \,. \tag{2}$$

Proof. The "only if" part follows from the definition. It remains to prove the "if" part. Let (indices are taken modulo 4)

$$V_{i,i+1} := V^{a_{i+2}}(a_i, a_{i+1}) \,,$$
$$V_{i,i+2} := V_{a_{i+1}}(a_i, a_{i+2}) = V^{a_{i+3}}(a_i, a_{i+2}) \,,$$
$$V_{i,i+3} := V_{a_{i+1}}(a_i, a_{i+3}) = V_{a_{i+2}}(a_i, a_{i+3}) \,,$$

where in the second line we used the hypothesis (1), while in the last we used the hypothesis (2). Hence by definition

$$V_{i,i+1} = V^*_{i+1,i} \,, V_{i,i+2} = V^*_{i+2,i} \,.$$

It thus follows that for all i and j,

$$V_{i,j} = V^*_{j,i} \,. \tag{3}$$

From the positivity of all subtriples (a_i, a_{i+1}, a_{i+2}) and the previous lemma, we get the inclusions

$$V_{i,i+1} \subset V_{i,i+2} \,, \ V_{i,i+1} \subset V_{i-1,i+1} \,. \tag{4}$$

From the positivity of the triple (a_i, a_{i+1}, a_{i+3}) we get the inclusions

$$V_{i,i+1} = V^{a_{i+3}}(a_i, a_{i+1}) \subset V_{a_{i+1}}(a_i, a_{i+3}) = V_{i,i+3} , \tag{5}$$

$$V_{i+1,i+3} = V^{a_i}(a_{i+1}, a_{i+3}) \subset V_{a_{i+1}}(a_i, a_{i+3}) = V_{i,i+3} . \tag{6}$$

Similarly the positivity of the triple (a_i, a_{i+2}, a_{i+3}) yields

$$V_{i+2,i+3} = V^{a_i}(a_{i+2}, a_{i+3}) \subset V_{a_{i+2}}(a_i, a_{i+3}) = V_{i,i+3} , \tag{7}$$

$$V_{i,i+2} = V^{a_{i+3}}(a_i, a_{i+2}) \subset V_{a_{i+2}}(a_i, a_{i+3}) = V_{i,i+3} . \tag{8}$$

All together the equation (3) as well as the inclusions (4), (5), (6), (7), and (8), prove that (a_0, a_1, a_2, a_3) is a positive quadruple. $\square$

3.1.2. *Deformation lemmas.* We need to prove some deformation lemmas.

Lemma 3.5 (DEFORMING TRIPLES). *Let $a(t)$, $b(t)$, and $c(t)$ be continuous arcs from $[0, 1]$ to $\mathbf{F}_\Theta$ so that*

 (1) *for all t in $[0, 1]$, the points in the triple $(a(t), b(t), c(t))$ are all pairwise transverse,*
 (2) *the triple $(a(0), b(0), c(0))$ is positive.*

Then, for all t, $(a(t), b(t), c(t))$ is a positive triple.

Proof. The hypothesis tells us that

$$c(t) \in \Omega_{a(t)} \cap \Omega_{b(t)} , \quad a(t) \in \Omega_{c(t)} \cap \Omega_{b(t)} , \quad b(t) \in \Omega_{a(t)} \cap \Omega_{c(t)} .$$

By hypothesis, denoting $V(e, d)$ a diamond with extremities e and d.

$$c(0) \in V(a(0), b(0)) , \quad a(0) \in V(c(0), b(0)) , \quad b(0) \in V(c(0), a(0)) .$$

We now use the fact that a diamond with extremities c and d is a connected component of $\Omega_c \cap \Omega_d$ by Proposition 2.5. Then for all t by continuity

$$c(t) \in V(a(t), b(t)) , \quad a(t) \in V(c(t), b(t)) , \quad b(t) \in V(c(t), a(t)) .$$

Thus the result follows from the definition. $\square$

 Similarly

Lemma 3.6 (DEFORMING QUADRUPLES). *Let γ and η be continuous arcs from $[0, 1]$ to $\mathbf{F}_\Theta$ so that there exists a and b in $\mathbf{F}_\Theta$ satisfying*

 (1) *for all t in $[0, 1]$, the points in the quadruple $(a, \gamma(t), b, \eta(t))$ are all pairwise transverse,*
 (2) *the quadruple $(a, \gamma(0), b, \eta(0))$ is positive.*

Then, for all t, $(a, \gamma(t), b, \eta(t))$ is a positive quadruple.

Proof. By applying Lemma 3.5, we obtain that all the subtriples of $(a, \gamma(t), b, \eta(t))$ are positive. By Lemma 3.4, we only need to check that

$$a \notin V_b(\gamma(t), \eta(t)) \,, \ b \notin V_a(\gamma(t), \eta(t)) \,,$$

$$\gamma(t) \notin V_{\eta(t)}(a, b) \,, \ \eta(t) \notin V_{\gamma(t)}(a, b) \,,$$

$$\gamma(t) \in V_a(\eta(t), b) \,, \ b \in V_{\gamma(t)}(a, \eta(t)) \,,$$

$$\eta(t) \in V_b(\gamma(t), b) \,, \ a \in V_{\eta(t)}(b, \gamma(t)) \,.$$

We use again the fact that a diamond with extremities c and d is a connected component of $\Omega_c \cap \Omega_d$ by Proposition 2.5. Then, the statement follows by continuity. □

Finally we also have as an immediate consequence of the connectedness of the positive cone:

Lemma 3.7 (CONNECTEDNESS). *Let a and b two transverse points. Let C be any positive circle through a and b.*
  (1) *Assume c is so that (a, c, b) is positive. Then there is an arc $t \mapsto c(t)$ from $[0, 1]$ to $V_c(a, b)$ connecting $c = c(0)$ to $c(1)$ so that $(a, c(1), b)$ is a positive triple on C.*
  (2) *Assume furthermore that d belongs to $V^a(c, b)$ then there are a path $t \mapsto c(t)$ as in the previous item and a path $t \mapsto d(t)$ from $[0, 1]$ to $V_c(a, b)$, so that $d(t) \in V^a(c(t), b)$ and $(a, c(1), d(1), b)$ are on C.*

Proof. Using the U-pinning s_b at b, we identify N with a positive semigroup N_b in U_b. Recall that we have $V_c(a, b) = \mathsf{N}_b \cdot a$. The first point just follows from the connectedness of the positive cone N_b. For use in the second point we take a path which is constant for $t > 1/2$.

Recall that $d = m_0 \cdot c$, with $m_0 \in \mathsf{N}_b$. Let then for $t \in [0, 1/2]$,

$$d(t) = m_0 \cdot c(t) \,,$$

then we have by the semigroup property $d(t) \in V^a(c(t), b)$. Observe also that $d(0) = d$. Then for $t \in [1/2, 1]$, we have $c(t) = c(1/2)$ and we choose, using the first part, a path $t \mapsto d(t)$ with $d(t) \in V^a(c(1/2), d)$, and so that $d(1)$ belongs to C. □

3.1.3. *Proof of the combinatorial properties.*

Proof of item (1) *of Proposition 3.1.* Assume (a, b, c) satisfies the hypothesis. Let $c(t)$ be path connecting c to a point d in $V_c(a, b)$ obtained by Lemma 3.7. On a circle, a triple is positive if and only if the three points are distinct, the result thus follows from Lemma 3.5. □

Proof of item (2) *of Proposition 3.1.* From the connectedness Lemma 3.7 we obtain deformation $c(t)$, $d(t)$ so that $(a, c(t), d(t), b)$ are pairwise

transverse, $c(0) = c$, $d(0) = d$, $(a, c(1), d(1), b)$ on a circle and $d(1) \in V_a(c(1), b)$. In particular $(a, c(1), d(1), b)$ is positive and thus by the deforming Lemma 3.6, (a, c, d, b) is positive.   □

Proof of item (3) *of Proposition 3.1.* The "only if" part follows from the definition. Then for the "if" part we find by the first part of Lemma 3.7, paths $x(t)$ and $y(t)$ in $V_{x_0}(a, b)$ and $V_{y_0}(a, b)$ respectively, so that $(x(0), y(0)) = (x_0, y_0)$ and $x(1)$, $y(1)$ are on a circle passing through a an b. Then $(a, x(1), b, y(1))$ is positive and so is (a, x_0, b, y_0) by Lemma 3.6, since $x(t)$ and $y(t)$ are transverse thanks to Proposition 2.5.   □

Proof of item (4) *of Proposition 3.1.* This is an immediate consequence of item (3) and the fact that in order to check the positivity of a configuration one only needs to check the positivity of subtriples and subquadruples.   □

Proof of item (5) *of Proposition 3.1.* If (a, b, c, d) is positive and (a, c, b, d) is positive we have strict inclusions $V^a(c, d) \subset V^a(b, d)$ and $V^a(b, d) \subset V^a(c, d)$, hence a contradiction.   □

Proof of item (6) *of Proposition 3.1.* Assume that $V(a, x_0, b, x_1)$ is positive. Let $V =:= V_{x_0}(a, b)$ and $V^* = V_{x_1}(a, b)$ the opposite to V. If both (a, x_1, b, x_2) and (a, x_0, b, x_2) are positive then we get that x_2 belongs to both V and $V*$, which is a contradiction.   □

Proof of the necklace property (Corollary 3.2). Let us first remark that from item (4) of Proposition 3.1, applied three times, the configuration

$$(a, \gamma, b, \alpha, c, \beta) \, ,$$

is positive. Thus (γ, α, β) is positive.   □

3.2. **Inclusion of diamonds.**

Proposition 3.8 (BOUNDED PROPERTY). *Let (a, b, d) be a positive triple and let $c \in V_b(a, d)$. Assume that there exists sequences $\{b_m\}_{m \in \mathbb{N}}$ and $\{c_m\}_{m \in \mathbb{N}}$, converging respectively to b and c and such that, for all m, (a, b_m, c_m, d) is a positive quadruple. then*

$$\lim_{m \to \infty} \left(\overline{V^d(b_m, c_m)} \right) \subset V_c(a, d) \, .$$

In particular,

Corollary 3.9 (INCLUSION). *Let (a, b, c, d) be a positive quadruple points in $\mathbf{F}_\Theta$. Then*

$$\overline{V^d(b, c)} \subset V_b(a, d).$$

Proposition 3.8 will be proved in Section 3.2.2.

3.2.1. *Preliminaries on circles.* Let then $V(a,d)$ be a diamond and C a positive circle passing through a and d so that C is an orbit of a positive H, isomorphic to $\mathsf{PSL}_2(\mathbb{R})$.

- Let $\delta = \{\delta_t \mid t \in \mathbb{R}\}$, be the 1-parameter group in H for which d is the attractive fixed point and a is the repulsive fixed point.
- Let $\gamma = C \smallsetminus \{a,d\}$.
- Let F be the set of fixed points of δ in $\mathbf{F}_\Theta$.

The result of this section is

Proposition 3.10. *For any e in γ, we have $\overline{V^a(e,d)} \subset \Omega_a$.*

This proposition implies Corollary 3.9:

Proof of the Corollary 3.9. Applying Proposition 3.10, we get that

$$\overline{V^a(b,c)} \subset \overline{V^a(b,d)} \cap \overline{V^d(c,a)} \subset \Omega_d \cap \Omega_a .$$

Since $V_c(a,d)$ is a connected component of $\Omega_d \cap \Omega_a$ containing $V^a(b,c)$ it follows that

$$\overline{V^a(b,c)} \subset V_c(a,d) . \qquad \square$$

Recall that by definition S_a is the complementary to Ω_a. In order to prove to prove Proposition 3.10, we introduce the following sets

$$J(e):=\overline{V^a(e,d)} \cap S_a ,$$
$$F(e):=J(e) \cap F ,$$
$$O^+ := \{x \in \mathbf{F}_\Theta \mid \lim_{t\to\infty} \delta_t(x) = d\} ,$$
$$O^- := \{x \in \mathbf{F}_\Theta \mid \lim_{t\to-\infty} \delta_t(x) = a\} .$$

We will first prove that the sets $J(e)$ and $F(e)$ are empty. We first prove the following lemma

Lemma 3.11. *For any e in γ,*
(1) *$J(e)$ is invariant by the semigroup $\delta^+:=\{\delta_t \mid t > 0\}$, i.e. $\delta_t(J(e)) \subset J(e)$ for all $t > 0$,*
(2) *$F(e)$ is independent of the choice of e,*
(3) *if $J(e)$ is not empty, so is $F(e)$,*
(4) *for all c and b in $\mathbf{F}_\Theta$ so that (a,c,d,b) is a positive quadruple and*

$$V(c,d) \cap \gamma \not{\emptyset}$$

where $V(c,d)$ is the only diamond in $V_c(a,d)$ obtained by Lemma 2.7, then

$$F(e) \subset S_c \cup S_b .$$

Proof. We prove the first point. By Proposition 2.12 for $t > 0$

$$V^a(\delta_t(e), d) \subset V^a(e, d) .$$

This implies that $V^a(e, d)$ is invariant by δ^+ and so is $J(e)$.

The second point is a consequence that F is pointwise fixed by δ_t:

$$J(\delta_t(e)) \cap F = \delta_t(J(e)) \cap F = J(e) \cap F .$$

The third point is a consequence of the compactness of $\mathbf{F}_\Theta$ which implies that any non-empty closed set in $\mathbf{F}_\Theta$ invariant by the semigroup δ^+ has a non-empty intersection with F.

Let us prove the last point now, wich will require several steps. Let be (a, c, d, b) as in the hypothesis. Thus

$$V(c, d) \cap \gamma \neq \emptyset .$$

Let then $V(e, d)$ the unique diamond in $V(c, d)$ obtained by Lemma 2.7, and similarly $V(c, b)$ in $V_c(a, b)$ and observe that

$$F(e) \subset J(e) = \left(\overline{V^a(e, d)} \cap S_a \right) \subset \left(\overline{V^a(c, b)} \cap S_a \right) . \tag{9}$$

Now, we remark that by positivity of (a, c, d, b),

$$V(c, b) \subset \Omega_a.$$

Thus

$$\overline{V(c, b)} \cap S_a = \left(\overline{V(c, b)} \smallsetminus V(c, b) \right) \cap S_a. \tag{10}$$

But since $V^a(c, d)$ is a connected component of the open set

$$\Omega_c \cap \Omega_d = \mathbf{F}_\Theta \smallsetminus (S_c \cup S_d) ,$$

we get

$$\left(\overline{V^a(c, b)} \smallsetminus V^a(c, b) \right) \subset (S_c \cup S_b) . \tag{11}$$

Combining inclusions (9), (10) and (11), we get that

$$F(e) \subset (S_c \cup S_b) . \qquad \square$$

We can now prove Proposition 3.10, in other words that $J(e)$ is empty. By item (3) of Lemma 3.11, it suffices to show that $F(e)$ is empty. The fact that $F(e)$ is empty follows from item (3) of Lemma 3.11 and the following result.

Lemma 3.12. *Let Q be a subset of $\mathbf{F}_\Theta$. Assume that there exists nonempty open sets U and V so that for all c in U, and all b in V,*

$$Q \subset S_c \cup S_b,$$

then Q is empty.

Proof. Let q be in Q and set $Z := S_q$. Then Z is a proper closed Zariski subset of $\mathbf{F}_\Theta$. Observe that if $u \notin Z$, then

$$q \notin S_u.$$

On the other hand we can find c in the nonempty set $U \smallsetminus Z$ and b in the nonempty set $V \smallsetminus Z$, and by hypothesis $q \in (S_u \cup S_v)$. This shows that $q \neq q$ and concludes the proof. $\qquad\square$

3.2.2. *Proof of the boundedness Proposition 3.8.*

Proof. We use the notation of the previous paragraph. Let C be a circle though a and d. Let $\gamma := V_c(a, d) \cap C$. Since being positive is an open condition for quadruples, we can find e and f in γ so that (e, b_m, c_m, f) is positive for m large enough as well as (a, e, f, d). Thus

$$V^a(b_m, c_m) \subset V^a(e, f)$$

Applying Proposition 3.10, we get that

$$\overline{V^a(b_m, c_m)} \subset \overline{V^a(e, f)} \subset V_c(a, d) \,,$$

which easily implies the result. $\qquad\square$

3.3. **Left and right limits of positive maps.** Our main result is

Proposition 3.13 (Existence of left and right limits)**.** *Let S be a totally ordered set and ϕ be a positive map from S to $\mathbf{F}_\Theta$.*

Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of points in S so that $a < b_n \leqslant b_{n+1} \leqslant b < c$, for a, b, and c in S.

Then $\{\phi(b_n)\}_{n\in\mathbb{N}}$ converges to a point y in $V_b(a, c)$. Symmetrically, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of points so that $c < a \leqslant a_{n+1} \leqslant a_n < b$. Then $\{\phi(a_n)\}_{n\in\mathbb{N}}$ converges to a point y in $V_a(c, b)$.

As an immediate corollary, we show that positive maps defined on dense subsets extend to positive maps. More precisely:

Corollary 3.14 (Extension of positive maps)**.** *Let A be dense subset in $[0, 1]$. Assume that we have a positive map ξ from A to $\mathbf{F}_\Theta$. Then there exists*

- *a unique left-continuous positive map ξ_+ from $[0, 1]$ to $\mathbf{F}_\Theta$ so that ξ coincide with ξ_+ on a dense subset of A,*
- *a unique right-continuous positive map ξ_- from $[0, 1]$ to $\mathbf{F}_\Theta$ so that ξ coincide with ξ_- on a dense subset of A.*

Moreover,

- *for any triple of pairwise distinct points (x, y, z) in $[0, 1]$*

$$(\xi_a(x), \xi_b(y), \xi_c(z)) \,,$$

  is a positive triple for any choice of a, b, and c in $\{+, -\}$,

- *if $\{x_m\}_{m\in\mathbb{N}}$, $\{z_m\}_{m\in\mathbb{N}}$ are sequences in $[0,1]$ converging to y, so that $x_m < y < z_m$, then*

$$\lim_{m\to\infty} \xi_{a_m}(x_m) = \xi_-(y) \, , \, \lim_{m\to\infty} \xi_{b_m}(z_m) = \xi_+(y) \, ,$$

  for any sequences $\{a_m\}_{m\in\mathbb{N}}$ and $\{b_m\}_{m\in\mathbb{N}}$ in $\{+,-\}$.

Proof of Proposition 3.13. Let us define $x_n = \phi(b_n)$. We can write $x_n = m_n \cdot x_{n-1}$, with $m_n \in \mathsf{N}_{\phi(c)}$. Thus by induction we have

$$x_n = m_n \cdots m_1 \cdot x_0 \, .$$

But we know that $V^{\phi(c)}(x_0, \phi(b))$ is a relatively compact region of $\Omega_{\phi(c)}$ by Proposition 3.8. Thus $\{x_n\}_{n\in\mathbb{N}}$ is a relatively compact region of $\Omega_{\phi(c)}$. It follows that

$$\pi_n = m_n \cdots m_1,$$

is a bounded sequence in $\mathsf{N}_{\phi(c)}$. We now prove that this sequence converges. Assume that we have subsequences that converge to different limits u and v. After extracting further subsequence, we may find subsequences

$$q_i = \pi_{n_i}, \ \ p_i = \pi_{m_i}, \ \text{with } n_i \leqslant m_i$$

such that $\{q_i\}_{i\in\mathbb{N}}$ converges to u and $\{p_i\}_{i\in\mathbb{N}}$ converges to v. It follows that $u = w_1 \cdot v$ with $w_1 \in \overline{\mathsf{N}}_{\phi(c)}$. Symmetrically, $v = w_0 \cdot u$ with $w_0 \in \overline{\mathsf{N}}_{\phi(c)}$. It follows that $w_0 \cdot w_1 = 1$, thus w_0 and w_1 are invertible in the closed semigroup $\overline{\mathsf{N}}_{\phi(c)}$, hence equal to the identity. In particular $u = v$ and $\{x_n\}_{n\in\mathbb{N}}$ converges.

The proof for the sequence $\{a_n\}_{n\in\mathbb{N}}$ is symmetric. $\qquad\square$

3.3.1. *Positivity and continuity.* In some cases, it suffices to show that the image of every triple is positive

Proposition 3.15 (TRIPLES SUFFICE). *Let ϕ be a continuous map from an interval I to $\mathbf{F}_\Theta$ so that the image of every ordered triple is positive, then ϕ is positive.*

Proof. To lighten the notation of this proof, let us write $u_0 := \phi(u)$ for every u in C. Let z in $]x, y[$ and $W_z(x, y) = V_{z_0}(x_0, y_0)$. Since $[x, y]$ is connected, $W_z(x, y)$ does not depend on z and we denote it $W(x, y)$: More precisely, fixing t, the set of z in $]x, y[$, so that $\phi(t)$ belongs to $W_z(x, y)$ is connected since ϕ is continuous.

Let $V(t, y)$ the unique diamond with extremities t_0 and y_0 – obtained in Lemma 2.7 – so that

$$V(t, y) \subset W(x, y) \, . \tag{12}$$

MAIN STEP: We first prove that if $x < t < y$, then

$$W(t, y) = V(t, y) \subset W(x, y) \, . \tag{13}$$

Let us consider

$$U = \{t \in]x, y[| \ W(t, y) = V(t, y)\}$$

Let us write $W(x, y) = \mathsf{N} \cdot x$, where N is an open semigroup in U_{y_0}. We can thus write $\phi(t) = n_t \cdot x$, with $t \mapsto n_t$ is a proper continuous map with values in N, satisfying $n_x = \mathrm{id}$. Then $V(t, y) = n_t W(x, y)$. We now proceed to the proof and show that U is open, non empty and closed.

(1) The set U is also the set of t so there exists s, with $t < s < y$ such that s_0 is in $V(t, y)$. In other words, $n_t^{-1} n_s$ belongs to N. Thus U is open.

(2) Since N is open, given s, for all t close enough to x we have $n_t^{-1} n_s$ is in N. Thus $n_s \in n_t \mathsf{N}$, hence $s_0 \in V(t, y)$. Thus U is non empty.

(3) If t belongs to U, then for all $s > t$, s_0 belongs to $W(t, y) = V(t, y)$. Hence for all $s > t$, $n_t^{-1} n_s$ belongs to N. Since $u \mapsto n_u$ is proper, given any t in U, we can find s, such that $n_t^{-1} n_s$ belongs to a compact set in N not containing the identity. Thus if $\{t\}_{m \in \mathbb{N}}$ is a sequence in U converging to t, we can produce a sequence $\{s\}_{m \in \mathbb{N}}$ converging s with $n_t^1 n_s$ different from the identity and belonging to $\bar{\mathsf{N}}$. Since $s_0 = n_s \cdot y$ is transverse to $t_0 = n_t \cdot y$, it follows that $n_t^1 n_s$ actually belongs to N. Hence that s_0 belongs to $V(t, y)$, hence that $V(t, y) = W(t, y)$. We have completed the proof that U is closed.

The proof of the Assertion (13) is now complete.

CONCLUSION: Let (a, b, c, d) so that $a < c < b < d$, with all subtriples of (a_0, b_0, c_0, d_0) positive. By item (2) of Proposition 3.1, we only have to prove that

$$V_{c_0}(a, b) = V_{d_0}^*(a, b) \, ,$$

Observe that

$$W(a, b) \subset W(a, d) = V_{b_0}(a_0, d_0) \, ,$$

by Assertion (13). Thus d_0 does not $W(a, b)$ and hence belongs to $W^*(a, b)$ by Lemma 2.7. We thus have

$$V_{d_0}(a_0, b_0) = W^*(a, b) = V_c^*(a_0, b_0) \, .$$

This completes the proof of the positivity of the quadruple (a_0, b_0, c_0, d_0), hence of the proposition.

$\square$

4. Triples, Tripods and metrics

In this section, we construct for every positive triple (a, b, c) a complete metric on the diamond $V_c(a, b)$ in Definition 4.9. We also show that this metric satisfy contraction properties (Propositions 4.11 and 4.12).

We first do it for special triples that we call tripods.

4.1. **Tripods and metrics.**

Definition 4.1. A *tripod* is a triple of pairwise distinct triple of points on a positive circle. A tripod is always positive. If $\tau = (x, t, y)$ is a tripod, we write

$$\tau^- = x \, , \ \tau^0 = t \, , \ \tau^+ = y \ , \overline{\tau} := (y, t, x) \, , V_\tau := V_{\tau^0}(\tau^-, \tau^+) \, .$$

Let $\mathcal{T}_0$ be the set of tripods. Observe that the stabilizer of any tripod is compact and that G acts transitively on the space of tripods. In particular G acts properly on the space of tripods . Let then d be a G-invariant metric on $\mathcal{T}_0$.

4.1.1. *Tripods and the parameterization.* Let us consider as in Paragraph 2.2.1, the L°-equivariant parameterization $\Psi \colon \mathsf{C} \to \mathsf{N}$ where C is a convex cone in $\mathfrak{u}_\Theta^N$ for some N (and Ψ is constructed from the exponential map). Note that Ψ extends continuously to a map $\overline{\mathsf{C}} \to \mathsf{U}$ that is also L°-equivariant.

Let h be the element of C corresponding to the unipotent associated to the preferred $\mathsf{SL}_2(\mathbb{R})$ — see Proposition 2.12. Let K_h be the stabilizer of h in $\mathsf{L}^\circ = \mathsf{L}_\Theta^\circ$. Since the stabilizer of a positive triple is compact, it follows that K_h is compact.

If now x and y are transverse points in $\mathbf{F}_\Theta$ and σ is an isomorphism of U with U_y (*i.e.* σ is a U-pinning at y), then the map

$$\Psi^\sigma \colon \mathsf{C} \mapsto \mathbf{F}_\Theta, \quad u \mapsto \sigma \circ \Psi(u) \cdot x \, ,$$

is a parameterization of the diamond $V_t(x, y)$ with $t := \Psi^\sigma(h)$. We then define

Definition 4.2. Given a tripod $\tau = (x, t, y)$ a *parameterization* of the diamond V_τ, is a map Ψ_τ of the form Ψ^σ so that $\Psi^\sigma(h) = t$.

From the definition follows

Proposition 4.3. *Given a tripod τ, a parameterization of the diamond exists and is unique up to post-composition by the stabilizer of τ, or up to precomposition by K_h.*

The next proposition is crucial; it insures that a sequence of parameterizations of diamonds associated with tripods converges to the constant map as soon as one sequence in the image converges, precisely

Proposition 4.4 (CONTRACTION IN CORNERS). *Let $\{\tau_m\}_{m\in\mathbb{N}}$ be a sequence of tripods, with $\tau_m = (x_m, t_m, y)$.*

Assume that $\{x_m\}_{m\in\mathbb{N}}$ converges to a point x_0 transverse to y. Assume that there exists a converging sequence $\{k_m\}_{m\in\mathbb{N}}$ in the cone C, so that

$$\lim_{m\to\infty} \Psi_{\tau_m}(k_m) = x_0 . \tag{14}$$

Then for any convergent sequence $\{k'_m\}_{m\in\mathbb{N}}$ in C,

$$\lim_{m\to\infty} \Psi_{\tau_m}(k'_m) = x_0 . \tag{15}$$

Proof. Let $\{\tau_m\}_{m\in\mathbb{N}}$, $\{x_m\}_{m\in\mathbb{N}}$, and $\{t_m\}_{m\in\mathbb{N}}$ as in the statement. Since x_0 is transverse to y, by replacing τ_m by $u_m\tau_m$ where $\{u_m\}_{m\in\mathbb{N}}$ is a converging sequence in U_y, we may as well assume that $\{x_m\}_{m\in\mathbb{N}}$ is constant and equal to x.

Using the fact that G acts transitively on tripods, let us write $t_m = g_m \cdot t_0$, with g_m in the connected component of the identity of $\mathsf{L}_{x,y} = \mathsf{P}_x \cap \mathsf{P}_y$. Thus

$$\Psi_{\tau_m}(h) = g_m \cdot t_0 = g_m \cdot \Psi_{\tau_0}(h) = g_m \sigma(\Psi(h)) \cdot x .$$

Note that the U-pinning $\sigma \colon \mathsf{U} \to \mathsf{U}_y$ extends to an isomorphism σ of P with P_y. We denote by g_m^0 the element of L° such that $\sigma(g_m^0) = g_m$. Up to maybe precomposing by an element of K_h, we may assume that Ψ_{τ_m} is the map $k \mapsto \sigma(\Psi(g_m^0 \cdot k)) \cdot x$.

Therefore we have, for any $\{\ell_m\}_{m\in\mathbb{N}}$ in $\overline{\mathsf{C}}$, that $\{\Psi_{\tau_m}(\ell_m)\}_{m\in\mathbb{N}}$ converges to $x = x_0$ if and only if the sequence $\{g_m^0 \cdot \ell_m\}_{m\in\mathbb{N}}$ converges to 0 in $\overline{\mathsf{C}}$.

For any y in $\overline{\mathsf{C}}$, let

$$K(y) := \overline{\mathsf{C}} \cap \left(y - \overline{\mathsf{C}}\right),$$
$$\text{where } u - A := \{u - x \mid x \in A\} .$$

Since $\overline{\mathsf{C}}$ is salient, $K(y)$ is compact for any y.

From the previous discussion, we get that the sequence $\{c_m = g_m^0 \cdot k_m\}_{m\in\mathbb{N}}$ converges to 0. Thus, using again the fact that $\overline{\mathsf{C}}$ is salient, for every positive real R, the sequence of compact sets $\{K(R \cdot c_m)\}_{m\in\mathbb{N}}$ converges to $\{0\}$.

Let now $\{k'_m\}_{m\in\mathbb{N}}$ be a sequence converging in C. Since by hypothesis $\{k_m\}_{m\in\mathbb{N}}$ converges in C, there exists a positive real R such that, for all m, $R \cdot k_m - k'_m$ belongs to $\overline{\mathsf{C}}$. In other words: k'_m belongs to $R \cdot k_m - \overline{\mathsf{C}}$. Thus,

for all m, $g_m^0 \cdot k_m'$ belongs to $K(R \cdot c_m)$. Hence the sequence $\{g_m^0 \cdot k_m'\}_{m \in \mathbb{N}}$ converges to 0. This means that the sequence $\{\Psi_{\tau_m}(k_m')\}_{m \in \mathbb{N}}$ converges to $x = x_0$ as wanted. $\qquad\square$

4.1.2. *Tripod metrics.* We choose once and for all a Euclidean distance d_0, associated to the Riemannian g_0, on C which is invariant by K_h. Note that this distance extends to $\overline{\mathsf{C}}$.

Definition 4.5. Given a tripod $\tau = (x, t, y)$, let Ψ_τ be a parameterization of V_τ, let

$$g_\tau^+ := \Psi_\tau^* g_0 \,, \quad g_\tau^- := \Psi_{\bar\tau}^* g_0 \,, \quad g_\tau = g_\tau^+ + g_\tau^- \,,$$

as well as d_τ^+, d_τ^-, and d_τ the associated distance so that

$$d_\tau^\pm \leqslant d_\tau \leqslant d_\tau^+ + d_\tau^- \,.$$

The metric g_τ is the *diamond metric* (for tripods) on V_τ. while d_τ is the *diamond distance*.

We then have

Proposition 4.6 (COMPLETENESS). *The diamond metric is independent of the choice of the parameterization and only depends on τ. Moreover, d_τ is complete on V_τ.*

Proof of Proposition 4.6. The independence on the parameterization is a consequence of Proposition 4.3 and the fact that d_0 itself is invariant under the group K_h.

Let us now prove the completeness. Let $\{u_m\}_{m \in \mathbb{N}}$ be a Cauchy sequence for d_τ, then it is a Cauchy sequence for both d_τ^+ and d_τ^-. It follows that $\{v_m\}_{m \in \mathbb{N}}$ and $\{w_m\}_{m \in \mathbb{N}}$ are both Cauchy sequences, where

$$v_m = \Psi_\tau^{-1}(u_m), \; w_m = \Psi_{\bar\tau}^{-1}(u_m) \,.$$

Since $\overline{\mathsf{C}}$ is complete with respect to the metric d_0, there exist v and w in $\overline{\mathsf{C}}$ so that

$$\lim_{m \to \infty} v_m = v, \; \lim_{m \to \infty} w_m = w \,.$$

Since by construction Ψ_τ and $\Psi_{\bar\tau}$ extend continuously to the closure of C,

$$\Psi_\tau(v) = \Psi_{\bar\tau}(w) = \lim_{m \to \infty} u_m =: u \,.$$

Obviously u belongs to $\overline{V_\tau}$. Thus by construction $u = n_y \cdot x$ and $u = n_x \cdot y$ where n_y belongs to U_y and n_x belongs to U_x. Since x and y are transverse, it follows that u is transverse to both y and x. Thus u belongs to V_τ. Hence d_τ is complete. $\qquad\square$

Proposition 4.7 (Contraction for tripods). *Let $\{\tau_m\}_{m \in \mathbb{N}}$ be a sequence of tripods. Assume that $V_{\tau_{m+1}} \subset V_{\tau_m}$ and that*

$$\bigcap_{m \in \mathbb{N}} V_{\tau_m} = \{z\} . \tag{16}$$

For any positive R, let $V_{\tau_m}(R)$ be the ball of radius R and center τ_m^0 with respect to d_{τ_m}. Then on $V_{\tau_m}(R)$, we have

$$g_{\tau_0} \leqslant k_m \cdot g_{\tau_m} ,$$

with $\{k_m\}_{m \in \mathbb{N}}$ converging to zero.

Proof. Since G acts transitively on the space of tripods $\mathcal{T}_0$, it follows that $\tau_m = h_m \tau_0$, for some h_m in G. Since our construction is G-equivariant, we observe that $g_{\tau_m} = h_m^* g_{\tau_0}$. Then, we take

$$\begin{aligned}
k_m &= \sup \left\{ \frac{g_{\tau_0}(w,w)}{g_{\tau_m}(w,w)} \mid w \in TV_{\tau_m}(R) \right\} \\
&= \sup \left\{ \frac{g_{\tau_0}(w,w)}{g_{\tau_0}(Th_m^{-1}w, Th_m^{-1}w)} \mid w \in TV_{\tau_m}(R) \right\} \\
&= \sup \left\{ \frac{g_{\tau_0}(Th_m(v), Th_m(v))}{g_{\tau_0}(v,v)} \mid v \in TV_{\tau_0}(R) \right\} .
\end{aligned} \tag{17}$$

The hypothesis (16) says that h_m converges uniformly on every compact set of V_{τ_0} to the constant map. It follows that $\{h_m\}_{m \in \mathbb{N}}$ also converges C^1 to zero on any compact set in V_{τ_0}. Thus, equality (17) shows that $\{k_m\}_{m \in \mathbb{N}}$ converges to zero. $\qquad\square$

4.2. Positive triples, tripods and metric.

Our goal is to construct a complete metric on the diamond associated to a positive triple and to prove a generalization of the contraction property (Proposition 4.4).

4.2.1. *Approximating triples: the tripod norm.* We will first approximate in a rough sense positive triples by tripods. For any positive triple $t = (x, z, y)$, let

$$\mathrm{K}(t) := \inf \{ d_\tau(z, \tau^0) \mid \tau \in \mathcal{T}_0 , (\tau^+, \tau^-) = (x, y) \} .$$

We call $\mathrm{K}(t)$ the *tripod norm*. Observe that $\mathrm{K}(t)$ depends continuously on t, and that the tripod norm vanishes for tripods. Let also

$$\begin{aligned}
D(t, K_0) &:= \{ \tau \in \mathcal{T}_0 \mid (\tau^+, \tau^-) = (x, y) , d_\tau(z, \tau^0) \leqslant \mathrm{K}(t) \} , \\
D(t) &:= D(t, \mathrm{K}(t)) .
\end{aligned}$$

Proposition 4.8.   (1) *Given $K_0 \geqslant \mathrm{K}(t)$, the set $D(t, K_0)$ is compact and non-empty.*
  (2) $\mathrm{K}(t) = 0$ *if and only if t is a tripod,*

(3) *For any K_0 there exists a constant A, such that if t is a tripod with $K(t) \leqslant K_0$, then for τ_0 and τ_1 in $D(t)$, then*

$$g_{\tau_0} \leqslant A g_{\tau_1} \ .$$

Proof. Let $t = (a, b, c)$ be a positive triple. Let $\{\tau_m\}_{m \in \mathbb{N}}$ be a sequence of tripods such that $(\tau_m^-, \tau_m^+) = (a, c)$ and

$$\{d_{\tau_m}(b, \tau_m^0)\}_{m \in \mathbb{N}} \ ,$$

is bounded. Let $\{g_m\}_{m \in \mathbb{N}}$ be a sequence of elements in $\mathsf{L}_{a,c} = \mathsf{P}_a \cap \mathsf{P}_c$ so that $\{g_m^{-1}(\tau_m^0)\}_{m \in \mathbb{N}}$ is constant and equal to τ^0. Let $\tau := (a, \tau^0, c)$. It follows that

$$\{d_\tau(g_m^{-1}(b), \tau^0)\}_{m \in \mathbb{N}} \ ,$$

is bounded. Since d_τ is a complete metric, and in particular every bounded set is compact, the sequence $\{g_m^{-1}(b)\}_{m \in \mathbb{N}}$ — after extracting a subsequence — converges to e with (a, e, c) positive. Since G acts properly on the space of tripods, it follows that $\{g_m\}_{m \in \mathbb{N}}$ is bounded. Thus after taking a subsequence $\{\tau_m\}_{m \in \mathbb{N}}$ converges to a tripod τ_∞, with τ_∞^0 in V_t. Since $D(t, K_0)$ is non-empty for $K_0 > K(t)$, it follows that $D(t, K_0)$ is compact. The result for $K_0 = K(t)$ follows from the fact that

$$D(t) = \bigcap_{K_0 > K(t)} D(t, K_0) \ .$$

The second assertion is an immediate consequence of the first. The third follows from the first as a consequence of the G-equivariance of the assignment $\tau \mapsto d_\tau$.   □

4.2.2. *The diamond metric for triples.* The following definition is one of the goal of this section

Definition 4.9. Let t be a positive triple – which is not a tripod. The *diamond metric* (for triples) g_t is the Riemannian metric on V_t defined by

$$g_t := \frac{1}{\mathrm{Vol}(D(t))} \int_{D(t)} g_\tau \, \mathrm{d} \, \mathrm{Vol}(\tau) \ ,$$

where Vol is a G-invariant volume form on $\mathcal{T}_0$. The associated distance is the *diamond metric* d_t.

As an immediate corollary of Proposition 4.8 and Proposition 4.6, we have

Corollary 4.10. *The diamond metric takes finite values and is complete. Moreover if the sequence of positive triples $\{t_m\}_{m \in \mathbb{N}}$ converges to a tripod τ, then $\{g_{t_m}\}_{m \in \mathbb{N}}$ converges to g_τ on every compact of the diamond V_τ.*

The following are two contractions properties of the diamond metric that we shall use in the sequel,

Proposition 4.11 (CONTRACTION). *Let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence of positive triples, with $t_m = (a_m, b_m, c_m)$. Assume that the sequence $\{\mathrm{K}(t_m)\}_{m\in\mathbb{N}}$ of tripod norms is bounded. Assume that $V_{t_{m+1}} \subset V_{t_m}$ and that*

$$\bigcap_{m\in\mathbb{N}} V_{t_m} = \{z\} . \tag{18}$$

For any positive R, let $V_{t_m}(R)$ be the ball of radius R and center a_m with respect to d_{t_m}. Then on $V_{t_m}(R)$, we have

$$g_{t_0} \leqslant k_m \cdot g_{t_m} ,$$

with $\{k_m\}_{m\in\mathbb{N}}$ converges to zero.

Proposition 4.12 (CONTRACTION IN CORNERS). *Let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence of positive triples, where $t_m = (a_m, b_m, c_m)$. Assume that*
  (1) *the sequence $\{\mathrm{K}(t_m)\}_{m\in\mathbb{N}}$ of tripod norms is bounded,*
  (2) *$\{a_m\}_{m\in\mathbb{N}}$ and $\{c_m\}_{m\in\mathbb{N}}$ converge to transverse points a and c,*
  (3) *$\{u_m\}_{m\in\mathbb{N}}$ is a sequence of elements of $\mathbf{F}_\Theta$, so that u_m belongs to V_{t_m}, the sequence $\{d_{t_m}(b_m, u_m)\}_{m\in\mathbb{N}}$ uniformly bounded, and $\lim_{m\to\infty} u_m = a$.*
Then $\lim_{m\to\infty}(b_m) = a$.

Proof of Proposition 4.11. By the Definition 4.9 of the diamond metric for triples, and Proposition 4.8 it follows that we can find a constant A, such that for all m, we can find a tripod τ_m, with

$$d_{\tau_m}(\tau_m^0, b_m) \leqslant A , \quad \frac{1}{A} g_{\tau_m} \leqslant g_{t_m} \leqslant A g_{\tau_m} .$$

The result now follows from the corresponding proposition for tripods: Proposition 4.7. □

Proof of Proposition 4.12. Let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence of positive triples satisfying the hypothesis of the proposition where $t_m = (t_m^-, t_m^0, t_m^+)$. By the first hypothesis and Proposition 4.8, we can find a constant A, a sequence of tripods $\{\tau_m\}_{m\in\mathbb{N}}$ with $\tau_m^\pm = t_m^\pm$ and so that

$$d_{\tau_m} \leqslant A d_{t_m} .$$

In particular, we have that $\{d_{\tau_m}(t_m, \tau_m^0)\}_{m\in\mathbb{N}}$ and $\{d_{\tau_m}(u_m, \tau_m^0)\}_{m\in\mathbb{N}}$ are uniformly bounded. The result now follows from a double application of Proposition 4.4. Indeed, since $\{d_{\tau_m}(t_m, \tau_m^0)\}_{m\in\mathbb{N}}$ is uniformly bounded, it follows that $t_m = \Psi_{\tau_m}(k_m)$ with $\{k_m\}_{m\in\mathbb{N}}$ bounded. Hence by Proposition 4.4, with $k_m' = h$, yields that

$$\lim_{m\to\infty}(\tau_m^0) = a .$$

Applying again Proposition 4.4 to $\{k'_m\}_{m\in\mathbb{N}}$ with $\Psi_{\tau_m}(k'_m) = u_m$ yields that

$$\lim_{m\to\infty}(u_m) = a .$$

This concludes the proof.   □

5. Positive representations are Anosov

In this section we introduce the notion of positive representations of a surface group. We then show that any Θ-positive representation is Θ-Anosov, establishing Theorem B and Corollary C from the introduction.

Definition 5.1 (Positive representations). Let G be a simple Lie group admitting a Θ-positive structure. A representation $\rho\colon \pi_1(S) \to \mathsf{G}$ is Θ-*positive* if there exists a non-empty $\pi_1(S)$-invariant subset A of $\partial_\infty \pi_1(S)$ and a positive ρ-equivariant map ξ from A to $\mathbf{F}_\Theta$. The map ξ is called the *positive boundary map* of ρ and the image of ξ is the *positive limit curve* of ρ.

We also say that a representation is positive if it is Θ-positive for some Θ.

To establish the Anosov property we first extend the positive boundary map to a left- and a right-continuous boundary map using Corollary 3.14. We prove that these extensions are continuous, and then deduce the Anosov property using the contraction property of the diamond metric (Proposition 4.11).

5.1. **Properness.** The following definition will be used several times in the sequel: an application f defined on a subset A of a topological set X, with value in some topological set Y is *bounded* if for every compact set K in X, $f(A \cap K)$ is empty or relatively compact.

Lemma 5.2. *Let A be a dense set in the circle. Let ϕ be a positive map from A to $\mathbf{F}_\Theta$. Let A_+^3 be the intersection of A^3 with the set $\mathcal{T}_{S^1}$ of pairwise distinct triple in S^1. Then ϕ^3 is bounded as a map from A_+^3 in $\mathcal{T}$.*

Proof. Let $\chi = (x_1, x_2, y_1, y_2, z_1, z_2)$ be an oriented sextuplet in S_1. Let I_χ be the subset of $(S^1)^3$ given by

$$I_\chi = \{(X, Y, Z) \mid x_1 < X < x_2 < y_1 < Y < y_2 < z_1 < Z < z_2\} .$$

Let ϕ be a positive map and

$$K := \phi^3\left(I_\chi \cap A\right).$$

It is enough to show that $\overline{K} \subset \mathcal{T}$, where the closure is taken in $\mathbf{F}_\Theta^3$.

Let us find by density triples (a_0, a_1, a_2), (b_0, b_1, b_2), and (c_0, c_1, c_2) in A, so that

$$(x_1, x_2, a_0, a_1, a_2, y_1, y_2, b_0, b_1, b_2, z_1, z_2, c_0, c_1, c_2) ,$$

is cyclically oriented. To simplify our notation, we use the same notation for any point in the above 15-tuples and its image by ϕ.

From the positivity of the map and thus the image of of the 15-tuple defined above, it follows that if (x, y, z) belongs to K, then

$$x \in V^{a_1}(c_2, a_0) , y \in V^{b_1}(a_2, b_0) , z \in V^{c_1}(b_2, c_0) .$$

Thus if (a, b, c) belongs to $\overline{K}$, then

$$a \in \overline{V^{a_1}(c_2, a_0)} , b \in \overline{V^{b_1}(a_2, b_0)} , c \in \overline{V^{c_1}(b_2, c_0)} .$$

Thus using the inclusion Corollary 3.9, we get

$$a \in V^{b_1}(c_1, a_1) , b \in V^{c_1}(a_1, b_1) , c \in V^{c_1}(b_1, c_1) .$$

Thus by the necklace Corollary 3.2, (a, b, c) is a positive triple and thus belongs to $\mathcal{T}$. This concludes the proof. $\qquad \square$

Let ρ be a Θ-positive representation, A a non-empty $\pi_1(S)$-invariant subset of $\partial_\infty \pi_1(S)$ and $\xi \colon A \to \mathbf{F}_\Theta$ the positive ρ-equivariant boundary map. Then, by Corollary 3.14, there exists a unique right-continuous ρ-equivariant boundary map $\xi_+ \colon \partial_\infty \pi_1(S) \to \mathbf{F}_\Theta$ and a unique left-continuous ρ-equivariant boundary map $\xi_- \colon \partial_\infty \pi_1(S) \to \mathbf{F}_\Theta$, coinciding with the map ξ on some dense subset of A.

Proposition 5.3. *Let $\rho \colon \pi_1(S) \to \mathsf{G}$. Let ξ_+ be a positive $\pi_1(S)$-invariant map from $\partial_\infty \pi_1(S)$ to $\mathbf{F}_\Theta$.*

Let $\mathcal{T}_{\pi_1(S)}$ be the set of pairwise distinct triple in $\partial_\infty \pi_1(S)$, $\mathcal{T}$ the set of positive triples in $\mathbf{F}_\Theta$. Let Ξ_+ be the map from $\mathcal{T}_{\pi_1(S)}$ to $\mathcal{T}/\mathsf{G}$, defined by

$$\Xi_+(x, y, z) := [\xi_+(x), \xi_+(y), \xi_+(z)] .$$

Then the image of Ξ_+ is relatively compact.

Proof. The map Ξ_+ is invariant by the diagonal action of $\pi_1(S)$. The result follows then from Lemma 5.2 using the fact that $\pi_1(S)$ acts cocompactly on $\mathcal{T}_{\pi_1(S)}$. $\qquad \square$

5.2. **An *a priori* bound on the tripod norm.** For any positive triple t, let K(t) be the real constant and $D(t)$ the set of tripods, both defined in paragraph 4.2. Then Proposition 5.3 implies an a priori bound on the tripod norm.

Proposition 5.4. *Let $\rho\colon \pi_1(S) \to \mathsf{G}$. Let ξ be a ρ-equivariant positive map from a $\pi_1(S)$-invariant dense subset of $\partial_\infty \pi_1(S)$ to $\mathbf{F}_\Theta$. Then there exists a constant A so that for all triple of pairwise distinct points t in the closure of $\xi(\partial_\infty \pi_1(S))$, we have*

$$K(t) \leqslant A .$$

Proof. This is an immediate consequence of Proposition 5.3 and the fact that K is a continuous function on $\mathcal{T}^+$.   □

5.3. **Continuity of equivariant positive maps.** Let $\xi_+\colon \partial_\infty \pi_1(S) \to \mathbf{F}_\Theta$ the right-continuous ρ-equivariant boundary map, and $\xi_-\colon \partial_\infty \pi_1(S) \to \mathbf{F}_\Theta$ the left-continuous ρ-equivariant boundary agreeing on a dense subset of A.

Let $\mathcal{T}_{\pi_1(S)}$ be the set of triples of pairwise distinct points of $\partial_\infty \pi_1(S)$. For $t = (x, y, z)$ in $\mathcal{T}_{\pi_1(S)}$, let us define

$$\tau(t) = (\xi_+(x), \xi_+(y), \xi_+(z)) .$$

Lemma 5.5. *The $\pi_1(S)$-invariant function f defined by*

$$f(x, y, z) = d_{\tau(t)}(\xi_+(y), \xi_-(y)) .$$

is bounded.

Proof. The function f is $\pi_1(S)$-invariant, it thus suffices to check that f is bounded on a fundamental domain U for the action of $\pi_1(S)$ on $\partial_\infty \pi_1(S)^{3+}$.

We know that ξ_+^3 is bounded. Thus there exists a compact K in $\mathcal{T}$ which contains $\xi_+^3(U)$ and thus $\xi_+^3(U)$ and $\xi_-^3(U)$.

It follows that for any complete distance d_0 on $\mathbf{F}_\Theta$ there exists $\epsilon > 0$ such that for any $t = (x, z, y)$ in U

$$d_0(\partial V_{\tau(t)}, \xi_-(z)) \geqslant \epsilon ,$$
$$d_0(\partial V_{\tau(t)}, \xi_+(z)) \geqslant \epsilon .$$

Recall now that $d_{\tau(t)}$ is complete on $V_y(x, z)$. Since K is compact, and $\tau(t)$ belongs to K, it follows that all the metrics $d_{\tau(t)}$ are equivalent on compact sets of the diamond. This implies that for all t in U,

$$d_{\tau(t)}(\xi_+(y), \xi_-(y)) < A ,$$

where A depends on K and ϵ_0. In other words, f is bounded by A.   □

Lemma 5.6. *The map ξ_+ is left-continuous.*

Proof. Let $\{x_m\}_{m \in \mathbb{N}}$ be a sequence of points of $\partial_\infty \pi_1(S)$, as well as points x and y so that (x_m, x, y) is oriented with respect to the orientation on $\partial_\infty \pi_1(S)$, and that $\{x_m\}_{m \in \mathbb{N}}$ converges to x. Let $t_m = (\xi_+(x_m), \xi_+(x), \xi_+(y))$.

Recall that by Corollary 3.14, $\{\xi_+(x_m)\}_{m\in\mathbb{N}}$ converges to $\xi_-(x)$. We now apply Proposition 4.12 to the following setting:

$$a_m = \xi_+(x_m) \, , \ b = \xi_+(x) \, , \ u_m = \xi_-(x) \, , \ c_m = \xi_+(y) \, .$$

Since

$$\{d_{t_m}(b_m, u_m)\}_{m\in\mathbb{N}} = \{d_{t_m}(\xi_+(x), \xi_-(x))\}_{m\in\mathbb{N}} \, ,$$

is bounded by Lemma 5.5 and $\{K(t_m)\}_{m\in\mathbb{N}}$ is bounded by Proposition 5.4, we get that

$$\lim_{m\to\infty} \xi_+(x_m) = \xi_+(x) \, .$$

This proves that ξ_+ is left-continuous. $\qquad\qquad\qquad\qquad\qquad\square$

As a consequence we obtain

Proposition 5.7. *Let $\rho\colon \pi_1(S) \to \mathsf{G}$ be a positive representation and ξ the positive ρ-invariant boundary map from a $\pi_1(S)$-invariant dense subset of $\partial_\infty \pi_1(S)$ to $\mathbf{F}_\Theta$. Then ξ extends to a ρ-equivariant positive continuous map.*

5.4. **The Anosov property.** We are now in the position that we can prove Theorem B in the introduction. More precisely we show

Proposition 5.8. *Let G admit a Θ-positive structure. Let $\rho\colon \pi_1(S) \to \mathsf{G}$ be a positive representation and $\xi\colon \partial_\infty\pi_1(S) \to \mathbf{F}_\Theta$ the ρ-equivariant continuous positive boundary map. Then ρ is Θ-Anosov and its boundary map is ξ.*

Let us start with a more general lemma

Lemma 5.9. *Let $\{b_m^0\}_{m\in\mathbb{N}}$ and $\{b_m^1\}_{m\in\mathbb{N}}$ be two sequences of elements of $\mathbf{F}_\Theta$ converging to c. Let (a_0, a_0, c, d_1, d_0) be positive quintuple of points in $\mathbf{F}_\Theta$ so that*

$$(a_0, d_0, b_m^0, b_m^1, d_1, a_1)$$

is a positive configuration. Let V_m be the unique diamond with extremities (b_m^0, b_m^1) so that V_m is a subset of the diamond $V_{d_0}(d_0, d_1)$. Then

$$\lim_{m\to\infty} V_m = \{c\} \, .$$

Proof. Let z_m so that z_m belongs to V_m, we want to prove that

$$\lim_{m\to\infty} z_m = c \, .$$

Thanks to Corollary 3.9, we can extract a subsequence so that

$$\lim_{m\to\infty} z_m = p \, ,$$

with p in $\overline{V(d_0, d_1)}$, where $V(d_0, d_1)$ is the diamond with extremities d_0 and d_1 included in $V_{d_0}(a_0, a_1)$. In particular p belongs to $\Omega_{a_0} \cap \Omega_{a_1}$. It follows that

$$z \in \overline{V_{d_0}(a_0, c)} \cap \overline{V_{d_1}(a_1, c)} \cap \Omega_a \, .$$

Let $V = V_{d_0}(a_0, c)$, and recall that by Lemma 2.7,

$$V_{d_1}(a_1, c) \subset V^* .$$

Finally remark that

$$\overline{V} \cap \Omega = \overline{\mathsf{N}} \cdot c , \overline{V^*} \cap \Omega = \overline{\mathsf{N}^{-1}} \cdot c ,$$

for some semigroup N in U, which satisfies $\mathsf{N} = \exp(C)$ where C is salient (see Section 2.2.1). Thus

$$\mathsf{N} \cap \overline{\mathsf{N}^{-1}} = \{\mathrm{id}\} .$$

Thus $z = c$, which is what we wanted to prove.   □

Proof. Let us choose an hyperbolization of S which defines a $\pi_1(S)$-invariant cross ratio on $\partial_\infty \pi_1(S)$. Let us also define an orientation on $\partial_\infty \pi_1(S)$. For any oriented positive triple $t = (x, y, z)$, let us consider the harmonic (with respect to the cross ratio) quadruple $q(s) = (x, y, z, w(s))$. Let then

$$Y_t := V_{\xi(z)}(\xi(y), \xi(w(s))) .$$

By construction Y_t is an open neighborhood of $\xi(z)$. Moreover if (x, y_1, y_0, z) is an oriented positive quadruple,

$$Y_{x, y_0, z} \subset Y_{x, y_1, z} . \tag{19}$$

Finally, since ξ is continuous , by Lemma 5.9

$$\lim_{y \to z} Y_{x, y, z} = \{\xi(z)\} . \tag{20}$$

We now show the Anosov property from Assertion (20). Let us spell out the details. Recall that we have chosen a uniformisation of the surface. Let us now identify the space of triples in the boundary at infinity with the unit tangent bundle of the universal cover $\mathsf{U}\mathbf{H}^2$. Let $\{\phi_s\}_{s \in \mathbb{R}}$ be the geodesic flow on $\mathsf{U}\mathbf{H}^2$. Let $\mathcal{F}$ be the trivial bundle $\mathbf{F}_\Theta \times \mathsf{U}\mathbf{H}^2$. The action of $\pi_1(S)$ on $\mathsf{U}\mathbf{H}^2$ and $\mathbf{F}_\Theta$ — through ρ — gives rise to an action of $\pi_1(S)$ on $\mathcal{F}$.

Let $\mathcal{U}$ be the subbundle with open fibers given by

$$\mathcal{U} = \{(x, v) \in \mathcal{F} \mid v \in \mathsf{U}\mathbf{H}^2, \ x \in Y_v\} .$$

The bundle $\mathcal{U}$ is invariant by the $\pi_1(S)$-action, moreover it has a canonical section σ_0 given by

$$\sigma_0(x, y, z) = \xi(z) .$$

Let us lift the flow $\{\phi_s\}_{s \in \mathbb{R}}$ to a flow $\{\Phi_s\}_{s \in \mathbb{R}}$ on $\mathcal{F}$ acting trivially on the first factor. By assertion (19), for all positive s

$$\Phi_{-s}(\mathcal{U}) \subset \mathcal{U} .$$

Moreover the section σ_0 is invariant by $\{\Phi_s\}_{s\in\mathbb{R}}$.

The diamond metric g_t and the diamond distance d_t on each Y_t gives a metric on each fiber of $\mathcal{U}$ which depends continuously on the base and is equivariant under the action of $\pi_1(S)$.

For any R, let $\mathcal{U}(R)$ the neigbourhood of the image of the section σ_0, given by

$$\mathcal{U}(R) = \{(x,v) \in \mathcal{F} \mid v \in \mathsf{UH}^2, \ d_v(x,\sigma_0(v)) \leqslant R\} \ .$$

It now follows for assertion (20) and Proposition 4.11, that for every u in $\mathsf{UH}^2/\pi_1(S)$, there is s_u so that, for all (x,u) in $\mathcal{U}(R)$

$$\text{for all } s \geqslant s_u, \ g_{\Phi_s(x,u)} \circ \mathsf{T}_{(x,u)}\Phi_{-s} \leqslant \frac{1}{2}g_{(x,u)} \quad . \tag{21}$$

Let now s be the real valued function on UH^2 defined by

$$s(u) = \inf\{s_u \mid s_u \text{ satisfies assertion (21) on } \mathcal{U}(R)\} \ .$$

The function $u \mapsto s(u)$ is upper semicontinuous and invariant under the action of $\pi_1(S)$. Thus by compactness of $\mathsf{UH}^2/\pi_1(S)$ the function has an upper bound s_0. Then, for all s greater than s_0, for all w in $\mathcal{U}(R)$

$$\Phi_s^* g \leqslant \frac{1}{2}g \ ,$$

on $\mathcal{U}(R)$. In other words, the action of $\{\Phi_{-s}\}_{s\in\mathbb{R}}$ is contracting on $\mathcal{U}$ and σ_0 is an invariant section.

Thus ρ is $\mathbf{F}_\Theta$-Anosov and $z \mapsto \xi(z)$ is the limit curve. $\qquad\square$

Now Corollary C in the introduction follows now directly from the openness of the set of Θ-Anosov representation.

Remark 5.10. *The definition of Θ-positive representations can be made in more generality for non-elementary word hyperbolic group Γ whose boundary admits a cyclic ordering. This holds if Γ is a surface group, but also if Γ is virtually free. For example, an appropriate extension of the arguments in this section show that a representation of Γ is Θ-Anosov if it admits a ρ-equivariant positive boundary map $\xi\colon \partial_\infty\Gamma \to \mathbf{F}_\Theta$.*

6. Closedness

In this section we consider the space $\mathrm{Hom}^*(\pi_1(S), \mathsf{G})$ of homomorphisms from $\pi_1(S)$ to G, which (even when restricted to a finite index subgroup) do not factor through a proper parabolic subgroup of G. We show that the set of Θ-positive representations $\mathrm{Hom}_{\Theta-\mathrm{pos}}(\pi_1(S), \mathsf{G})$ is an open and closed subset of $\mathrm{Hom}^*(\pi_1(S), \mathsf{G})$, hence a union of connected components.

Proposition 6.1. *The set of Θ-positive representations is a subset of* $\mathrm{Hom}^*(\pi_1(S), \mathsf{G})$.

Proof. Let us first note that since the centralizer of a positive triple is compact, the centralizer of a positive representation is compact as well. Let $\rho\colon \pi_1(S) \to \mathsf{G}$ be a positive representation with ρ-equivariant positive boundary map $\xi\colon \partial_\infty \pi_1(S) \to \mathbf{F}_\Theta$. Then ρ is Θ-Anosov with boundary map ξ. This remains true when restricting the representation to a finite index subgroup. To argue by contradiction we can thus assume that without loss of generality $\rho(\pi_1(S))$ is contained in a proper parabolic subgroup of G. We consider the semi-simplification ρ^{ss} of ρ, which is its projection to a Levi factor of the parabolic subgroup. Then by [17, Proposition 2.39] the semi-simplification is Θ-Anosov with ρ^{ss}-equivariant boundary map ξ^{ss}. The G-orbit of ρ^{ss} in $\mathrm{Hom}(\pi_1(S), \mathsf{G})$ is the unique closed orbit in the closure of the G-orbit of ρ; there exists thus a path $g_t \in \mathsf{G}$ such that ρ^{ss} is the limit of $g_t \rho g_t^{-1}$. Since the boundary map ξ^{ss} is transverse, Lemma 3.5 implies that the boundary map ξ^{ss} is also positive as well. But this is a contradiction because the centralizer of ρ^{ss} in G contains the center of the Levi factor of the parabolic subgroup which is non-compact. $\quad\square$

By a classical result of Borel and Tits [4, Corollaire 3.3] (proved also by Morozov [31] in characteristic zero), the set $\mathrm{Hom}^*(\pi_1(S), \mathsf{G})$ is contained in the set of reductive homomorphisms, *i.e.* representations $\rho\colon \pi_1(S) \to \mathsf{G}$, whose Zariski closure is reductive. Thus a direct consequence of Theorem 6.1 is

Corollary 6.2. *Let $\rho\colon \pi_1(S) \to \mathsf{G}$ be a Θ-positive representation, then the Zariski closure of $\rho(\pi_1(S))$ is reductive.*

We expect that the list of possible Zariski closures of Θ-positive representations is indeed restrictive. Classifications of the Zariski closures for maximal representations were given in [9, 10, 23, 24] and for Hitchin representations in [18, 34].

Since the set of Θ-positive representations is open in $\mathrm{Hom}(\pi_1(S), \mathsf{G})$ by Corollary C, it is also open in $\mathrm{Hom}^*(\pi_1(S), \mathsf{G})$

We will now show

Theorem 6.3. *The set of Θ-positive homomorphisms is closed in the set* $\mathrm{Hom}^*(\pi_1(S), \mathsf{G})$.

This implies Theorem D in the introduction. We will first prove the following proposition of independent interest

Proposition 6.4. *Let $\{\rho_m\}_{m\in\mathbb{N}}$ be a sequence of Θ-positive representations converging to a representation ρ_∞. Let ξ_m be the limit curve of ρ_m. Assume*

that we can find x_0 and y_0 in $\partial_\infty \pi_1(S)$ so that $\{\xi_m(x_0), \xi_m(x_0)\}_{m \in \mathbb{N}}$ converges to a transverse pair, then ρ_∞ is positive.

6.1. **Proof of proposition 6.4.** We fix a countable set A in $\partial_\infty \pi_1(S)$, invariant by $\pi_1(S)$ and containing x_0 and y_0. We may now assume, by the Cantor diagonal argument, that $\{\xi_m|_A\}_{m \in \mathbb{N}}$ converges simply to a map ξ_∞ from A to $\mathbf{F}_\Theta$. By hypothesis $\xi_\infty(x_0)$ and $\xi_\infty(y_0)$ are transverse.

For any pair of distinct points x, y in A, let c so that (c, x, y) are pairwise distinct and

$$W_\infty(x, y) := \lim_{n \to \infty} \overline{V^{\xi_n(c)}(\xi_n(x), \xi_n(y))} \, ,$$

the convergence being for the Hausdorff topology, and using again the Cantor diagonal extraction, we can assume that all those sequences converge of all (x, y) distinct in A. Observe that $W_\infty(x, y)$ only depends on x, y and the choice of a connected component of $\partial_\infty \pi_1(S) \setminus \{x, y\}$. Also, the following equivariance property holds: $\rho_\infty(\gamma) W_\infty(x, y) = W_\infty(\gamma \cdot x, \gamma \cdot y)$.

Lemma 6.5. *Assume that $\xi_\infty(x)$ and $\xi_\infty(y)$ are transverse then $W_\infty(x, y)$ is a closure of a diamond with vertices x and y and is Zariski dense.*

Proof. Since $\xi_\infty(x)$ and $\xi_\infty(y)$ are transverse, $W_\infty(x, y)$ is the closure of a diamond (see Proposition 3.8). It thus contains an open set, and in particular is Zariski dense. $\qquad\square$

Lemma 6.6. *For every pairs of distinct points (x, y) and (z, t) in A, one has*

$$\overline{W_\infty(x, y)}^Z = \overline{W_\infty(z, t)}^Z$$

where $\overline{M}^Z$ denotes the Zariski closure of a set M. In particular, for all distinct x and y, $W_\infty(x, y)$ is Zariski dense.

Observe that only the last assertion depends on the assumption that $\xi_\infty(x_0)$ and $\xi_\infty(y_0)$ are transverse.

Proof. We shall use freely the following fact. If γ is an algebraic automorphism of a variety V, if B is a Zariski closed subset so that $\gamma(B) \subset B$ then $\gamma(B) = B$.

We first prove that if $[u, v] \subset [w, s]$, then we have

$$\overline{W_\infty(u, v)}^Z = \overline{W_\infty(w, s)}^Z \, . \tag{22}$$

Let us prove this fact. We can always find an element γ of $\pi_1(S)$ such that

$$\gamma[w, s] \subset [u, v] \, .$$

Thus
$$\rho_\infty(\gamma)\left(\overline{W_\infty(w,s)}^Z\right) \subset \overline{W_\infty(u,v)}^Z \subset \overline{W_\infty(w,s)}^Z .$$
Thus by the initial observation we get that
$$\overline{W_\infty(w,s)}^Z \subset \overline{W_\infty(u,v)}^Z \subset \overline{W_\infty(w,s)}^Z ,$$
and thus the assertion (22) follows. Let now γ in $\pi_1(S)$ so that
$$\gamma[x,y] \subset [x,y] , \quad \gamma[x,y] \cup [z,t] \neq \partial_\infty\pi_1(S) .$$
We can then find distinct points u and v so that
$$(\gamma[x,y] \cup [z,t]) \subset [u,v] .$$
Thus, applying thrice assertion (22), we have
$$\overline{W_\infty(x,y)}^Z = \overline{W_\infty(\gamma\cdot x, \gamma\cdot y)}^Z = \overline{W_\infty(u,v)}^Z = \overline{W_\infty(z,t)}^Z .$$
The last assertion follows from the fact that $\xi_\infty(x_0)$ and $\xi_\infty(y_0)$ are transverse and thus $W_\infty(x_0,y_0)$ is Zariski dense by lemma 6.5.   □

We are now in the position to show that ρ_∞ is Θ-positive. This follows from the following proposition:

Proposition 6.7. *For any triple of points (x,y,z), the triple*
$$(\xi_\infty(x), \xi_\infty(y), \xi_\infty(z)),$$
consists of pairwise transverse points. For any positive quadruple of points (x,y,z,w), the quadruple
$$(\xi_\infty(x), \xi_\infty(y), \xi_\infty(z), \xi_\infty(w)) ,$$
is positive.

Proof. Let x, y, and z be a triple of pairwise distinct points in $\partial_\infty\pi_1(S)$. Let us denote for simplicity $x_n = \xi_n(x)$, $y_n = \xi_n(y)$ and $z_n = \xi_n(z)$ for n in $\mathbb{N} \cup \{\infty\}$. We choose the diamonds by letting
$$V_n^0 = V^{z_n}(x_n,y_n), \quad V_n^1 = V^{y_n}(x_n,z_n) \quad V_n^2 = V^{y_n}(z_n,y_n).$$
Since $W_\infty(x,y)$, $W_\infty(x,y)$ and $W_\infty(x,y)$ are Zariski dense, we can pick three points a, b, and c so that

(1) $a \in W_\infty(x,y)$, $b \in W_\infty(y,z)$, $c \in W_\infty(z,x)$,
(2) a,b,c are pairwise transverse,
(3) any point in $\{a,b,c\}$ is transverse to any point in $\{x_\infty, y_\infty, z_\infty\}$.

Let now pick sequences $\{a_m\}_{m\in\mathbb{N}}$, $\{b_m\}_{m\in\mathbb{N}}$, and $\{c_m\}_{m\in\mathbb{N}}$ in V_n^0, V_n^1, and V_n^2 respectively and converging to a, b, and c respectively.

We will now apply the necklace property several times. By Proposition 3.2, (a_n, b_n, c_n) is a positive triple and since (a,b,c) are pairwise transverse it follows that (a,b,c) is a positive triple.

Then, since x_n belongs to $V^{b_n}(a_n, c_n)$, it follows that x_∞ belongs to $\overline{V^b(a, c)}$. Since x_∞ is transverse to both a and c, x_∞ belongs to $V^b(a, c)$. Symmetrically y_∞ belongs to $V_c^*(a, b)$, z_∞ belongs to $V^a(c, b)$. Applying Proposition 3.2 again, $(x_\infty, y_\infty, z_\infty)$ is a positive triple.

The fact that the quadruple $(x_\infty, y_\infty, z_\infty, w_\infty)$ is positive, now follows from Proposition 3.1. □

6.2. **Proof of Theorem 6.3.** We consider a sequence $\{\rho_m\}_{m\in\mathbb{N}}$ of Θ-positive representations converging to a representation ρ_∞. Let $\{\xi_m\}_{m\in\mathbb{N}}$ be the corresponding sequence of positive limit maps.

We fix a countable set A in $\partial_\infty \pi_1(S)$, invariant by $\pi_1(S)$. We may now assume applying the Cantor diagonal argument, that $\{\xi|_A\}_{m\in\mathbb{N}}$ converges simply to a map ξ_∞ from A to $\mathbf{F}_\Theta$. We then have two possible cases:

(1) Either there exists $x, y \in A$ such that $\xi_\infty(x)$ is transverse to $\xi_\infty(y)$,
(2) Or for all $x, y \in A$ the flags $\xi_\infty(x)$ and $\xi_\infty(y)$ are not transverse.

If we are in case (2), Theorem A.1 applied to $\mathsf{H} = \rho_\infty(\pi_1(S))$ implies that $\rho_\infty(\pi_1(S))$ is contained in a proper parabolic subgroup of G. This contradicts that ρ_∞ lies in $\mathrm{Hom}^*(\pi_1(S), \mathsf{G})$. Thus case (2) is not possible and we are in case (1). We can apply Proposition 6.4 to obtain that ρ_∞ is positive.

7. Connected Components

In this section, we prove that Theorem D implies the existence of components consisting solely of discrete faithful representations. for any simple Lie group G admitting a Θ-positive structure. More precisely, we show that the unions of the connected components $\mathcal{P}_e(\Sigma, \mathsf{G})$ of $\mathrm{Rep}^+(\pi_1(S), \mathsf{G})$, (where Σ is a Riemann surface modelled on S) introduced in [6] using methods from the theory of Higgs bundles consist entirely of Θ-positive representations. In particular they consist solely of discrete faithful representations.

Let us recall that for real split Lie groups G, the Hitchin component was originally defined by Hitchin as the image of the Hitchin section Φ which assigns to a tuple of holomorphic differentials on a Riemann surface Σ a G-Higgs bundle on Σ. Let us denote the image of Φ by $\mathcal{P}(\Sigma, \mathsf{G})$. Through the non-abelian Hodge correspondence the set $\mathcal{P}(\Sigma, \mathsf{G})$ corresponds to a subset of the representation variety $\mathrm{Rep}^+(\pi_1(S), \mathsf{G})$, which we denote by the same symbol. Hitchin showed that $\mathcal{P}(\Sigma, \mathsf{G})$ is open and closed and the map Φ gives an explicit parameterization of the components of $\mathcal{P}(\Sigma, \mathsf{G})$. In the case of maximal

representations a similar but more complicated parameterization of the space of maximal representations was obtained in several works by Bradlow, García-Prada and Gothen. For simple Lie groups admitting a Θ-positive structure, the authors of [2] (see also [6] for classical groups and [13] for $\mathsf{Sp}(2n)$) in a similar way define subsets $\mathcal{P}_e(\Sigma, \mathsf{G})$ of the moduli space of G-Higgs bundles by giving precise parameterizations of the set. They prove that the sets $\mathcal{P}_e(\Sigma, \mathsf{G})$ are open and closed in $\mathrm{Rep}^+(\pi_1(S), \mathsf{G})$. They further prove that all representations in $\mathcal{P}_e(\Sigma, \mathsf{G})$ have compact centralizer and do not factor through a proper parabolic subgroup. This implies that $\mathcal{P}_e(\Sigma, \mathsf{G})$ is contained in $\mathrm{Rep}^*(\pi_1(S), \mathsf{G})$. They further show that these sets $\mathcal{P}_e(\Sigma, \mathsf{G})$ contain an open set of Θ-positive representations.

Thus Theorem D implies

Theorem 7.1. *The open and closed subsets $\mathcal{P}_e(\Sigma, \mathsf{G})$ in $\mathrm{Hom}^+(\pi_1(S), \mathsf{G})/\mathsf{G}$ consist entirely of Θ-positive representations.*

Proof. It is an immediate consequence of Theorem D and the fact that $\mathcal{P}_e(\Sigma, \mathsf{G}) \subset \mathrm{Rep}^*(\pi_1(S), \mathsf{G})$ that if a connected component of $\mathcal{P}_e(\Sigma, \mathsf{G}) \subset \mathrm{Rep}^*(\pi_1(S), \mathsf{G})$ contains a Θ-positive representation, then the entire component of $\mathcal{P}_e(\Sigma, \mathsf{G}) \subset \mathrm{Rep}^*(\pi_1(S), \mathsf{G})$ consists of Θ-positive representations. Here is one way to construct a Θ-positive representations. Consider an embedding of $\mathsf{SL}_2(\mathbb{R})$, such that the induced map from $\mathsf{PSL}_2(\mathbb{R})$ to $\mathbf{F}_\Theta$ is a positive circle, then the corresponding Fuchsian representation is positive. These Fuchsian representations can now in addition be twisted by a representation of $\pi_1(S)$ into the centralizer of this $\mathsf{SL}_2(\mathbb{R})$ in G. This is called a twisted positive Fuchsian representation. We call a component of $\mathcal{P}_e(\Sigma, \mathsf{G})$ standard if it contains a twisted positive Fuchsian representation.

When G is not locally isomorphic to $\mathsf{Sp}_4(\mathbb{R})$, or $\mathsf{SO}(p, p+1)$, every component of $\mathcal{P}_e(\Sigma, \mathsf{G})$ is standard [6]. For $\mathsf{Sp}_4(\mathbb{R})$ there exist exceptional connected components which do not contain any twisted positive Fuchsian representation [16], but embedding $\mathsf{Sp}_4(\mathbb{R})$ diagonally into $\mathsf{Sp}_8(\mathbb{R})$ these exceptional components are sent into a standard component for $\mathsf{Sp}_8(\mathbb{R})$. Thus the expectional components for $\mathsf{Sp}_4(\mathbb{R})$ consist entirely Θ-positive representations. For $\mathsf{SO}(p, p+1)$ there also exist exceptional connected components which do not contain any twisted positive Fuchsian representation, but embedding $\mathsf{SO}(p, p+1) \to \mathsf{SO}(p, p+2)$, these components are sent to standard components for $\mathsf{SO}(p, p+2)$ and thus they consist also entirely of Θ-positive representations [6]. $\qquad \square$

As a corollary we obtain

Corollary 7.2. *The subsets $\mathcal{P}_e(\Sigma, \mathsf{G})$ of $\mathrm{Hom}^+(\pi_1(S), \mathsf{G})/\mathsf{G}$ only contain discrete faithful representations.*

We expect that the sets $\mathcal{P}_e(\Sigma, \mathsf{G})$ coincide precisely with the set of Θ-positive representations. We also wonder whether positivity is the only phenomenon for representation varieties of surface groups that leads to entire components made of discrete faithful representations.

Appendix A. Transversality in the space of flags

Our goal is to prove the following result of independent interest:

Theorem A.1. *Let P be a parabolic and P^* an opposite parabolic. Let H be a subgroup of G. Assume that no element in a given orbit of H in G/P is transverse to P^*. Then there is a finite index subgroup of H which is included in some parabolic.*

This result is also sketched in [12, Lemma 7.9], but we prefer to give an independent and simple proof based on Tits systems.

A.0.1. *Bruhat cell decomposition.* We use freely the following standard facts on Bruhat decompositions, see for example [26, 35].

Let G be a semisimple Lie group over $\mathbb{R}$. Let B be a minimal parabolic subgroup, A a maximal split real torus, W the Weyl group, Δ a choice of simple roots, that we identify with the corresponding reflections on the Lie algebra of A. Then we have the *Bruhat cell decomposition*

$$\mathsf{G} = \bigsqcup_{w \in \mathsf{W}} \mathsf{B}w\mathsf{B} .$$

Moreover, let us denote by ℓ the word metric on W coming from the generators Δ, calling $\ell(w)$ the *length* of w. Let $C(w)$ the Zariski closure of $\mathsf{B}w\mathsf{B}$ then

$$C(w) = \bigsqcup_{h \in W(w)} \mathsf{B}w\mathsf{B} ,$$

where $W(w)$ is a subset of W containing w so that for any h in $W(w)$ different from w,

$$\ell(h) < \ell(w) .$$

There exists a unique element w_0 of longest length in W and the cell $\mathsf{B}w_0\mathsf{B}$ is precisely the union of all subgroups B' conjugate to B and transverse to it.

If J is a proper subset of Δ, let W_J be the subgroup of W generated by the reflections in J, then

$$\mathsf{P}_J := \bigsqcup_{w \in \mathsf{W}_J} \mathsf{B}w\mathsf{B}\,,$$

is a *standard parabolic subgroup* of G.

We shall need the following lemma which is true for general Coxeter groups by [33]. For Weyl groups, it follows from [5, Chap. VI, §1, Exercise 17-b].

Lemma A.2. *Every involution in W different from w_0, is conjugated to an involution of W_J for some proper subset J of Δ.*

Proof of Theorem A.1. After conjugation we may assume that the parabolic P contains the minimal parabolic subgroup B. Let us first notice that if π be the projection from G/B to G/P and π^* be the projection on G/P^* where P^* is the opposite parabolic. Then if x and y in G/B are so that $\pi(x)$ and $\pi^*(y)$ are not transverse, then x and y are not transverse. Thus it is enough to prove the proposition when $\mathsf{P} = \mathsf{B}$.

Let Z be the Zariski closure of H and Z_0 the irreducible component (as an algebraic variety) of Z containing the identity. Recall that Z_0 is a finite index subgroup of Z [3, Paragraph 1.5]. It is thus enough to prove the result for H an irreducible (as an algebraic variety) algebraic group of G.

We will use the algebraic irreducibility for the following fact, which follows at once from the definition of irreducibility as an algebraic variety:

$(*)$ *If H is included in a finite union of $C(w)$ for w in a subset A of W then H is included in $C(a)$ for some a in A.*

The hypothesis now translates into the fact that H does not intersect $\mathsf{B}w_0\mathsf{B}$, where w_0 is the longest element in W. Thus by $(*)$ and the Bruhat cell decomposition H is included in $C(w)$ for w in W different from w_0. Let s be the element in W with the shortest length amongst the w so that $\mathsf{H} \subset C(w)$. Since H is invariant under taking the inverse, H is also included in $C(s^{-1})$. We now show by contradiction that s is an involution. If s is different from s^{-1}, then

$$\mathsf{H} \subset C(s) \cap C(s^{-1}) = \bigcup_{a \in A_s} C(a)\,, \tag{23}$$

where for all a in A_s, $\ell(a) < \ell(s)$. Using $(*)$ again, equation (23) implies that $\mathsf{H} \subset C(a)$ for some a with $\ell(a) < \ell(s)$, a contradiction.

Thus s is an involution different from w_0. Then Lemma A.2 implies that $C(s)$ – hence H – is included in a parabolic. The result is proved.   □

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