POSITIVITY AND REPRESENTATIONS OF SURFACE GROUPS

OLIVIER GUICHARD, FRANÇOIS LABOURIE, AND ANNA WIENHARD

Abstract. In [23, 25] Guichard and Wienhard introduced the notion of Θ-positivity, a generalization of Lusztig's total positivity to real Lie groups that are not necessarily split.

Based on this notion, we introduce in this paper Θ-*positive representations* of surface groups. We prove that Θ-positive representations of closed surface groups are Θ-Anosov. This implies that Θ-positive representations are discrete and faithful and that the set of Θ-positive representations is open in the representation variety.

We further establish important properties on limits of Θ-positive representations, proving that the set of Θ-positive representations are closed in the set of representations containing a Θ-proximal element. This is used in [3] to prove the closedness of the set of Θ-positive representations.

1. Introduction

An important feature of Teichmüller space, seen as a connected component of the space of representations of the fundamental group of a closed connected orientable surface S of genus at least 2 in $\mathsf{PSL}_2(\mathbb{R})$, is that it consists entirely of representations which are discrete and faithful. These representations are moreover quasi-isometries from $\pi_1(S)$ to $\mathsf{PSL}_2(\mathbb{R})$. This situation does not extend to the case of *any* semisimple group, notably for simply connected complex ones, where the representation variety is irreducible as an algebraic variety [35].

O.G. and F.L. acknowledge the funding of the ANR grant DynGeo ANR-11-BS01-013 and of the Institut Universitaire de France. A.W. acknowledges funding by the DFG, Germanys Excellence Strategy EXC-2181/1 - 390900948 (the Heidelberg STRUCTURES Cluster of Excellence), by the ERC under ERC-Consolidator grant 614733 and ERC-Advanced Grant 101018839. She thanks the Klaus Tschira Foundation, the Hector Fellow Academy, and the Clay Foundation for support. F.L., O.G., and A.W. acknowledge support from N.S.F grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties". F.L. and A.W. acknowledge support from N.S.F Grant No. 1440140 while in residence in Fall 2019, at the MSRI in Berkeley, F.L. acknowledges funding by the University of California at Berkeley and by the ERC under ERC-Advanced grant 101095722.

However, this phenomenon was shown to happen for *some* groups of higher rank. Two families of representation varieties of the fundamental group of S have been singled out as they contain connected components consisting entirely of discrete and faithful representations:

- Hitchin components when G is a real split group [17] (the Hitchin components were defined in [28]; the case of $\mathsf{SL}_3(\mathbb{R})$ was treated in [13] and the case of $\mathsf{SL}_n(\mathbb{R})$ in [30]),
- spaces of maximal representations, which are defined when G is Hermitian [12] ([19] proves the case of $\mathsf{SL}_2(\mathbb{R})$, [10] studies maximal representations into symplectic groups).

When G is $\mathsf{PSL}_2(\mathbb{R})$, the Hitchin component and the space of maximal representations both agree with the Teichmüller space.

The study of these two families is closely related to the theory of Anosov representations as introduced in [30, 24]. Being Anosov is a notion defined for any reductive Lie group and with respect to a choice of a parabolic subgroup. Every Anosov representation is in particular faithful, discrete and a quasi-isometric embedding [31, 24, 15].

Representations in the Hitchin components as well as maximal representations can be characterized in terms of equivariant curves from the boundary at infinity of $\pi_1(S)$ into an appropriate flag variety, which preserve some positivity. Lusztig's total positivity [33] was systematically used by Fock and Goncharov [17] in their study of moduli spaces of local systems which leads, together with insights of [30], to the characterization of Hitchin components in these terms. The case of maximal representations is based on the maximality of the Maslov index and related to Lie semigroups in G [12].

In [25, 23], Guichard and Wienhard introduced the notion of Θ-*positivity*. This notion extends Lusztig's total positivity to generalized flag manifolds associated with the parabolic defined by a set Θ of simple roots.

They classified all possible simple Lie groups that admit a Θ-positive structure. These include real split Lie groups, for which Θ-positivity is Lusztig's total positivity, Hermitian Lie groups of tube type, where Θ-positivity is given by the maximality of the Maslov index, but also two other families of Lie groups, namely the family of classical groups $\mathsf{SO}(p,q)$ —with $p \neq q$— and an exceptional family consisting of the real rank 4 form of F_4, E_6, E_7, and E_8 respectively. They conjectured that Θ-positivity provides the right underlying algebraic structure for the existence of components made solely of discrete and faithful representations [37, Conjecture 19].

A Θ-positive structure on G implies in particular the existence of a positive semigroup in the unipotent radical of the parabolic group P_Θ, which then leads to the notions of *positive triples* and *positive quadruples* (as well as positive tuples) in the flag variety $\mathbf{F}_\Theta \simeq \mathsf{G}/\mathsf{P}_\Theta$. In the basic example of $\mathsf{G} = \mathsf{PSL}_2(\mathbb{R})$ and $\mathbf{F}_\Theta = \mathbf{P}^1(\mathbb{R})$, a triple is positive if it consists of pairwise distinct points and a quadruple is positive if it is cyclically ordered.

Let us give a geometric picture of positivity in the flag variety $\mathbf{F}_\Theta$. For this let a and b be two points in $\mathbf{F}_\Theta$ which are transverse to each other. Then Θ-positivity provides the existence of preferred connected components of the set of all points in $\mathbf{F}_\Theta$ that are transverse to both a and b. These preferred components are called *diamonds* (with extremities a and b). They are several, at least two, disjoint diamonds with given extremities. The semigroup property alluded to before translates into a nesting property of diamonds: if c is a point in a diamond $V(a, b)$ with extremities a and b, then there is exactly one diamond $V(c, b)$ (with extremities c and b) included in $V(a, b)$. These nesting properties of diamonds play an important role in our arguments.

If a and b are transverse, and c belongs to a diamond with extremities a and b, we say the triple (a, b, c) is *positive*. Similarly, one can define positive quadruples using configurations of diamonds (see Figure 2 and Definition 2.10). We show in Section 3 that being positive is invariant under all permutations for a triple, and invariant under the dihedral group for a quadruple.

We define a map ξ from a cyclically ordered set A to $\mathbf{F}_\Theta$ to be *positive* if ξ maps triples of pairwise distinct points to positive triples and cyclically ordered quadruples to positive quadruples.

This allows us to define the notion of a Θ-*positive representation*: A representation $\rho\colon \pi_1(S) \to \mathsf{G}$ is Θ-*positive* if there exist a non-empty subset A of $\partial_\infty \pi_1(S)$, invariant by $\pi_1(S)$, and a ρ-equivariant positive boundary map from A to $\mathbf{F}_\Theta$.

We prove

Theorem A. *Let G be a semi-simple Lie group that admits a Θ-positive structure. Let ρ be a Θ-positive representation from $\pi_1(S)$ to G.*

Then ρ is a Θ-Anosov representation.

As a direct consequence we obtain that a Θ-positive representation is faithful with discrete image, its orbit map into the symmetric space is a quasi-isometric embedding and the boundary map extends uniquely to a Hölder map [31, 24, 15, 9].

Theorem A provides a general proof of the Anosov property for all Hitchin representations and all maximal representations. This is especially relevant for the case of the Hitchin component of $\mathsf{SO}(p,p)$ and of F_4, E_6, E_7, and E_8 and the case of maximal representations into the exceptional Hermitian Lie group of tube type, which cannot be tightly embedded into $\mathsf{Sp}_{2n}(\mathbb{R})$ [11, 26, 27]. The Anosov property was established for all maximal representations which tightly embed into $\mathsf{Sp}_{2n}(\mathbb{R})$ in [10] and for the Hitchin component of $\mathsf{SL}_n(\mathbb{R})$ in [30], from which follows the Anosov property for the Hitchin components of $\mathsf{Sp}_{2n}(\mathbb{R})$, $\mathsf{SO}(p,p+1)$, and G_2. Fock and Goncharov established a related key property: for every Hitchin representation, there exists a continuous, transverse (and positive) boundary map [17, Theorem 7.2]; from this, the Anosov property can be established for Zariski dense Hitchin representations using for example [24, Theorem 4.11].

Using their work on amalgamation of Anosov representations, Dey and Kapovich [16, Section 6] established also the Anosov property for all Hitchin components for all real split groups.

Using the openness of the set of Θ-Anosov representations, a further consequence of Theorem A is the following

Corollary B. *The set of Θ-positive representations* $\mathrm{Hom}_{\Theta-\mathrm{pos}}(\pi_1(S),\mathsf{G})$ *is an open subset in the set of all homomorphisms* $\mathrm{Hom}(\pi_1(S),\mathsf{G})$.

To show that the set of Θ-positive representation indeed give rise to higher Teichmüller spaces it remains to prove that the set of Θ-positive representations is closed. We establish essential steps in this direction. For this we consider the set $\mathrm{Hom}^{\Theta}(\pi_1(S),\mathsf{G})$ of homomorphisms ρ of $\pi_1(S)$ in G such that the image of ρ contains a Θ-loxodromic element (i.e. an element having both attracting and repelling fixed points in the flag variety $\mathbf{F}_\Theta$ associated to Θ). Observe that Proposition 6.3 clarifies the relation with the Zariski closure, and in particular the set $\mathrm{Hom}^{\Theta}(\pi_1(S),\mathsf{G})$ contains representations with Zariski dense images. We establish in Proposition 6.1 that $\mathrm{Hom}_{\Theta-\mathrm{pos}}(\Gamma,\mathsf{G})$ is a subset of $\mathrm{Hom}^{\Theta}(\pi_1(S),\mathsf{G})$. We show

Theorem C. *The set of Θ-positive representations* $\mathrm{Hom}_{\Theta-\mathrm{pos}}(\pi_1(S),\mathsf{G})$ *is a nonempty union of connected components of* $\mathrm{Hom}^{\Theta}(\pi_1(S),\mathsf{G})$.

In the case when G is locally isomorphic to $\mathsf{SO}(p,q)$, $p\leqslant q$, Beyrer and Pozzetti [4] recently proved the *closedness* of the set of Θ-positive Anosov representations in $\mathrm{Hom}(\pi_1(S),\mathsf{G})$, thus by Theorem A also the closedness of the set of Θ-positive representations. They derive this as a consequence of a family of collar lemmas and fine properties of the boundary maps they establish. In [3], together with Beyrer and

Pozzetti, we prove collar lemmas in full generality for Θ-positivity, which in combination with Theorem C establishes that the set of Θ-positive representations is closed in $\mathrm{Hom}(\pi_1(S), \mathsf{G})$. Thus with Corollary B and Theorem A that Θ-positive representations give rise to connected components consisting entirely of discrete and faithful representations.

Note that special Θ-positive representations arise from positive embeddings of $\mathsf{SL}_2(\mathbb{R})$ into G. These positive embeddings of $\mathsf{SL}_2(\mathbb{R})$ can be produced

explicitly using specific "positive" nilpotent element in the Lie algebra of G. They have the property that the embedding induces a positive map from $\mathbf{P}^1(\mathbb{R})$ into $\mathbf{F}_\Theta$. We call the image of such a map a *positive circle*. Every group G admitting a Θ-positive structure contains a special (conjugacy class of) Θ-principal $\mathsf{SL}_2(\mathbb{R})$. The circles associated to this Θ-principal $\mathsf{SL}_2(\mathbb{R})$ play an important role in some of our arguments. Precomposing a positive embedding $\mathsf{SL}_2(\mathbb{R})$ into G with a discrete embedding of $\pi_1(S)$ into $\mathsf{SL}_2(\mathbb{R})$, we obtain a Θ-positive representation.

Recently, Bradlow, Collier, García-Prada, Gothen, and Oliveira [7] developed the theory of magical $\mathfrak{sl}_2$-triples, which is very closely related to the theory of Θ-positivity. In fact a real simple Lie group is associated to a magical $\mathfrak{sl}_2$-triple if and only if it admits a Θ-positive structure. Using methods from the theory of Higgs bundles, they parametrize special connected components $\mathcal{P}_e(S, \mathsf{G})$, called Cayley components. We expect these connected components $\mathcal{P}_e(S, \mathsf{G})$ to consist entirely of Θ-positive representations, and furthermore to coincide with the set of Θ-positive representations. We discuss the relation between $\mathcal{P}_e(S, \mathsf{G})$ and Θ-positive representations in Section 7.

Acknowledgements: We thank Michel Brion, Steve Bradlow, Brian Collier, Beatrice Pozzetti, and Jérémy Toulisse for interesting discussions regarding the topics of this paper. We also thank Nicolas Tholozan and Tengren Zhang for pointing out mistakes in previous versions.

We thank the referees who pointed out some inaccuracies in the previous version.

Outline of the paper: In Section 2, we recall the necessary algebraic material from [25, 23] and introduce the main definitions: diamonds, positive configurations, positive circles and positive maps. In Section 3, we prove three propositions concerning combinatorial properties of configurations, proper inclusion of diamonds and extension of positive maps —some of the properties proved here are also in

[23], but the proofs in the present paper are geometric, while those in [23] are algebraic. In Section 4, we introduce the diamond metric on diamonds and establish its properties. With these preparations we prove Theorem A and Corollary B in Section 5, Theorem C in Section 6. In Section 7 we discuss the connection with the Cayley components introduced in [7] .

Contents

2. Definitions

2.1. **Lie algebra notations.** Let G be a semi-simple group.

The *roots* of G are the nonzero weights under the adjoint action of a Cartan subspace $\mathfrak{a}$ on the Lie algebra $\mathfrak{g}$ of G. They form a root system $\Sigma \subset \mathfrak{a}^*$ (nonreduced in some cases) and the choice of a linear ordering on $\mathfrak{a}^*$ gives rise to the set Σ^+ of positive roots, and to the set Δ of simple roots. The α-weight space will be denoted by $\mathfrak{g}_\alpha \subset \mathfrak{g}$.

The parabolic subgroups of G are the subgroups conjugated to one of the standard parabolic subgroups P_Θ (for Θ varying in the subsets of Δ); namely P_Θ is the normalizer in G of the Lie algebra $\mathfrak{u}_\Theta := \bigoplus_{\alpha \in \Sigma^+ \smallsetminus \mathrm{span}(\Delta \smallsetminus \Theta)} \mathfrak{g}_\alpha$. The unipotent radical of P_Θ is the subgroup $\mathsf{U}_\Theta = \exp(\mathfrak{u}_\Theta)$. A parabolic subgroup is its own normalizer so that the space $\mathbf{F}_\Theta$ of parabolic subgroups conjugated to P_Θ is isomorphic to $\mathsf{G}/\mathsf{P}_\Theta$.

The space $\mathbf{F}_\Theta$ is also naturally G-isomorphic to the G-orbit (for the adjoint action) of $\mathfrak{u}_\Theta$ in the space $\mathbf{L}$ of Lie subalgebras of $\mathfrak{g}$. The group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$ also acts on $\mathbf{L}$ and the actions of G and of $\mathrm{Aut}(\mathfrak{g})$ on this space are related via the adjoint action seen as an homomorphism $\mathsf{G} \to \mathrm{Aut}(\mathfrak{g})$. For ψ in $\mathrm{Aut}(\mathfrak{g})$ and for $\mathfrak{u}$ in the G-orbit of $\mathfrak{u}_\Theta$ (*i.e.* $\mathfrak{u}$ belongs to $\mathbf{F}_\Theta$), the algebra $\psi(\mathfrak{u})$ may not belong to $\mathbf{F}_\Theta$; in fact $\psi(u)$ belongs to $\mathbf{F}_{\psi_*(\Theta)}$ where $\psi_* \colon \Delta \to \Delta$ denotes the action of ψ on the set of simple roots (or on the Dynkin diagram). There is thus a subgroup $\mathrm{Aut}_0(\mathfrak{g})$ of $\mathrm{Aut}(\mathfrak{g})$ that acts (transitively) on $\mathbf{F}_\Theta$. This group $\mathrm{Aut}_0(\mathfrak{g})$ has better transitive properties than G, *e.g.* it will act transitively on the diamonds that are introduced later. We will therefore use several times $\mathrm{Aut}_0(\mathfrak{g})$ instead of G.

Two parabolic subgroups P and P' are called *transverse* or *opposite* if their intersection $\mathsf{P} \cap \mathsf{P}'$ is a reductive subgroup (*i.e.* the unipotent radical of this intersection is trivial); this is equivalent to having $\mathrm{UniRad}(\mathsf{P}) \cap \mathsf{P}' = \{1\}$. In that case, there exists $\Theta \subset \Delta$ such that the pair $(\mathsf{P}, \mathsf{P}')$ is conjugated to $(\mathsf{P}_\Theta, \mathsf{P}_\Theta^{\mathrm{opp}})$ where $\mathsf{P}_\Theta^{\mathrm{opp}}$ is the normalizer of $\bigoplus_{\alpha \in \Sigma^+ \smallsetminus \mathrm{span}(\Delta \smallsetminus \Theta)} \mathfrak{g}_{-\alpha}$. The intersection $\mathsf{L}_\Theta := \mathsf{P}_\Theta \cap \mathsf{P}_\Theta^{\mathrm{opp}}$ is a Levi factor of P_Θ (and of $\mathsf{P}_\Theta^{\mathrm{opp}}$).

We will always work with a parabolic subgroup $\mathsf{P} \simeq \mathsf{P}_\Theta$ such that P_Θ is conjugated to its opposite $\mathsf{P}_\Theta^{\mathrm{opp}}$; in this situation it makes sense to look at transverse elements in $\mathbf{F}_\Theta \simeq \mathsf{G}/\mathsf{P}_\Theta$. In particular we will use the following notation, for x in $\mathbf{F}_\Theta$,

$$
\begin{aligned}
\mathsf{P}_x &:= \mathrm{Stab}(x)\ , \\
\mathsf{U}_x &:= \mathrm{UniRad}(\mathsf{P}_x)\ , \\
\Omega_x &:= \{y \in \mathbf{F}_\Theta \mid y \text{ is transverse to } x\}\ , \\
S_x &:= \mathbf{F}_\Theta \smallsetminus \Omega_x\ .
\end{aligned}
$$

We will sometimes use that, if a and b are transverse points, then $\mathsf{L}_{a,b} := \mathsf{P}_a \cap \mathsf{P}_b$ is a Levi factor of P_a and P_b. Recall that Ω_x is an open orbit of U_x and that S_x is a proper algebraic subvariety of $\mathbf{F}_\Theta$.

Given a point a in $\mathbf{F}_\Theta$, a *unipotent pinning*, or *U-pinning* at a, is an identification s of U_Θ with U_a that exponentiates an isomorphism from $\mathfrak{u}_\Theta$ to $\mathfrak{u}_a$ which itself is induced by the restriction of an automorphism of the Lie algebra $\mathfrak{g}$ (*i.e.* an element of $\mathrm{Aut}_0(\mathfrak{g})$). Observe that there are finitely many U-pinnings up to the action of L_Θ.

2.2. Cones and semigroup.

Definition 2.1. [23, Theorem 12.2] A *positive structure* with respect to $\mathbf{F}_\Theta$ (or a Θ-*positive structure*) is a semigroup N of U_Θ such that, denoting x and y the points of $\mathbf{F}_\Theta$ whose stabilizers are P_Θ and $\mathsf{P}_\Theta^{\mathrm{opp}}$ respectively, $\mathsf{N} \cdot y$ is a connected component of $\Omega_x \cap \Omega_y$.

In this case, N is invariant by conjugation by the connected component L_Θ° of L_Θ and is a sharp semigroup: for any h, k in $\overline{\mathsf{N}}$, if $hk = 1$, then $h = k = 1$ (*i.e.* the only invertible element in $\overline{\mathsf{N}}$ is the identity element).

We shall see that given a and b transverse to each other in $\mathbf{F}_\Theta$ and an identification of U_Θ with U_a (*i.e.* a U-pinning) which sends N to a subgroup N_a of U_a, then $\mathsf{N}_a \cdot b$ is a connected component of $\Omega_a \cap \Omega_b$.

In [23] it is proved that, up to the action of $\mathrm{Aut}_0(\mathfrak{g})$, the semigroup N in the definition is unique.

We first present some conclusions of the construction of the semigroup N that we are going to use in this paper, then concentrate on the notions of *diamonds* and *positive configurations* that play a crucial role in this paper.

2.2.1. *The parametrization of the positive semigroup.* Theorem 4.5 of [25] and Theorem 1.3 of [23] give a precise description of the possible parametrizations of the semigroup N. We recall here the material necessary for our purpose.

Fact 2.2. *There exist $N \geqslant 1$ and C a L_Θ°-invariant cone in $(\mathfrak{u}_\Theta)^N$ such that the map*

$$\begin{aligned} (\mathfrak{u}_\Theta)^N &\longrightarrow \mathsf{U} \\ (x_1, \ldots, x_N) &\longmapsto \exp(x_1) \cdots \exp(x_N) \end{aligned}$$

induces by restriction a L_Θ°-equivariant diffeomorphism

$$\Psi \colon \mathsf{C} \longrightarrow \mathsf{N}\,.$$

Furthermore the stabilizer in L°_Θ of any point h in C, and therefore of any point n in N, is a compact subgroup of L°_Θ.

The closure $\overline{\mathsf{C}}$ is also L°_Θ-invariant and Definition 2.1 implies that the cone $\overline{\mathsf{C}}$ is salient, *i.e.* the intersection of $\overline{\mathsf{C}}$ and $-\overline{\mathsf{C}}$ is reduced to $\{0\}$.

Remark 2.3. More precisely, for every α in Θ an L°_Θ-invariant cone C_α has been chosen in the L-irreducible factor of $\mathfrak{u}_\Theta$ corresponding to α (and C_α is open in that factor) and we have that $\mathsf{C} = C_1 \times C_2 \times \cdots \times C_N$ where N is the length of the longest element in a finite Coxeter group associated with Θ and, for every $i = 1, \ldots, N$, C_i is one of the cones C_α [23, Theorem 1.3].

2.3. **Diamonds.** Let a and b be two transverse points in $\mathbf{F}_\Theta$.

Definition 2.4. A *diamond* with *extremities a and b*, associated with a U-pinning s_a at a, is the subset

$$s_a(\mathsf{N}) \cdot b \ .$$

The terminology *diamond* was coined in [32] in the context of $\mathsf{G} = \mathsf{SO}(2, n)$. To give an idea, in that context $\mathbf{F}_\Theta$ is covered by charts which are identified with the Minkowski space $\mathbb{R}^{1,n-1}$. Then a diamond is, in a suitable chart, the intersection of the future time cone F^+ of a, with the past time cone F^- of b.

In that case there are precisely two diamonds with given extremities. More generally, from [23, Corollary 13.5], it follows that the number of diamonds with given extremities is $2^{\sharp\Theta}$.

Remark 2.5. We observe that diamonds are semi-algebraic sets and make sense over a real closed field.

We list some first properties of diamonds that are direct consequences of the definition or are proved in [23, Section 13].

Proposition 2.6.
(1) *A diamond with extremities a and b is a connected component of $\Omega_a \cap \Omega_b$.*
(2) *Given a diamond $s_a(\mathsf{N}) \cdot b$, there exists a U-pinning s_b at b such that*
$$s_a(\mathsf{N}) \cdot b = s_b(\mathsf{N}) \cdot a \ .$$
(3) *Given any diamond $V(a, b) = s_a(\mathsf{N}) \cdot b$ then a belongs to the closure of $V(a, b)$.*

Proof. The first item is a consequence of [23, Theorems 1.3 and 1.4]. The second item is a consequence of [23, Proposition 13.1].

The third item follows from the fact that the identity belongs to the closure of N.

□

We also remark that

Proposition 2.7. *Given a diamond V there is a unique diamond V^* satisfying the following property: given any U-pinning s_b at b, if $V = s_b(\mathsf{N}) \cdot a$ then $V^* = s_b(\mathsf{N}^{-1}) \cdot a$. The diamond V^* is called the* opposite diamond *to V (one says also that the diamond V^* is opposite to V). A diamond and its opposite are disjoint, more precisely any point in V is transverse to any point in V^*.*

Proof. We just have to remark that the definition of the opposite diamond does not depend on the choices. More precisely, given two U-pinnings s_a and s_b, if

$$V = s_b(\mathsf{N}) \cdot a = s_a(\mathsf{N}) \cdot b \ ,$$

then

$$s_b(\mathsf{N}^{-1}) \cdot a = s_a(\mathsf{N}^{-1}) \cdot b \ ;$$

this holds by [23, Section 13].

The last point comes from [25, Remark 4.9] and from [23, Section 13.6]. In particular, if $x \in V$, then $x = s_b(n) \cdot a$ with $n \in \mathsf{N}$, while if $y \in V^*$, then $y = s_b(m^{-1}) \cdot a$ with $m \in \mathsf{N}$. Thus

$$x = s_b(nm) \cdot y \ .$$

Since N is a semigroup, this means that x belongs to a diamond with extremities y and b. By the first point of Proposition 2.6, x is transverse to y. □

As a consequence of the proposition, if c is an element in a diamond with extremities a and b, we will denote by

- $V_c(a, b)$ the unique diamond containing c with extremities a and b.

Note that, for any d in $V_c(a, b)$, one has $V_d(a, b) = V_c(a, b)$; also $V_c(b, a) = V_c(a, b)$.

In addition, $V_c^*(a, b)$ is the diamond opposite to the diamond containing c.

As an immediate consequence of the semigroup property we obtain the following result that we shall use freely:

Lemma 2.8 (Nesting property)**.** *Let c be a point in a diamond with extremities a and b.*

(1) Then there exists a unique diamond $V(a, c)$ with extremities a and c such that

$$V(a, c) \subset V_c(a, b) \ .$$

Furthermore there is a neighborhood U of a in $\mathbf{F}_\Theta$ such that

$$U \cap V(a, c) = U \cap V_c(a, b).$$

(2) *Moreover, if* $V(c,b)$ *is the unique diamond with extremities* c *and* b *included in* $V_c(a,b)$ *then*

$$V(a,c) \cap V(c,b) = \emptyset \ .$$

(3) *Finally* a *belongs to the opposite diamond* $V^*(c,b)$ *and the diamond* $V(a,c)$ *is contained in* $V^*(c,b)$.

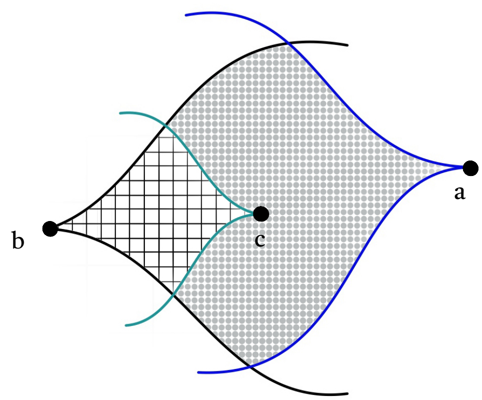

FIGURE 1. The nesting of $V(c,b)$ in $V_c(a,b)$

The proof of Lemma 2.8 will use the following statement.

Lemma 2.9. *Let* a *and* b *be two transverse points of* $\mathbf{F}_\Theta$ *and* V_0 *a diamond with extremities* a *and* b. *Then, there exists a basis* $\mathcal{B}$ *of neighborhoods of* a *such that for every* U *in* $\mathcal{B}$, *the intersection* $U \cap V_0$ *is connected and non empty.*

Proof. Up to acting by an element of G, we can assume that the stabilizer of b is P_Θ and that the stabilizer of a is $\mathsf{P}_\Theta^{\mathrm{opp}}$. The map from $\mathfrak{u}_\Theta$ to Ω_b given by $x \mapsto \exp(x) \cdot a$ is a L_Θ-equivariant diffeomorphism.

Consider the decomposition $\mathfrak{u}_\Theta = \bigoplus_i V_i$ into L_Θ-irreducible factors. Let us fix an auxiliary Euclidean norm $\|\cdot\|$ on $\mathfrak{u}_\Theta$ such that the previous decomposition is orthogonal. There is a one-parameter subgroup $\Lambda = \{\lambda_t\}_{t\in\mathbb{R}}$ of L_Θ such that, for all i and for all v in V_i, $\lambda_t \cdot v = e^{n_i t} v$ for some positive numbers n_i.

Let S be the unit sphere in $\mathfrak{u}_\Theta$ for $\|\cdot\|$. Then the map from $S \times \mathbb{R}$ to $\Omega_b \smallsetminus \{a\}$, given by

$$g\colon (v,t) \mapsto \exp(\lambda_t \cdot v) \cdot a \ ,$$

is a diffeomorphism satisfying that for all v in S, all real numbers t and s

$$g(v, t+s) = \lambda_s \cdot g(v,t) \ .$$

Thus since $\Omega_a \cap \Omega_b$ is Λ-invariant we have the following property: for all v in S and all t, t' in $\mathbb{R}$, $g(v,t)$ belongs to $\Omega_a \cap \Omega_b$ if and only if $g(v,t')$ belongs to $\Omega_a \cap \Omega_b$.

Thus, there is a connected open Ω_0 in S, such that the diamond V_0 —being a connected component of $\Omega_a \cap \Omega_b$ by the first item of Proposition 2.6— is the image of $\Omega_0 \times \mathbb{R}$ by the map g. Let finally O_t be the images of $S \times (-\infty, t)$ by g and $U_t = O_t \cup \{a\}$. Then $\{U_t\}_{t\in\mathbb{R}}$ is a family of neighborhoods of a with the wanted property. □

Proof of Lemma 2.8. Let us first construct diamonds $V^0(c,b)$ and $V^0(a,c)$ included in $V_c(a,b)$. Let us write $V_c(a,b) = \mathsf{N}_b \cdot a = \mathsf{N}_a \cdot b$ and consider the diamonds

$$V^0(c,b) = \mathsf{N}_b \cdot c\ ,\ \ V^0(a,c) = \mathsf{N}_a \cdot c\ .$$

By construction $c = n_b \cdot a = n_a \cdot b$ with $n_b \in \mathsf{N}_b$ and $n_a \in \mathsf{N}_a$. By the semigroup property

$$\mathsf{N}_b \cdot n_b \subset \mathsf{N}_b\ ,\ \ \mathsf{N}_a \cdot n_a \subset \mathsf{N}_a\ ,$$

which leads to the inclusions

$$V^0(c,b) \subset V_c(a,b)\ ,\ \ V^0(a,c) \subset V_c(a,b)\ .$$

We now prove that these specific diamonds are disjoint. By the construction and the inclusion above both $V^0(a,c)$ and $V^0(b,c)$ are connected components of $V_c(a,b) \smallsetminus S_c$. It follows that they are either equal or disjoint. By the sharpness property of N, the identity element does not belong to the closure of $\mathsf{N} \cdot n_a$. Let thus O be an open set in U_a containing the identity and with trivial intersection with $\mathsf{N} \cdot n_a$. Then $O \cdot b$ is a neighborhood of b that does not intersect $\mathsf{N}_a \cdot c = \mathsf{N}_a n_a \cdot b$. Thus b does not belong to the closure of $V^0(a,c)$. From the last item of Proposition 2.6, $V^0(a,c)$ is hence different from $V^0(c,b)$ and by the above discussion they are disjoint:

$$V^0(a,c) \cap V^0(b,c) = \emptyset\ .$$

This concludes item (2) of the lemma.

Let us prove next the existence of the neighborhood U. Denote for any open set V, $\partial V := \overline{V} \smallsetminus V$ and denote V^c the complementary of V and recall that

$$\begin{aligned}
\partial(V \cap W) \subset& \overline{V} \cap \overline{W} \smallsetminus (V \cap W)\\
=& \overline{V} \cap \overline{W} \cap (W^c \cup V^c)\\
=& (\overline{V} \cap \overline{W} \cap W^c) \cup (\overline{V} \cap \overline{W} \cap V^c)\\
=& (\partial V \cap \overline{W}) \cup (\partial W \cap \overline{V})\ .
\end{aligned}$$

Let $V(a,c)$ be any diamond with extremities a and c included in $V_c(a,b)$. Let U be a neighborhood of a such that

- the intersection of $\overline{U}$ with $S_c \cup S_b$ is empty,
- $V_c(a,b) \cap U$ is connected and non empty.

The existence of this open set U is guaranteed by Lemma 2.9. From the first item we have that

$$\partial(V(a,c) \cap U) \subset ((\partial V(a,c)) \cap \overline{U}) \cup (\overline{V}(a,c) \cap \partial U) \ \subset (S_a \cup \partial U)\ .$$

From the inclusion $V(a,c) \subset V_c(a,b)$ we have

$$\partial(V(a,c) \cap U) \subset \overline{V_c(a,b) \cap U}\ .$$

Since $S_a \cup \partial U$ is included in the complementary of $V_c(a,b) \cap U$ we furthermore have

$$(S_a \cup \partial U) \cap (\overline{V_c(a,b) \cap U}) \ \subset \partial(V_c(a,b) \cap U)\ .$$

Thus combining these inclusions, we get

$$\partial(V(a,c) \cap U) \subset (S_a \cup \partial U) \cap (\overline{V_c(a,b) \cap U}) \subset \partial(V_c(a,b) \cap U)\ .$$

Now a simple connectedness argument show that if A and B are two open sets, with B connected, $A \subset B$ and $\partial A \subset \partial B$, then $A = B$. Thus, in our case,

$$V(a,c) \cap U = V_c(a,b) \cap U \neq \emptyset.$$

Since this is true for all diamonds with extremities a and c included in $V_c(a,b)$ and since diamonds with the same extremities are either disjoint of equal, we finally conclude that there is a unique diamond with extremities a and c included in $V_c(a,b)$. This concludes item (1) of the lemma.

For the last item, observe that

$$a = n_b^{-1} c \in \mathsf{N}_b^{-1} c = V^*(c,b)\ .$$

Since a belongs to the closure of $V(a,c)$, we have hence $V(a,c) \cap V^*(c,b) \neq \emptyset$. Furthermore $V(a,c) \subset \Omega_c$ and $V(a,c) \subset V(a,b) \subset \Omega_b$; this means that the connected set $V(a,c)$ is contained in $\Omega_a \cap \Omega_b$. Therefore $V^*(b,c)$ is the connected component of $\Omega_a \cap \Omega_b$ containing $V(a,b)$: this is the sought for inclusion. ◻

2.4. **Positive configurations.** The following definition plays a central role in this article:

Let $p \geqslant 3$ and equip $\{1,\ldots,p\}$ with the usual cyclic order.

Definition 2.10 (Positive configuration)**.** We say that a configuration $(a_1,\ldots,a_p)$ in $\mathbf{F}_\Theta^p$ is *positive*, if there exist diamonds $V_{i,j}$ with extremities a_i and a_j for all $i \neq j$ such that

(1) $V_{i,j} = V^*_{j,i}$,
(2) a_j belongs to $V_{i,k}$, if (i, j, k) is cyclically oriented,
(3) we have $V_{i,j} \subset V_{i,k}$ and $V_{j,k} \subset V_{i,k}$, if (i, j, k) is cyclically oriented.

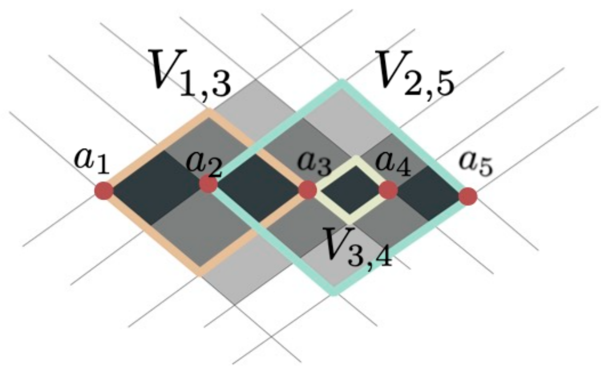

Figure 2. A positive 5-configuration and some diamonds

Proposition 3.1 will give easier criteria to understand positive triples and quadruples and will show that the definition is equivalent to the definition given in the introduction.

Observe that, by properties (2) and (1) above, the choice of $V_{i,k}$ among diamonds with extremities a_i and a_k is forced by the cyclic ordering. Furthermore, the fact that $V_{i,k}$ does not depend on the index j between i and k involves the positivity of a subquadruple. It thus follows that if $(a_1, \ldots, a_p)$ is such that every cyclically oriented subquadruple is positive then $(a_1, \ldots, a_p)$ is positive.

By construction, every subconfiguration of a positive configuration is positive. On the real projective line, a configuration of p points with $p > 3$ is positive exactly if it is cyclically oriented, and a triple is positive if it consists of pairwise distinct elements.

Moreover

Proposition 2.11. *Positivity of configurations is invariant under cyclic permutation and under the order reversing permutation. In particular*

(1) to be positive for a triple is invariant under all permutations,
(2) to be positive for a quadruple is invariant under the dihedral group.

Proof. The definition is invariant under cyclic transformations. If σ_0 is the reverse ordering, we choose the new diamonds $V^{\circ}_{i,j} = V^{*}_{\sigma_0(i),\sigma_0(j)}$. ◻

2.5. **Positive circles and** $\mathsf{PSL}_2(\mathbb{R})$**.** Let H be a subgroup in G locally isomorphic to $\mathsf{PSL}_2(\mathbb{R})$. An H-*circle* in $\mathbf{F}_\Theta$ is a closed H-orbit, it can be parametrized by a *circle map* which is a H-equivariant map from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}_\Theta$. The group H is *proximal* if it contains a proximal element in $\mathbf{F}_\Theta$, *i.e.* an element having an attracting fixed point on $\mathbf{F}_\Theta$.

Proposition 2.12 (H-circle)**.** *Given a positive structure, there exists $\mathcal{H}$, an* $\mathrm{Aut}_0(\mathfrak{g})$*-orbit of pairs* (H, C) *such that* H *is a subgroup of* G *locally isomorphic to* $\mathsf{PSL}_2(\mathbb{R})$*, C is an* H*-circle, satisfying the following properties*

(1) H *has a compact centralizer in* G*;*
(2) Given a diamond V with extremities a and b, there exists (H, C) *in $\mathcal{H}$ with C containing a and b, and such that C intersects the diamond V. Furthermore*
   - *If c is a point in C different from a and b, then (a, c, b) is a positive triple and*
   $$V_c(a, b) \cap C \ \text{ and } V^*_c(a, b) \cap C,$$
   are the two connected components of $C \smallsetminus \{a, b\}$.
   - *If d belongs to the connected component of $C \smallsetminus \{c, b\}$ not containing a, then*
   $$V_d(b, c) \subset V_d(a, b)\ .$$

(3) Given any three pairwise distinct points a, b, and c in $\mathbf{F}_\Theta$. Then there is at most one element (H, C) *of $\mathcal{H}$ such that C contains a, b, and c.*

Note that in point (2) V needs to be equal to $V_c(a, b)$ or $V^*_c(a, b)$.

Proof. Let s_b be a U-pinning at b such that $V = s_b(\mathsf{N}) \cdot a$. One just picks the Lie subgroup associated with an $\mathfrak{sl}_2$-triple given by the Jacobson–Morozov theorem applied to a nilpotent element x chosen so that $\mathsf{N}_2 = s_b(\exp(\mathbb{R}_{>0}x))$ is included in N. The corresponding $\mathfrak{sl}_2$-triple is the Θ-principal $\mathfrak{sl}_2$-triple introduced in [23, Section 7]. Item 1 is immediate.

For item (2), the existence is immediate by $\mathrm{Aut}_0(\mathfrak{g})$-transitivity. The two connected components of $C \smallsetminus \{a, b\}$ are $\mathsf{N}_2 \cdot a$ and $\mathsf{N}_2^{-1} \cdot a$ and are thus included in diamonds opposite to each other.

Moreover, for the last statement in item 2, let us write $c = n \cdot a$ with n in N_2. Observe that, by a deformation argument

$$V_{n\cdot d}(c, b) = V_d(c, b) \text{ and } V_d(a, b) = V_c(a, b) .$$

Then $V_d(a, b) = \mathsf{N} \cdot a$ and

$$V_d(c, b) = V_{n\cdot d}(c, b) = nV_d(a, b) = n\mathsf{N} \cdot a \subset \mathsf{N} \cdot a = V_d(a, b) ,$$

where the inclusion holds by the semigroup property.

For the item (3), let us consider Z the stabilizer of a, b, and c. Then Z is precisely the stabilizer in $\mathsf{L}_{a,b}$ of the element n in N_2 such that $c = n \cdot a$. Since H is determined by N_2, this implies that Z is in fact the centralizer of H. This concludes the proof. □

Remark 2.13. More detail on the construction of the Θ-principal $\mathfrak{sl}_2$-triple can be found in [23, Section 7]. Note that there are others $\mathfrak{sl}_2$-triples which induce positive maps from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}_\Theta$. For example, if G is a split real Lie group, the principal $\mathfrak{sl}_2$-triple gives rise to such a map.

We fix once and for all such an $\mathrm{Aut}_0(\mathfrak{g})$-orbit $\mathcal{H}$.

As an important example of positive configuration, we have

Proposition 2.14. *Let (H, C) be in $\mathcal{H}$. Any cyclically ordered configuration of points on C is positive.*

Proof. It is enough to prove the results for triples and quadruples.

Let first (a_0, a_1, a_2) be a triple on C. By item (2) in Proposition 2.12, a_{i+1} belongs to a diamond with extremities a_i and a_{i+2}. Let us define (where indices are taken modulo 3)

$$V_{i,i+2} \coloneqq V_{a_{i+1}}(a_i, a_{i+2}) \text{ , } V_{i,i+1} \coloneqq V^*_{i+1,i} \text{ .}$$

Then the properties of Definition 2.10 are obviously satisfied and the triple is positive.

Let now consider (a_0, a_1, a_2, a_3) a quadruple on C, such that a_{i+1} and a_{i+3} belongs to different components of $C \smallsetminus \{a_i, a_{i+2}\}$. Observe that by a deformation argument we have

$$V_{a_{i+2}}(a_i, a_{i+3}) = V_{a_{i+1}}(a_i, a_{i+3}) \text{ .}$$

We now define

$$\begin{aligned} V_{i,i+2} &\coloneqq V_{a_{i+1}}(a_i, a_{i+2}) \text{ ,} \\ V_{i,i+3} &\coloneqq V_{a_{i+2}}(a_i, a_{i+3}) = V_{a_{i+1}}(a_i, a_{i+3}) \text{ ,} \\ V_{i,i+1} &\coloneqq V^*_{i+1,i} \text{ .} \end{aligned}$$

It then follows from item (2) of Proposition 2.12 that $V_{a_{i+3}}(a_i, a_{i+2}) = V^*_{a_{i+1}}(a_i, a_{i+2})$, and thus that $V_{i+2,i} = V^*_{i,i+2}$.

From the equality $V_{i,i+1} = V^*_{i+1,i}$ and from item (2) of Proposition 2.12, $V_{i+1,i+2} \cap C$ is the connected component of $C \smallsetminus \{a_{i+1}, a_{i+2}\}$ not containing a_i and a_{i+3}. Let d be in $V_{i+1,i+2} \cap C$, then

$$V_{i+1,i+2} = V_d(a_{i+1}, a_{i+2}) \subset V_d(a_i, a_{i+3}) = V_{i,i+3} ,$$

where, for the inclusion, we applied twice the last part of the item (2) of Proposition 2.12.

This concludes the proof. □

2.6. **Positive maps.** Let S be a cyclically ordered set containing at least three points.

Definition 2.15 (Positive map)**.** A map f from S to $\mathbf{F}_\Theta$ is *positive* if the image of every cyclically ordered quadruple is a positive quadruple, and the image of every cyclically ordered triple is a positive triple.[1]

Observe then that the image of every cyclically ordered configuration by a positive map is a positive configuration.

By Proposition 2.14, for any (H, C) in $\mathcal{H}$, C —seen as a map from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}_\Theta$— is positive.

3. Properties of positivity

We prove in this section, three main propositions concerning positivity:

- The first one, Proposition 3.1, gives various combinatorial properties of positive triples, quadruples and configurations;
- The second one, Proposition 3.8, gives information about the limit of diamonds included in a given diamond;
- The last one, Proposition 3.13, shows that positive maps share the property of monotone maps: they coincide on a dense subset with a left-continuous positive map.

We also establish that certain elements in G are $\mathbf{F}_\Theta$-proximal using positivity.

Several of the combinatorial properties of positive configurations have been addressed in [23] with a more algebraic approach, for reader's convenience, we provide here geometric proofs using the nesting properties of diamonds.

[1]When S has more than three points, the second requirement follows from the first

3.1. **Combinatorics of positivity.** The next proposition gives fundamental properties of positive triples and quadruples.

- The first one gives an easy criterion for positivity of triples, while the second and third concern quadruples. In particular, this shows that the definition of positivity given in the introduction is equivalent to Definition 2.10.
- The fourth one gives a recursive way to build positive configuration.
- The fifth and sixth give "exclusion" properties that are important in the study of positivity though they are not used in this paper.

We are going to prove this proposition and its corollary in the context of a group defined over $\mathbb{R}$, although by Tarski–Seidenberg Theorem, the statements will be true over every real closed field.

Proposition 3.1 (Combinatorial properties)**.** *(1) Assume that a and b are transverse and that c belongs to a diamond with extremities a and b, then (a,b,c) is positive.*

(2) Assuming (a,x_0,b) and (a,y_0,b) are positive then (a,x_0,b,y_0) is positive if and only if $V_{x_0}(a,b)=V^*_{y_0}(a,b)$.

(3) Assuming (a,c,b) is positive and d belongs to $V^*_a(c,b)$, then (a,c,d,b) is positive.

(4) More generally, assume that $(x_0,x_1,\ldots,x_p)$ is a positive configuration and that $y\in V^*_{x_2}(x_0,x_1)$ then

$$(x_0,y,x_1,\ldots,x_p)\ ,$$

is a positive configuration.

(5) If (a,b,c,d) is positive, then (a,c,b,d) is not positive.

(6) Let x_0, x_1, and x_2 be three points such that (a,x_i,b) is positive ($i=0,1,2$), then the three quadruples (a,x_0,b,x_1), (a,x_1,b,x_2), and (a,x_2,b,x_0) cannot all be positive.

Finally we have,

Corollary 3.2 (Necklace property)**.** *Let (a,b,c) be a positive triple. Let α, β, and γ be elements of $V_a(b,c)$, $V_b(a,c)$, and $V_c(a,b)$ respectively. Then the triple (α,β,γ) is positive.*

The proof of this proposition and of Corollary 3.2 will be given in Section 3.1.3. It is important to remark that all these properties are true for configurations in $\mathbf{P}^1(\mathbb{R})$.

3.1.1. *Triples and quadruples.* Before addressing the proof of Proposition 3.1, we establish a number of preliminary statements.

Lemma 3.3. *A triple* (a_0, a_1, a_2) *is positive if and only if* a_0, a_1, a_2 *belong to diamonds with extremities* (a_1, a_2), (a_2, a_0) *and* (a_0, a_1) *respectively.*

Proof. We just need to prove the "if" part. Let, for $i = 0, 1, 2$ (indices are taken modulo 3)

$$V_{i,i+1} \coloneqq V^*_{a_{i+2}}(a_i, a_{i+1}) \ , \ V_{i,i+2} \coloneqq V_{a_{i+1}}(a_i, a_{i+2}).$$

Observe that

$$V_{i,i+1} = V^*_{i+1,i} \ .$$

Then Lemma 2.8.(3) provides all the necessary inclusions needed to prove that the triple is positive. ◻

The following lemma gives a way to go from positive triples to positive quadruples.

Lemma 3.4. *Let* (a_0, a_1, a_2, a_3) *be a quadruple. Assume that all subtriples are positive. Then the quadruple* (a_0, a_1, a_2, a_3) *is positive, if and only if, for all* i *(indices are taken modulo* 4*)*

$$a_i \in V^*_{a_{i+2}}(a_{i+1}, a_{i+3}) \ , \tag{1}$$

$$a_{i+2} \in V_{a_{i+1}}(a_i, a_{i+3}) \ . \tag{2}$$

Proof. The "only if" part follows from the definition. It remains to prove the "if" part. Let

$$\begin{aligned} V_{i,i+1} &\coloneqq V^*_{a_{i+2}}(a_i, a_{i+1}) = V^*_{a_{i+3}}(a_i, a_{i+1}) \ , \\ V_{i,i+2} &\coloneqq V_{a_{i+1}}(a_i, a_{i+2}) = V^*_{a_{i+3}}(a_i, a_{i+2}) \ , \\ V_{i,i+3} &\coloneqq V_{a_{i+1}}(a_i, a_{i+3}) = V_{a_{i+2}}(a_i, a_{i+3}) \ , \end{aligned}$$

where in the second line we used the hypothesis (1), while in the first and last lines we used the hypothesis (2) and the fact that if d belongs to $V_a(b, c)$ then $V_d(b, c) = V_a(b, c)$. Hence by definition

$$V_{i,i+1} = V^*_{i+1,i} \ , V_{i,i+2} = V^*_{i+2,i} \ .$$

It thus follows that for all i and j,

$$V_{i,j} = V^*_{j,i} \ . \tag{3}$$

From the positivity of the subtriple (a_i, a_{i+1}, a_{i+2}), we get the inclusions

$$V_{i,i+1} \subset V_{i,i+2} \ , \ \ V_{i+1,i+2} \subset V_{i,i+2} \ . \tag{4}$$

From the positivity of the triple (a_i, a_{i+1}, a_{i+3}) we get the inclusions

$$V_{i,i+1} = V^*_{a_{i+3}}(a_i, a_{i+1}) \subset V_{a_{i+1}}(a_i, a_{i+3}) = V_{i,i+3} \ , \tag{5}$$

$$V_{i+1,i+3} = V^*_{a_i}(a_{i+1}, a_{i+3}) \subset V_{a_{i+1}}(a_i, a_{i+3}) = V_{i,i+3} \ . \tag{6}$$

Similarly the positivity of the triple (a_i, a_{i+2}, a_{i+3}) yields

$$V_{i+2,i+3} = V^*_{a_i}(a_{i+2}, a_{i+3}) \subset V_{a_{i+2}}(a_i, a_{i+3}) = V_{i,i+3} \ , \tag{7}$$

$$V_{i,i+2} = V^*_{a_{i+3}}(a_i, a_{i+2}) \subset V_{a_{i+2}}(a_i, a_{i+3}) = V_{i,i+3} \ . \tag{8}$$

All together the equation (3) as well as the inclusions (4), (5), (6), (7), and (8) prove that (a_0, a_1, a_2, a_3) is a positive quadruple. □

3.1.2. *Deformation lemmas.* We need to prove some deformation lemmas.

Lemma 3.5 (Deforming triples). *Let $a(t)$, $b(t)$, and $c(t)$ be continuous arcs from $[0,1]$ to $\mathbf{F}_\Theta$ such that*

(1) for all t in $[0,1]$, $a(t)$, $b(t)$, and $c(t)$ are pairwise transverse,
(2) the triple $(a(0), b(0), c(0))$ is positive.

Then, for all t, $(a(t), b(t), c(t))$ is a positive triple.

Proof. The hypothesis tells us that

$$c(t) \in \Omega_{a(t)} \cap \Omega_{b(t)} \ , \ \ a(t) \in \Omega_{c(t)} \cap \Omega_{b(t)} \ , \ \ b(t) \in \Omega_{a(t)} \cap \Omega_{c(t)} \ .$$

By hypothesis, there are diamonds $V(a(0), b(0))$, $V(c(0), b(0))$, and $V(c(0), a(0))$ such that

$$c(0) \in V(a(0), b(0)) \ , \ \ a(0) \in V(c(0), b(0)) \ , \ \ b(0) \in V(c(0), a(0)) \ .$$

We can extend these to continuous maps $t \mapsto V(a(t), b(t))$, $t \mapsto V(c(t), b(t))$, and $t \mapsto V(c(t), a(t))$ in the space of diamonds (one always has that $V(e,d)$ is a diamond with extremities e and d). We now use the fact that a diamond with extremities e and d is a connected component of $\Omega_e \cap \Omega_d$ (Proposition 2.6). Then by continuity, for all t

$$c(t) \in V(a(t), b(t)) \ , \ \ a(t) \in V(c(t), b(t)) \ , \ \ b(t) \in V(c(t), a(t)) \ .$$

Thus the result follows from Lemma 3.3. □

Similarly

Lemma 3.6 (Deforming quadruples). *Let γ and η be continuous arcs from $[0,1]$ to $\mathbf{F}_\Theta$ such that there exist a and b in $\mathbf{F}_\Theta$ satisfying*

(1) for all t in $[0,1]$, $a, \gamma(t), b, \eta(t)$ are pairwise transverse,
(2) the quadruple $(a, \gamma(0), b, \eta(0))$ is positive.

Then, for all t, $(a, \gamma(t), b, \eta(t))$ is a positive quadruple.

Proof. By applying Lemma 3.5, we obtain that all the subtriples of $(a, \gamma(t), b, \eta(t))$ are positive. By Lemma 3.4, we only need to check that

$$\begin{gathered} a \in V_b^*(\gamma(t), \eta(t)) \ , \ b \in V_a^*(\gamma(t), \eta(t)) \ , \\ \gamma(t) \in V_{\eta(t)}^*(a, b) \ , \ \eta(t) \in V_{\gamma(t)}^*(a, b) \ , \\ \gamma(t) \in V_a(\eta(t), b) \ , \ b \in V_{\gamma(t)}(a, \eta(t)) \ , \\ \eta(t) \in V_b(\gamma(t), b) \ , \ a \in V_{\eta(t)}(b, \gamma(t)) \ . \end{gathered}$$

Using again the fact that a diamond with extremities c and d is a connected component of $\Omega_c \cap \Omega_d$ (Proposition 2.6), the statement follows by continuity. □

Finally we also have as an immediate consequence of the connectedness of the positive cone:

Lemma 3.7 (Connectedness)**.** *Let a and b be two transverse points.*

(1) Assume c is so that (a, c, b) is positive. Then there is (H, C) in $\mathcal{H}$ such that a and b belong to C, and there is a path $t \mapsto c(t)$ from $[0, 1]$ to $V_c(a, b)$ connecting $c = c(0)$ to $c(1)$ so that $(a, c(1), b)$ is a positive triple on C.

(2) Assume furthermore that d belongs to $V_a^*(c, b)$ then there are (H, C) in $\mathcal{H}$, a path $t \mapsto c(t)$ as in the previous item, and a path $t \mapsto d(t)$ from $[0, 1]$ to $V_c(a, b)$, so that $d(t) \in V_a^*(c(t), b)$ and $(a, c(1), d(1), b)$ are on C.

Proof. Using a U-pinning at b, we identify N with a positive semigroup N_b in U_b such that we have $V_c(a, b) = \mathsf{N}_b \cdot a$. The first point just follows from the connectedness of the positive semigroup N_b. For use in the second point we take a path $t \mapsto c(t)$ which is constant for $t > 1/2$.

Recall that $d = m_0 \cdot c$, with $m_0 \in \mathsf{N}_b$. Let us define, for $t \in [0, 1/2]$,

$$d(t) = m_0 \cdot c(t) \ ,$$

then we have by the semigroup property $d(t) \in V_a^*(c(t), b)$. Observe also that $d(0) = d$. Then for $t \in [1/2, 1]$, we have $c(t) = c(1/2)$ and we choose, using that C contains elements of $V_a^*(c(1/2), d)$ (*cf.* Proposition 2.12.(2)), a path $t \mapsto d(t)$ with $d(t) \in V_a^*(c(1/2), d)$, and such that $d(1)$ belongs to C. □

3.1.3. *Proof of the combinatorial properties.*

Proof of item (1) of Proposition 3.1. Assume (a, b, c) satisfies the hypothesis. Let (H, C) in $\mathcal{H}$ and $t \mapsto c(t)$ obtained in Lemma 3.7. On C, a triple is positive if and only if the three points are pairwise distinct, the result thus follows from Lemma 3.5. □

Proof of item (2) of Proposition 3.1. The "only if" part follows from the definition. Then for the "if" part we find, by Lemma 3.7, (H, C) in $\mathcal{H}$ and paths $t \mapsto x(t)$ and $t \mapsto y(t)$ in $V_{x_0}(a,b)$ and $V_{y_0}(a,b)$ respectively, such that $(x(0), y(0)) = (x_0, y_0)$ and $x(1)$, $y(1)$ are on C, the H-circle passing through a and b. Then $(a, x(1), b, y(1))$ is positive and so is (a, x_0, b, y_0) by Lemma 3.6, since $x(t)$ and $y(t)$ are transverse thanks to Proposition 2.7. □

Proof of item (3) of Proposition 3.1. From the connectedness Lemma 3.7 we obtain (H, C) in $\mathcal{H}$ and paths $t \mapsto c(t)$, $t \mapsto d(t)$ such that $a, c(t), d(t), b$ are pairwise transverse, $c(0) = c$, $d(0) = d$, $(a, c(1), d(1), b)$ on C and $d(1) \in V_a(c(1), b)$. In particular $(a, c(1), d(1), b)$ is positive and thus by the deformation Lemma 3.6, (a, c, d, b) is positive. □

Proof of item (4) of Proposition 3.1. This is an immediate consequence of item (3) and the fact that in order to check the positivity of a configuration one only needs to check the positivity of subtriples and subquadruples. □

Proof of item (5) of Proposition 3.1. If (a, b, c, d) is positive, we have the strict inclusion $V_a(b, d) \subset V_a(c, d)$ and if (a, c, b, d) is positive, we have the strict inclusion $V_a(c, d) \subset V_a(b, d)$. Hence a contradiction. □

Proof of item (6) of Proposition 3.1. Assume that (a, x_0, b, x_1) is positive. Then $V = V_{x_0}(a, b)$ and $V_{x_1}(a, b)$ are opposite diamonds. If both (a, x_1, b, x_2) and (a, x_0, b, x_2) are positive then we get that x_2 belongs to both V and V^*, which is a contradiction. □

Proof of the necklace property (Corollary 3.2). Let us first remark that from item (4) of Proposition 3.1, applied three times, (and using cyclic invariance of positivity) the configuration

$$(a, \gamma, b, \alpha, c, \beta) \, ,$$

is positive. Thus (γ, α, β) is positive. □

3.2. **Inclusion of diamonds.**

Proposition 3.8 (Boundedness property)**.** *Let (a, b, d) be a positive triple and let $c \in V_b(a, d)$. Assume that there exist sequences $\{b_m\}_{m \in \mathbb{N}}$ and $\{c_m\}_{m \in \mathbb{N}}$, converging respectively to b and c and such that, for all m, (a, b_m, c_m, d) is a positive quadruple. Then the sequence $(\overline{V}^*_d(b_m, c_m))$ converges in the Hausdorff topology and*

$$\lim_{m \to \infty} \left(\overline{V}^*_d(b_m, c_m) \right) \subset V_c(a, d) \, .$$

In particular,

Corollary 3.9 (Inclusion)**.** *Let (a, b, d, e) be a positive quadruple in $\mathbf{F}_\Theta$. Then*

$$\overline{V}^*_e(b, d) \subset V_b(a, e) \ .$$

Let (a, b, c, d, e) be a positive quintuple of points in $\mathbf{F}_\Theta$. then

$$\overline{V}_c(b, d) \subset V_c(a, e).$$

Proposition 3.8 will be proved in Section 3.2.2. Corollary 3.9 will be proved in Section 3.2.1 as the consequence of an intermediate statement.

3.2.1. *Preliminaries on circles.* Let $V(a, d)$ be a diamond with extremities a and b, and let (H, C) be in $\mathcal{H}$ such that a and d belong to C and $C \cap V(a, d) \neq \emptyset$.

- Let $\delta = \{\delta_t \mid t \in \mathbb{R}\}$, be the 1-parameter group in H for which d is the attracting fixed point and a is the repelling fixed point for the element δ_t $(t > 0)$. The corresponding basins of attraction/repulsion are denoted by
$$O^+ := \{x \in \mathbf{F}_\Theta \mid \lim_{t\to\infty} \delta_t(x) = d\} \ ,$$
$$O^- := \{x \in \mathbf{F}_\Theta \mid \lim_{t\to-\infty} \delta_t(x) = a\} \ .$$
- Let $\gamma = C \setminus \{a, d\}$.
- Let F be the set of fixed points of δ in $\mathbf{F}_\Theta$.

The result of this section is

Proposition 3.10. *For any e in γ, we have $\overline{V}^*_a(e, d) \subset \Omega_a$.*

This proposition implies Corollary 3.9:

Proof of the Corollary 3.9. Applying Proposition 3.10, we get that

$$\overline{V}^*_a(b, d) \subset \overline{V}^*_a(b, e) \cap \overline{V}^*_e(d, a) \subset \Omega_e \cap \Omega_a \ .$$

Since $V_d(a, e)$ is a connected component of $\Omega_e \cap \Omega_a$ containing $V^*_a(b, d)$ it follows that

$$\overline{V}^*_a(b, d) \subset V_d(a, e) \ .$$

This proves the first part of the corollary.

Suppose now that (a, b, c, d, e) is positive. Then the equalities $V_b(a, e) = V_c(a, e)$ and $V_c(b, d) = V^*_e(b, d)$ together with the first part imply the inclusion $\overline{V}^*_e(b, d) \subset V_c(a, e)$. ∎

Recall that S_a is complementary to Ω_a. In order to prove Proposition 3.10, we introduce the following sets for e in γ

$$\begin{aligned} J(e) &:= \overline{V}^*_a(e,d) \cap S_a \ , \\ F(e) &:= J(e) \cap F \ . \end{aligned}$$

We will first prove that the sets $J(e)$ and $F(e)$ are empty. We first prove the following lemma

Lemma 3.11. *Let γ_1 be one of the two connected components of γ . For any e in γ_1,*

1. *$J(e)$ is invariant by the semigroup $\delta^+ := \{\delta_t \mid t > 0\}$, i.e. $\delta_t(J(e)) \subset J(e)$ for all $t > 0$,*
2. *$F(e)$ is independent of the choice of e in γ_1,*
3. *if $J(e)$ is not empty, so is $F(e)$,*
4. *for all c and b in $\mathbf{F}_\Theta$ such that (a, c, d, b) is a positive quadruple and*
$$V^*_a(c,d) \cap \gamma_1 \neq \emptyset \ ,$$
then
$$F(e) \subset S_c \cup S_b \ .$$

Proof. We prove the first point. By Proposition 2.14 for $t > 0$

$$V^*_a(\delta_t(e), d) \subset V^*_a(e,d) \ .$$

This implies that $V^*_a(e,d)$ is invariant by δ^+ and so is $J(e)$.

The second point is a consequence that F is pointwise fixed by δ_t:

$$J(\delta_t(e)) \cap F = \delta_t(J(e)) \cap F = J(e) \cap F \ .$$

The third point is a consequence of the hyperbolic nature of the subgroup δ: indeed, using a linear representation of G we can assume that δ is a one-parameter subgroup of diagonal matrices and that $J(e)$ is a non-empty closed subset of the projective space invariant by the semigroup δ^+; in this case, every ray $(\delta_t(x))_{t \geqslant 0}$ (for x in $J(e)$) has a limit that is a fixed point of δ.

Let us prove now the last point. Let (a, c, d, b) be as in the hypothesis. Thus $V^*_a(c,d) \cap \gamma_1 \neq \emptyset$ and by point (2) we can choose e in this intersection. By Proposition 3.1.(4), (a, c, e, d, b) is a positive configuration and hence $V^*_a(e,d) \subset V^*_a(c,b)$. From this we get

$$F(e) \subset J(e) = \left(\overline{V}^*_a(e,d) \cap S_a\right) \subset \left(\overline{V}^*_a(c,b) \cap S_a\right) \ . \tag{9}$$

Now, we remark that since (a, c, d, b) is positive, we have

$$V^*_a(c,b) \subset \Omega_a.$$

Thus

$$\overline{V}^*_a(c,b) \cap S_a \subset \overline{V}^*_a(c,b) \setminus V^*_a(c,b). \tag{10}$$

But since $V_a^*(c,b)$ is a connected component of the open set

$$\Omega_c \cap \Omega_b = \mathbf{F}_\Theta \smallsetminus (S_c \cup S_b) \ ,$$

we get

$$\left(\overline{V}_a^*(c,b) \smallsetminus V_a^*(c,b)\right) \subset (S_c \cup S_b)\,. \tag{11}$$

Combining inclusions (9), (10) and (11), we get that

$$F(e) \subset (S_c \cup S_b) \ . \qquad \square$$

We can now prove Proposition 3.10, in other words that $J(e)$ is empty. By item (3) of Lemma 3.11, it suffices to show that $F(e)$ is empty. The fact that $F(e)$ is empty follows from item (4) of Lemma 3.11 and the following result.

Lemma 3.12. *Let Q be a subset of $\mathbf{F}_\Theta$. Assume that there exist nonempty open sets U and V such that for all c in U, and for all b in V,*

$$Q \subset S_c \cup S_b,$$

then Q is empty.

Proof. Let q be in Q and set $Z := S_q$. Then Z is a proper closed Zariski subset of $\mathbf{F}_\Theta$. Observe that if $u \notin Z$, then

$$q \notin S_u.$$

On the other hand we can find u in the nonempty set $U \smallsetminus Z$ and v in the nonempty set $V \smallsetminus Z$, and by hypothesis $q \in S_u \cup S_v$. This shows that $q \neq q$ and concludes the proof. $\square$

3.2.2. *Proof of the boundedness Proposition 3.8.* We use the notation of the previous paragraph.

First note that $\overline{V}_a^*(b_m,c_m) = \overline{V}_d^*(b_m,c_m) = \overline{V}_a^*(b_m,d) \cap \overline{V}_d^*(a,c_m)$. Since the sequences $\{b_m\}_{m\in\mathbb{N}}$ and $\{c_m\}_{m\in\mathbb{N}}$ converge, the sequences of closures of diamonds $(\overline{V}_a^*(b_m,d))$ and $(\overline{V}_d^*(a,c_m))$ also converge and thus the sequence $(\overline{V}_a^*(b_m,c_m))$ converges. Let C be an H-circle through a and d such that $\gamma := V_c(a,d) \cap C$ is not empty. Since being positive is an open condition for quadruples, we can find e and f in γ so that (e,b_m,c_m,f) is positive for m large enough as well as (a,e,f,d). Thus

$$V_a^*(b_m,c_m) \subset V_a^*(e,f)$$

Applying Proposition 3.10, we get that

$$\overline{V}_a^*(b_m,c_m) \subset \overline{V}_a^*(e,f) \subset V_c(a,d)\ ,$$

which easily implies the result.

3.3. **Left and right limits of positive maps.** Our main result is

Proposition 3.13 (Existence of left and right limits). *Let S be a totally ordered set and ϕ be a positive map from S to $\mathbf{F}_\Theta$.*

Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence in S such that there exist a, b, and c in S with $a < b_n \leqslant b_{n+1} \leqslant b < c$, for all n. Then $\{\phi(b_n)\}_{n\in\mathbb{N}}$ converges to a point y in $V_b(a,c)$.

Symmetrically, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of points such that $c < a \leqslant a_{n+1} \leqslant a_n < b$. Then $\{\phi(a_n)\}_{n\in\mathbb{N}}$ converges to a point y in $V_a(c,b)$.

Remark 3.14. In the case of Lusztig's total positivity, this statement was proved in [17, Theorem 7.4], this is also proved for $\mathsf{SL}_n(\mathbb{R})$ in [30, Section 5].

As an immediate corollary, we show that positive maps defined on dense subsets extend to positive maps. More precisely:

Corollary 3.15 (Extension of positive maps). *Let A be dense subset in $[0,1]$. Assume that we have a positive map ξ from A to $\mathbf{F}_\Theta$. Then there exist*

- *a unique left-continuous positive map ξ_+ from $[0,1]$ to $\mathbf{F}_\Theta$ such that ξ coincide with ξ_+ on a dense subset of A,*
- *a unique right-continuous positive map ξ_- from $[0,1]$ to $\mathbf{F}_\Theta$ such that ξ coincide with ξ_- on a dense subset of A.*

Moreover,

- *for any ordered quadruple (x,y,z,t) of pairwise distinct points in $[0,1]$*
$$(\xi_\epsilon(x), \xi_\eta(y), \xi_\nu(z), \xi_\beta(t)) ,$$
is a positive quadruple for any choice of ϵ, η, ν, and β in $\{+,-\}$,
- *if $\{x_m\}_{m\in\mathbb{N}}$, $\{z_m\}_{m\in\mathbb{N}}$ are sequences in $[0,1]$ converging to y, with for all n, $x_m < y < z_m$, then*
$$\lim_{m\to\infty} \xi_{\epsilon_m}(x_m) = \xi_+(y) \ , \ \lim_{m\to\infty} \xi_{\eta_m}(z_m) = \xi_-(y) \ ,$$
for any sequences $\{\epsilon_m\}_{m\in\mathbb{N}}$ and $\{\eta_m\}_{m\in\mathbb{N}}$ in $\{+,-\}$.

Proof of Proposition 3.13. Let us define $x_n = \phi(b_n)$. We can write $x_n = m_n \cdot x_{n-1}$, with $m_n \in \mathsf{N}_{\phi(c)}$. Thus by induction we have
$$x_n = m_n \cdots m_1 \cdot x_0 \ .$$

But we know that $V^*_{\phi(c)}(x_0, \phi(b))$ is a relatively compact region of $\Omega_{\phi(c)}$ by Proposition 3.8. Thus $\{x_n\}_{n\in\mathbb{N}}$ is relatively compact in $\Omega_{\phi(c)}$. It follows that
$$\pi_n = m_n \cdots m_1,$$
is a bounded sequence in $\mathsf{N}_{\phi(c)}$. We now prove that this sequence converges. Assume that we have subsequences that converge to

different limits u and v. After extracting further subsequence, we may find subsequences

$$q_i = \pi_{n_i}, \quad p_i = \pi_{m_i}, \text{ with } n_i \leqslant m_i$$

such that $\{q_i\}_{i\in\mathbb{N}}$ converges to u and $\{p_i\}_{i\in\mathbb{N}}$ converges to v. It follows that $u = w_1 \cdot v$ with $w_1 \in \overline{\mathsf{N}}_{\phi(c)}$. Symmetrically, $v = w_0 \cdot u$ with $w_0 \in \overline{\mathsf{N}}_{\phi(c)}$. It follows that $w_0 \cdot w_1 = 1$, thus w_0 and w_1 are invertible in the closed semigroup $\overline{\mathsf{N}}_{\phi(c)}$, hence equal to the identity. In particular $u = v$ and $\{x_n\}_{n\in\mathbb{N}}$ converges.

The proof for the sequence $\{a_n\}_{n\in\mathbb{N}}$ is symmetric. □

3.3.1. *Positivity and continuity.* In some cases, it suffices to show that the image of every triple is positive

Proposition 3.16 (Triples suffice). *Let ϕ be a continuous map from an interval I to $\mathbf{F}_\Theta$ such that the image of every ordered triple is positive, then ϕ is positive.*

Proof. Since $[x, y]$ is connected and ϕ is continuous, $V_{\phi(z)}(\phi(x), \phi(y))$ does not depend on z in $]x, y[$ and we denote it $W(x, y)$.

Let $V(t, y)$ be the unique diamond with extremities $\phi(t)$ and $\phi(y)$ obtained in Lemma 2.8 so that

$$V(t, y) \subset W(x, y) . \tag{12}$$

Main step: We first prove that if $x < t < y$, then

$$W(t, y) = V(t, y) . \tag{13}$$

Let us consider

$$U = \{t \in]x, y[\mid W(t, y) = V(t, y)\}$$

Let us write $W(x, y) = \mathsf{N} \cdot \phi(x)$, where N is an open semigroup in $\mathsf{U}_{\phi(y)}$. We can thus write $\phi(t) = n_t \cdot \phi(x)$, with $t \mapsto n_t$ a continuous map defined on $]x, y[$ with values in N; the limit of n_t when t tends to x is equal to id. Then $V(t, y) = n_t W(x, y)$. We now proceed to the proof and show that U is open, non empty and closed.

1. (1) The set U is also the set of t for which there exists s, with $t < s < y$ such that $\phi(s)$ is in $V(t, y)$. In other words, $n_t^{-1} n_s$ belongs to N. Thus U is open.
2. (2) Since N is open, given s, for all t close enough to x we have $n_t^{-1} n_s$ is in N. Thus $n_s \in n_t\mathsf{N}$, hence $\phi(s) \in V(t, y)$. Therefore U is non empty.

(3) Let t be in the closure of U. Let s be $> t$ and let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence in U converging to t. Thus $\{n_{t_m}^{-1}n_s\}_{m\in\mathbb{N}}$ converges to $n_t^{-1}n_s$. Since, for m big enough, we have $s > t_m$, the element $n_{t_m}^{-1}n_s$ belongs to N; hence $n_t^{-1}n_s$ belongs to $\overline{\mathsf{N}}$, *i.e.* $\phi(s)$ belongs to $\overline{V}(t,y)$. As the map ϕ is transverse, $\phi(t)$ is transverse to $\phi(s)$ and this implies that $\phi(s)$ belongs to $V(t,y)$ and $n_t^{-1}n_s$ belongs to N. Therefore t belongs to U and we have completed the proof that U is closed.

The proof of the Equation (13) is now complete.

Conclusion: Let (a,b,c,d) so that $a < b < c < d$, with all subtriples of $(\phi(a),\phi(b),\phi(c),\phi(d))$ positive. By item (3) of Proposition 3.1, we only have to prove that

$$V_{\phi(c)}(\phi(a),\phi(b)) = V^*_{\phi(d)}(\phi(a),\phi(b)) ,$$

Observe that

$$W(a,b) \subset W(a,d) = V_{\phi(b)}(\phi(a),\phi(d)) ,$$

by Equation (13). Thus $\phi(d)$ does not belong to $W(a,b)$ and hence belongs to $W^*(a,b)$ by Lemma 2.8. We thus have

$$V_{\phi(d)}(\phi(a),\phi(b)) = W^*(a,b) = V^*_c(\phi(a),\phi(b)) .$$

This completes the proof that the quadruple $(\phi(a),\phi(b),\phi(c),\phi(d))$ is positive, hence of the proposition.

□

3.4. **Proximal elements.** In this section we show that positive equivariant maps give rise to proximal elements.

We first prove the following proposition for elements in G:

Proposition 3.17. *Let g be in G and let g^-, g^+, and a be in $\mathbf{F}_\Theta$ such that g^- and g^+ are fixed by the action of g and that the quadruple $(g^-,a,g\cdot a,g^+)$ is positive.*

Then the action of g on $\mathbf{F}_\Theta$ is proximal, its attracting fixed point is g^+, and its repelling fixed point is g^-.

Proof. Up to the action of $\mathrm{Aut}_0(\mathfrak{g})$ we can assume that g^- is the point in $\mathbf{F}_\Theta$ with stabilizer equal to P_Θ, g^+ is the point with stabilizer $\mathsf{P}_\Theta^{\mathrm{opp}}$, and that $a = n\cdot g^+$ with n in $\mathsf{N}\subset\mathsf{U}_\Theta$. One then has g belonging to L_Θ and $g\cdot a = n'\cdot g^+$ with n' in N equal to gng^{-1}.

For every α in Θ, we denote by $\pi_\alpha\colon \mathfrak{u}_\Theta \to \mathfrak{u}_\alpha$ the L_Θ-equivariant projection and by C_α the salient invariant open convex cone in $\mathfrak{u}_\alpha$ defining positivity (cf. Remark 2.3).

Let α be in Θ. The positivity of the quadruple $(g^-, a, g\cdot a, g^+)$ implies that the elements $x = \pi_\alpha(\log n)$ and $x' = \pi_\alpha(\log n')$ are both in C_α and that $x - x'$ also belongs to C_α. Let A be the automorphism of $\mathfrak{u}_\alpha$ given by the restriction of the adjoint action of g to the subspace $\mathfrak{u}_\alpha$; one has thus $A(C_\alpha) = C_\alpha$ and $A(x) = x'$. This implies that the set $B := (-x + C_\alpha) \cap (x - C_\alpha)$ is sent into $(-x' + C_\alpha) \cap (x' - C_\alpha)$ by A (where $x - C_\alpha = \{x - y \mid y \in C_\alpha\}$). Therefore the element A is contracting for the norm on $\mathfrak{u}_\alpha$ whose unit ball is the open convex set B.

We deduce from this that all the eigenvalues of the adjoint action of g on $\bigoplus_{\alpha\in\Theta} \mathfrak{u}_\alpha$ are of modulus less than 1. Since this subspace generates $\mathfrak{u}_\Theta$ [29], we have also that all the eigenvalues of g on $\mathfrak{u}_\Theta$ are of modulus < 1. This means precisely that g^+ is an attracting fixed point for the action of g on $\mathbf{F}_\Theta$ and thus g is proximal. For the same reasons, g^- is the repelling fixed point of g. ◻

From this, we immediately get:

Proposition 3.18. *Let γ be a homeomorphism of the circle S^1 having one attracting fixed point γ^+ and one repelling fixed γ^- in S^1. Let $S \subset S^1$ be an infinite γ-invariant set containing γ^+ and γ^- and let ξ be positive map from S to $\mathbf{F}_\Theta$. Assume that there exists an element g in G such that, for all s in S,*

$$\xi(\gamma(s)) = g \cdot \xi(s) ,$$

Then g is proximal and $\xi(\gamma^+)$ is the attracting fixed point of g and similarly $\xi(\gamma^-)$ is the repelling fixed point of g.

4. Triples, Tripods and metrics

In this section, we construct for every positive triple (a, b, c) a complete metric on the diamond $V_c(a, b)$ (Definition 4.9). We also show that this family of metrics satisfies contraction properties (Propositions 4.11 and 4.12).

We first do it for triples of a special type that we call tripods.

4.1. Tripods and metrics.

Recall that in Proposition 2.12, we fixed $\mathcal{H}$ a class of pairs (H, C) where the subgroups H are isogenic to $\mathsf{PSL}_2(\mathbb{R})$ and C is an H-orbit, isomorphic to $\mathbf{P}^1(\mathbb{R})$ and called an H-*circle*.

Definition 4.1. A *tripod* is a triple of pairwise distinct points on C for some (H, C) in $\mathcal{H}$.

A tripod is always positive. If $\tau = (x, t, y)$ is a tripod, we write

$$\tau^- = x\,,\ \tau^0 = t\,,\ \tau^+ = y\,,\ \overline{\tau} := (y, t, x)\,,\ V_\tau := V_{\tau^0}(\tau^-, \tau^+)\,.$$

More generally, for a positive triple $t = (a, b, c)$, we write $t^- = a$, $t^0 = b$, $t^+ = c$ and $V_t \coloneqq V_b(a, c)$.

Let $\mathcal{T}_0$ be the set of tripods. Observe that $\mathrm{Aut}_0(\mathfrak{g})$ acts transitively on the left on the space of tripods, and that the positive circle containing a tripod is unique.

The stabilizer of any tripod is compact (*cf.* Section 3.1.1 below), in particular $\mathrm{Aut}_0(\mathfrak{g})$ acts properly on the space of tripods and we can then fix d an $\mathrm{Aut}_0(\mathfrak{g})$-invariant Riemannian metric on the set of tripods $\mathcal{T}_0$.

4.1.1. *Tripods and the parametrization.* Let us consider (as in Paragraph 2.2.1), the convex cone C and the parametrization $\Psi\colon \mathsf{C} \to \mathsf{N}$ equivariant with respect to L°_Θ (Ψ is given by the product of exponential maps). Note that Ψ extends continuously to a map $\overline{\mathsf{C}} \to \mathsf{U}_\Theta$ that is also L°_Θ-equivariant.

Let h be the element of C corresponding to the unipotent associated to the preferred $\mathsf{SL}_2(\mathbb{R})$ — see the proof of Proposition 2.12. Let K_h be the stabilizer of h in L°_Θ. Since the stabilizer of a positive triple is compact, it follows that K_h is compact.

If now x and y are transverse points in $\mathbf{F}_\Theta$ and $\sigma\colon \mathsf{U}_\Theta \to \mathsf{U}_y$ is a U-pinning at y, then the map

$$\Psi^\sigma\colon \mathsf{C} \mapsto \mathbf{F}_\Theta,\ u \mapsto \sigma \circ \Psi(u) \cdot x\,,$$

is a parametrization of the diamond $V_t(x, y)$ with $t \coloneqq \Psi^\sigma(h)$. We then define

Definition 4.2. Given a tripod $\tau = (x, t, y)$ a *τ-parametrization of the diamond V_τ*, is a map Ψ_τ of the form Ψ^σ so that $\Psi^\sigma(h) = t$.

From the definition follows

Proposition 4.3. *Given a tripod τ, a τ-parametrization of the diamond exists and is unique up to postcomposition by the stabilizer of τ (equivalently up to precomposition by K_h).*

The next proposition is crucial; it insures that a sequence of parametrizations of diamonds associated with tripods converges to the constant map as soon as one sequence in the image converges, precisely

Proposition 4.4 (Contraction in corners)**.** *Let $\{\tau_m\}_{m\in\mathbb{N}}$ be a sequence of tripods, with $\tau_m = (x_m, t_m, y)$, and Ψ_{τ_m} a τ_m-parametrization for all m.*

Assume that $\{x_m\}_{m\in\mathbb{N}}$ converges to a point x transverse to y. Assume that there exists a converging sequence $\{k_m\}_{m\in\mathbb{N}}$ in the cone C, and such that

$$\lim_{m\to\infty} \Psi_{\tau_m}(k_m) = x\,. \tag{14}$$

Then for any sequence $\{k'_m\}_{m\in\mathbb{N}}$ *in* C *that is bounded in* $\overline{\mathsf{C}}$*,*

$$\lim_{m\to\infty} \Psi_{\tau_m}(k'_m) = x \,. \tag{15}$$

Proof. Since x is transverse to y, by replacing τ_m by $u_m\tau_m$ where $\{u_m\}_{m\in\mathbb{N}}$ is a converging sequence in U_y, we may as well assume that $\{x_m\}_{m\in\mathbb{N}}$ is constant and equal to x.

Using the fact that $\mathrm{Aut}_0(\mathfrak{g})$ acts transitively on tripods, let us write $t_m = g_m \cdot t_0$, with g_m fixing x and y. Thus

$$\Psi_{\tau_m}(h) = g_m \cdot t_0 = g_m \cdot \Psi_{\tau_0}(h) = g_m\sigma(\Psi(h)) \cdot x \,.$$

Note that the U-pinning $\sigma\colon \mathsf{U}_\Theta \to \mathsf{U}_y$ comes from an element σ of $\mathrm{Aut}_0(\mathfrak{g})$. Denote $g^0_m = \sigma^{-1}\circ g_m \circ \sigma$; it is an element of $\mathrm{Aut}_0(\mathfrak{g})$ stabilizing the standard unipotent subalgebras $\mathfrak{u}_\Theta$ and $\mathfrak{u}^{\mathrm{opp}}_\Theta$. Up to maybe precomposing by an element of K_h, we may assume that Ψ_{τ_m} is the map $k \mapsto \sigma(\Psi(g^0_m\cdot k))\cdot x$.

Therefore we have, for any $\{\ell_m\}_{m\in\mathbb{N}}$ in $\overline{\mathsf{C}}$, that $\{\Psi_{\tau_m}(\ell_m)\}_{m\in\mathbb{N}}$ converges to x if and only if the sequence $\{g^0_m\cdot \ell_m\}_{m\in\mathbb{N}}$ converges to 0 in $\overline{\mathsf{C}}$.

For any y in $\overline{\mathsf{C}}$, let

$$K(y) \coloneqq \overline{\mathsf{C}} \cap \left(y - \overline{\mathsf{C}}\right).$$

Since $\overline{\mathsf{C}}$ is salient, $K(y)$ is compact for any y.

From the previous discussion, we get that the sequence $\{c_m = g^0_m\cdot k_m\}_{m\in\mathbb{N}}$ converges to 0. Thus, using again the fact that $\overline{\mathsf{C}}$ is salient, for every positive real R, the sequence of compact sets $\{K(R\cdot c_m)\}_{m\in\mathbb{N}}$ converges to $\{0\}$.

Let now $\{k'_m\}_{m\in\mathbb{N}}$ be a sequence in C, bounded in $\overline{\mathsf{C}}$. Since by hypothesis $\{k_m\}_{m\in\mathbb{N}}$ converges in C, there exists a positive real R such that, for all m, $R\cdot k_m - k'_m$ belongs to $\overline{\mathsf{C}}$. In other words: k'_m belongs to $R\cdot k_m - \overline{\mathsf{C}}$. Thus, for all m, $g^0_m\cdot k'_m$ belongs to $K(R\cdot c_m)$. Hence the sequence $\{g^0_m\cdot k'_m\}_{m\in\mathbb{N}}$ converges to 0. This means that the sequence $\{\Psi_{\tau_m}(k'_m)\}_{m\in\mathbb{N}}$ converges to x as wanted. □

4.1.2. *Diamond metrics for tripods.* We choose once and for all a Euclidean distance d_0 on the convex cone C, associated with the Riemannian g_0 induced by a K_h-invariant scalar product on $\mathfrak{u}_\Theta$. This distance d_0 is K_h-invariant and extends to $\overline{\mathsf{C}}$.

Definition 4.5. Given a tripod $\tau = (x,t,y)$, let Ψ_τ be a τ-parametrization of V_τ, let

$$g^+_\tau \coloneqq (\Psi_\tau)_* g_0\ ,\quad g^-_\tau \coloneqq (\Psi_{\overline{\tau}})_* g_0\ ,\quad g_\tau = g^+_\tau + g^-_\tau\ ,$$

as well as d_τ^+, d_τ^-, and d_τ the associated distances so that

$$d_\tau^\pm \leqslant d_\tau \leqslant d_\tau^+ + d_\tau^- \ .$$

The metric g_τ is the *diamond metric* (for the tripod τ) on V_τ, while d_τ is the *diamond distance*.

The terminology is justified by

Proposition 4.6 (Uniqueness and Completeness). *The diamond metric is independent of the choice of the τ-parametrization and only depends on τ. Moreover, d_τ is complete and proper on V_τ.*

There exists a function $F\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\lim_0 F = 1$ and such that the following holds: For any tripods τ and τ' with the same extremities τ_- and τ_+, if $d_\tau(\tau^0, \tau'^0) \leqslant \varepsilon$ then

$$F(\varepsilon)^{-1} d_{\tau'} \leqslant d_\tau \leqslant F(\varepsilon)\, d_{\tau'} \ .$$

Proof. The independence on the parametrization is a consequence of Proposition 4.3 and the fact that d_0 itself is invariant under the group K_h.

Let us now prove completeness. Let $\{u_m\}_{m\in\mathbb{N}}$ be a Cauchy sequence for d_τ, then it is a Cauchy sequence for both d_τ^+ and d_τ^-. It follows that the sequences $\{v_m\}_{m\in\mathbb{N}}$ and $\{w_m\}_{m\in\mathbb{N}}$ defined by

$$v_m = \Psi_\tau^{-1}(u_m),\ w_m = \Psi_{\overline{\tau}}^{-1}(u_m) \ ,$$

are both Cauchy sequences. Since $\overline{\mathsf{C}}$ is complete with respect to the metric d_0, there exist v and w in $\overline{\mathsf{C}}$ such that

$$\lim_{m\to\infty} v_m = v,\ \lim_{m\to\infty} w_m = w \ .$$

Since by construction Ψ_τ and $\Psi_{\overline{\tau}}$ extend continuously to the closure of C,

$$\Psi_\tau(v) = \Psi_{\overline{\tau}}(w) = \lim_{m\to\infty} u_m =: u \ .$$

Obviously u belongs to $\overline{V}_\tau$. By construction $u = n_y \cdot x$ and $u = n_x \cdot y$ where n_y belongs to U_y and n_x belongs to U_x. Since x and y are transverse, it follows that u is transverse to both y and x. Thus u belongs to V_τ. Hence d_τ is complete.

Let us prove the last part. Observe now by hypothesis, there exists ℓ in L_Θ such that $\Psi_{\tau'} = \Psi_\tau \circ \ell$. Thus

$$g_{\tau'}^+ = (\Psi_\tau)_*(\ell_* g_0) \ .$$

Recall that g_0 is induced by a scalar product on $\mathfrak{u}_\Theta$, and that L_Θ acts linearly on $\mathfrak{u}_\Theta$, it then follows that there is a function F_0 on L_Θ such

that
$$F_0(\ell)g_0 \leqslant \ell_*(g_0) \leqslant F_0(\ell)g_0 \ ,$$
and with $F_0(\ell) \underset{\ell\to\mathsf{K}_\Theta}{\longrightarrow} 1$ where K_Θ is a maximal compact subgroup L_Θ. Pushing forward by Ψ_τ we have
$$F_0(\ell)g^+_\tau \leqslant g^+_{\tau'} \leqslant F_0(\ell)g^+_\tau \ .$$
The same holds for g^-_τ and $g^-_{\tau'}$. Hence the same inequality holds for $g_\tau = g^+_\tau + g^-_\tau$ and $g_{\tau'}$; this concludes the proof with the remark that $d_\tau(\tau^0, \tau'^0) \longrightarrow 0$ implies that $\ell \longrightarrow \mathsf{K}_\Theta$. Precisely, we can define
$$F(\epsilon) = \sup\{F_0(h) \mid d_\tau(\tau^0, h(\tau^0)) \leqslant \epsilon\} \ ,$$
and observing that by equivariance F does not depend on the choice of τ. □

Proposition 4.7 (Contraction for tripods)**.** *Let $\{\tau_m\}_{m\in\mathbb{N}}$ be a sequence of tripods. Assume that, for all m in $\mathbb{N}$, $V_{\tau_{m+1}} \subset V_{\tau_m}$ and that*
$$\bigcap_{m\in\mathbb{N}} V_{\tau_m} = \{z\} \ . \tag{16}$$
For any positive R, let $V_{\tau_m}(R)$ be the ball of radius R and center τ^0_m with respect to d_{τ_m}. Then on $V_{\tau_m}(R)$, we have
$$g_{\tau_0} \leqslant k_m \cdot g_{\tau_m} \ ,$$
with $\{k_m\}_{m\in\mathbb{N}}$ converging to zero.

Proof. Since $\mathrm{Aut}_0(\mathfrak{g})$ acts transitively on the space of tripods $\mathcal{T}_0$, it follows that $\tau_m = h_m\cdot\tau_0$, for some h_m in $\mathrm{Aut}_0(\mathfrak{g})$. Since the construction of the tripod metrics is $\mathrm{Aut}_0(\mathfrak{g})$-equivariant, we observe that $g_{\tau_m} = h^*_m g_{\tau_0}$. Then, we take
$$k_m = \sup\left\{\frac{g_{\tau_0}(w,w)}{g_{\tau_m}(w,w)} \;\middle|\; w \in \mathsf{T}V_{\tau_m}(R)\right\} \tag{17}$$
$$= \sup\left\{\frac{g_{\tau_0}(w,w)}{g_{\tau_0}(\mathsf{T}h^{-1}_m w, \mathsf{T}h^{-1}_m w)} \;\middle|\; w \in \mathsf{T}V_{\tau_m}(R)\right\} \tag{18}$$
$$= \sup\left\{\frac{g_{\tau_0}(\mathsf{T}h_m(v), \mathsf{T}h_m(v))}{g_{\tau_0}(v,v)} \;\middle|\; v \in \mathsf{T}V_{\tau_0}(R)\right\} \ . \tag{19}$$
The hypothesis (16) says that $\{h_m\}_{m\in\mathbb{N}}$, seen as a sequences of diffeomorphisms of $\mathbf{F}_\Theta$ converges uniformly on every compact set of V_{τ_0} to the constant map. It follows that $\{h_m\}_{m\in\mathbb{N}}$ also converges C^1 to the constant map on any compact set in V_{τ_0} and hence $\{\mathsf{T}h_m\}_{m\in\mathbb{N}}$ converges to zero uniformly on every compact set of V_{τ_0}. Thus, equality (19) shows that $\{k_m\}_{m\in\mathbb{N}}$ converges to zero. □

4.2. **Positive triples, tripods and metrics.** Our goal is to construct a complete metric on the diamond associated with a positive triple and to prove a generalization of the contraction properties (Propositions 4.11 and 4.12).

4.2.1. *Approximating triples: the tripod defect.* We will first approximate in a rough sense positive triples by tripods. For any positive triple $t = (x, z, y)$, let

$$\mathrm{K}(t) := \inf\left\{ d_\tau(z, \tau^0) \mid \tau \in \mathcal{T}_0\,,\ z \in V_\tau\,,\ (\tau^-, \tau^+) = (x, y) \right\}.$$

We call $\mathrm{K}(t)$ the *tripod defect*.

Observe that $\mathrm{K}(t)$ depends continuously on t, and that the tripod defect vanishes for tripods. Let also

$$\begin{aligned} D(t, K_0) &:= \left\{ \tau \in \mathcal{T}_0 \mid (\tau^-, \tau^+) = (x, y)\,, d_\tau(z, \tau^0) \leqslant K_0 \right\}, \\ D(t) &:= D(t, \mathrm{K}(t))\,. \end{aligned}$$

Proposition 4.8. *(1) Given $K_0 \geqslant \mathrm{K}(t)$, the set $D(t, K_0)$ is compact and non-empty.*

(2) $\mathrm{K}(t) = 0$ if and only if t is a tripod.

(3) For any K_0, there exists a constant $A = A(K_0)$ such that if $t = (a, b, c)$ is a positive triple with $\mathrm{K}(t) \leqslant K_0$, then for every τ_0 and τ_1 in $D(t, K_0)$, we have, on $V_b(a, c)$

$$g_{\tau_0} \leqslant A g_{\tau_1}\,.$$

Furthermore $A(K_0)$ tends to 1 as K_0 goes to 0.

Proof. Let $t = (a, b, c)$ be a positive triple. Let $\{\tau_m\}_{m\in\mathbb{N}}$ be a sequence of tripods such that $(\tau_m^-, \tau_m^+) = (a, c)$ and

$$\{d_{\tau_m}(b, \tau_m^0)\}_{m\in\mathbb{N}}\,,$$

is bounded. Let $\{g_m\}_{m\in\mathbb{N}}$ be a sequence of elements in $\mathrm{Aut}_0(\mathfrak{g})$, stabilizing a and c and such that $\{g_m^{-1}(\tau_m^0)\}_{m\in\mathbb{N}}$ is constant and let τ^0 be this constant. Let $\tau := (a, \tau^0, c)$. It follows that

$$\{d_\tau(g_m^{-1}(b), \tau^0)\}_{m\in\mathbb{N}}\,,$$

is bounded. Since d_τ is a proper metric (*i.e.* every bounded set is relatively compact), the sequence $\{g_m^{-1}(b)\}_{m\in\mathbb{N}}$ — after extracting a subsequence — converges to e with (a, e, c) positive. Since $\mathrm{Aut}_0(\mathfrak{g})$ acts properly on the space of tripods, it follows that $\{g_m\}_{m\in\mathbb{N}}$ is bounded. Thus after taking a subsequence $\{\tau_m\}_{m\in\mathbb{N}}$ converges to a tripod τ_∞, with τ_∞^0 in V_t. This proves that, for all K_0, the set $D(t, K_0)$ is compact.

Since $D(t, K_0)$ is non-empty for $K_0 > \mathrm{K}(t)$, it follows that the decreasing intersection

$$D(t) = \bigcap_{K_0 > \mathrm{K}(t)} D(t, K_0)$$

is not empty.

The second assertion is an immediate consequence of the first.

The third follows from the first as a consequence of the second part of Proposition 4.6. □

4.2.2. *The diamond metric for triples.* The following definition is one of the goal of this section.

Definition 4.9. Let t be a positive triple. The *diamond metric* (for the triple t) g_t is the Riemannian metric on V_t defined as follows: for every x in V_t, the unit ball of $g_{t,x}$ is the John ellipsoid of the union of the unit balls of $g_{\tau,x}$ for τ varying in $D(t)$.

The associated distance is the *diamond metric* d_t.

Explicitely, one has $g_{\tau,x} \leqslant g_{t,x}$ for every τ in $D(t)$ and $g_{t,x} \leqslant g$ for every Euclidean scalar product g on $T_x V_t$ such that $g_{\tau,x} \leqslant g$ for every τ in $D(t)$. Furthermore $g_{t,x}$ is the unique minimizer among the Euclidean scalar products g satisfying the previous condition. It also follows immediately from point (3) of Proposition 4.8 that $g_t \leqslant A g_\tau$ for every τ in $D(t)$ with $A = A(K(t))$.

When t is a tripod, this definition agrees with the one of the previous paragraph thanks to the second item of Proposition 4.8.

As an immediate corollary of Proposition 4.8 and Proposition 4.6, we have

Corollary 4.10. *The diamond metric is complete. Moreover if a sequence of positive triples $\{t_m\}_{m\in\mathbb{N}}$ converges to a tripod τ, then $\{g_{t_m}\}_{m\in\mathbb{N}}$ converges to g_τ on every compact of the diamond V_τ.*

The following Propositions 4.11 and 4.12 are two contractions properties of the diamond metrics that we shall use in the sequel.

Proposition 4.11 (Contraction). *Let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence of positive triples, with $t_m = (a_m, b_m, c_m)$. Assume that the sequence $\{\mathrm{K}(t_m)\}_{m\in\mathbb{N}}$ of tripod defects is bounded. Assume that $V_{t_{m+1}} \subset V_{t_m}$ and that*

$$\bigcap_{m\in\mathbb{N}} V_{t_m} = \{z\} . \tag{20}$$

For any positive R, let $V_{t_m}(R)$ be the ball of radius R and center a_m with respect to d_{t_m}. Then on $V_{t_m}(R)$, we have

$$g_{t_0} \leqslant k_m \cdot g_{t_m} ,$$

with $\{k_m\}_{m\in\mathbb{N}}$ converges to zero.

Proof. By Definition 4.9 of the diamond metric for triples, and Proposition 4.8 it follows that we can find a constant A, such that for all m, we can find a tripod τ_m with the same extremities than t_m and with

$$d_{\tau_m}(\tau_m^0, b_m) \leqslant A\ ,\ \frac{1}{A} g_{\tau_m} \leqslant g_{t_m} \leqslant A g_{\tau_m}\ .$$

The result now follows from the corresponding proposition for tripods: Proposition 4.7. ☐

Proposition 4.12 (Contraction in corners)**.** *Let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence of positive triples, where $t_m = (t_m^-, t_m^0, t_m^+)$. Assume that*

(1) the sequence $\{\mathrm{K}(t_m)\}_{m\in\mathbb{N}}$ of tripod defects is bounded;
(2) the sequences $\{t_m^-\}_{m\in\mathbb{N}}$ and $\{t_m^+\}_{m\in\mathbb{N}}$ converge to transverse points a and c respectively;
(3) There exists $\{u_m\}_{m\in\mathbb{N}}$ a sequence of elements of $\mathbf{F}_\Theta$, such that u_m belongs to V_{t_m}, the sequence $\{d_{t_m}(t_m^0, u_m)\}_{m\in\mathbb{N}}$ uniformly bounded, and $\lim_{m\to\infty} u_m = a$.

Then $\lim_{m\to\infty} t_m^0 = a$.

Proof. By the first hypothesis and Proposition 4.8, we can find a constant A, a sequence of tripods $\{\tau_m\}_{m\in\mathbb{N}}$ with $\tau_m^\pm = t_m^\pm$ and such that

$$d_{\tau_m} \leqslant A\, d_{t_m}\ .$$

In particular, we have that $\{d_{\tau_m}(t_m^0, \tau_m^0)\}_{m\in\mathbb{N}}$ and $\{d_{\tau_m}(u_m, \tau_m^0)\}_{m\in\mathbb{N}}$ are uniformly bounded. The result now follows by applying twice Proposition 4.4. Indeed, since $\{d_{\tau_m}(t_m^0, \tau_m^0)\}_{m\in\mathbb{N}}$ is uniformly bounded, it follows that $t_m^0 = \Psi_{\tau_m}(k_m)$ with $\{k_m\}_{m\in\mathbb{N}}$ bounded. Hence by Proposition 4.4 (applied to any converging subsequence of $\{k_m\}_{m\in\mathbb{N}}$), with $k'_m = h$, yields that

$$\lim_{m\to\infty} \tau_m^0 = a\ .$$

Applying again Proposition 4.4 to $\{k'_m\}_{m\in\mathbb{N}}$ with $\Psi_{\tau_m}(k'_m) = u_m$ yields that

$$\lim_{m\to\infty} u_m = a\ .$$

This concludes the proof. ☐

5. Positive representations are Anosov

In this section we introduce the notion of positive representations of a surface group. We then show that any Θ-positive representation is Θ-Anosov, establishing Theorem A and Corollary B from the introduction. As in the introduction, S is a connected oriented closed surface of genus at least 2.

Definition 5.1 (Positive representations)**.** Let G be a semi-simple Lie group admitting a Θ-positive structure. A representation $\rho\colon \pi_1(S) \to \mathsf{G}$ is Θ-*positive* if there exist a non-empty $\pi_1(S)$-invariant subset A of $\partial_\infty\pi_1(S)$ and a positive ρ-equivariant map ξ from A to $\mathbf{F}_\Theta$.

The set A is necessarily dense since the action of $\pi_1(S)$ on $\partial_\infty\pi_1(S)$ is minimal. We will often say that a representation is positive if it is Θ-positive.

5.1. **Anosov representations.** Let us recall at this stage the definition of a Θ-Anosov representation from [30]. For simplicity we restrict ourselves to the case of representations of $\pi_1(S)$. Let us equip the surface S with an auxiliary hyperbolic metric. Let $\mathsf{U}S$ be the unit tangent bundle of S equipped with its geodesic flow ϕ_t. Let us also freely identify the space of cyclically oriented triples of $\partial_\infty\pi_1(S)$ with the unit tangent bundle $\mathsf{U}\mathbf{H}^2$ of the universal cover of S.

Let ρ be a representation of $\Gamma := \pi_1(S)$ in G. Let $\mathcal{F}_\Theta$ the flat $\mathbf{F}_\Theta$-bundle over $\mathsf{U}S$ associated with ρ, and Φ_t the parallel transport on $\mathcal{F}_\Theta$ along ϕ_t.

The representation ρ is Θ-*Anosov* if there exists a ρ-equivariant continuous map, called the *limit map*.

$$\xi\colon \partial_\infty\Gamma \to \mathbf{F}_\Theta \ ,$$

such that the corresponding section Ξ of $\mathcal{F}_\Theta$ (which is constant along the leaves of the weakly unstable foliation) satisfies the following *contraction property*: there exist an open neighborhood $\mathcal{V}$ of the image of Ξ that is a fiber bundle over $\mathsf{U}S$ with fiber $\mathcal{V}_x$ for x in $\mathsf{U}S$, a continuous family of Riemannian metric g_x on the fibers $\mathcal{V}_x$, and some positive number T such that $\Phi_{-T}(\mathcal{V}) \subset \mathcal{V}$, and, for all x in $\mathcal{V}$,

$$(\Phi_T)^* g_x \leqslant \frac{1}{2} g_{\phi_T(x)} \ ,$$

where g_x is g restricted to $\mathcal{V}_x$.

Note that there is another section Ξ^*, which is constant along the leaves of the weakly stable foliation and is contracted under Φ_{-T}.

Observe that, in general, the existence of a continuous equivariant map, even sending distinct points to transverse points, is weaker than the condition of being Anosov.

To establish the Anosov property for positive representations, we first extend the positive boundary map to a left-continuous boundary map and to a right-continuous boundary map using Corollary 3.15. We prove then that these extensions are continuous (and thus coincide), and then deduce the Anosov property using the contraction property of the diamond metrics (Proposition 4.11).

5.2. **Properness.** The following definition will be used several times in the sequel: an application f defined on a subset A of a topological set X, with values in some topological set Y is *bounded* if for every compact set K in X, $f(A \cap K)$ is relatively compact.

Lemma 5.2. *Let A be a dense set in the circle. Let $\phi_\pm$ be positive maps from A to $\mathbf{F}_\Theta$. We assume that, for all cyclically oriented quadruple (x, y, z, t) in A and for any choice of ε, η, ν, and β in $\{+, -\}$, the quadruple*

$$(\phi_\varepsilon(x), \phi_\eta(y), \phi_\nu(z), \phi_\beta(t))$$

is a positive. Let A^3_+ be the set of triples of pairwise distinct elements of A.

Then, for any ε, η, and ν in $\{+, -\}$, $\phi_\varepsilon \times \phi_\eta \times \phi_\nu$ is bounded as a map from A^3_+ to the space $\mathcal{T}$ of positive triples in $\mathbf{F}_\Theta$.

Proof. Let $\chi = (x_1, x_2, y_1, y_2, z_1, z_2)$ be a cyclically oriented sextuplet in S^1. Let I_χ be the subset of $(S^1)^3$ given by

$$I_\chi = \{(X, Y, Z) \mid x_1 < X < x_2 < y_1 < Y < y_2 < z_1 < Z < z_2\}\,.$$

Observe that I_χ consists of cyclically oriented triples. Let

$$K = \phi_\epsilon \times \phi_\eta \times \phi_\nu \left(I_\chi \cap A^3_+\right).$$

It is enough to show that $\overline{K} \subset \mathcal{T}$, where the closure is taken in $\mathbf{F}^3_\Theta$.

Let us fix, by density, a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, and c_2 in A such that

$$(x_1, x_2, a_0, a_1, a_2, y_1, y_2, b_0, b_1, b_2, z_1, z_2, c_0, c_1, c_2)$$

is cyclically oriented. To lighten notation, we set $\alpha_i = \phi_\epsilon(a_i)$, $\beta_i = \phi_\eta(b_i)$, and $\gamma_i = \phi_\nu(c_i)$ for $i = 0, 1, 2$.

From the positivity of the maps and thus of the image of the 15-tuple defined above, it follows that if (x, y, z) belongs to K, then

$$x \in V^*_{\alpha_1}(\gamma_2, \alpha_0)\,,\ y \in V^*_{\beta_1}(\alpha_2, \beta_0)\,,\ z \in V^*_{\gamma_1}(\beta_2, \gamma_0)\,.$$

Thus if (a, b, c) belongs to $\overline{K}$, then

$$a \in \overline{V}^*_{\alpha_1}(\gamma_2, \alpha_0)\,,\ b \in \overline{V}^*_{\beta_1}(\alpha_2, \beta_0)\,,\ c \in \overline{V}^*_{\gamma_1}(\beta_2, \gamma_0)\,.$$

Using the inclusion Corollary 3.9, we get

$$a \in V^*_{\alpha_1}(\gamma_1, \alpha_1)\,,\ b \in V^*_{\beta_1}(\alpha_1, \beta_1)\,,\ c \in V^*_{\gamma_1}(\beta_1, \gamma_1)\,.$$

By the necklace Corollary 3.2, (a, b, c) is a positive triple, *i.e.* it belongs to $\mathcal{T}$. This concludes the proof. □

Proposition 5.3. *Let $\rho\colon \pi_1(S) \to \mathsf{G}$. Let ξ be a positive $\pi_1(S)$-invariant map from $\partial_\infty\pi_1(S)$ to $\mathbf{F}_\Theta$.*

Let $\mathcal{T}_{\pi_1(S)}$ be the set of triples of pairwise distinct elements in $\partial_\infty\pi_1(S)$, and let $\mathcal{T}$ be the set of positive triples in $\mathbf{F}_\Theta$. Let Ξ be the map from $\mathcal{T}_{\pi_1(S)}$ to $\mathcal{T}/\mathsf{G}$, defined by

$$\Xi(x,y,z) := [\xi(x), \xi(y), \xi(z)] \ .$$

Then the image of Ξ is relatively compact.

Proof. The map Ξ is invariant by the diagonal action of $\pi_1(S)$. The result follows then from Lemma 5.2 using the fact that $\pi_1(S)$ acts cocompactly on $\mathcal{T}_{\pi_1(S)}$. □

5.3. **An *a priori* bound on the tripod defect.** For any positive triple t, let $\mathrm{K}(t)$ be the tripod defect introduced in paragraph 4.2. Then Proposition 5.3 implies an a priori bound on the tripod defect.

Proposition 5.4. *Let $\rho\colon \pi_1(S) \to \mathsf{G}$. Let ξ be a ρ-equivariant positive map from a $\pi_1(S)$-invariant dense subset A of $\partial_\infty\pi_1(S)$ to $\mathbf{F}_\Theta$. Then there exists a constant K_0 such that for all triple t of pairwise distinct points in the closure of $\xi(A)$, we have*

$$\mathrm{K}(t) \leqslant K_0 \ .$$

Proof. This is an immediate consequence of Proposition 5.3 and the fact that K is a continuous function on $\mathcal{T}$. □

5.4. **Continuity of equivariant positive maps.** Let ρ be a Θ-positive representation, A a non-empty $\pi_1(S)$-invariant subset of $\partial_\infty\pi_1(S)$ and $\xi\colon A \to \mathbf{F}_\Theta$ the positive ρ-equivariant boundary map. Then, by Corollary 3.15, there exist a unique right-continuous ρ-equivariant boundary map $\xi_+\colon \partial_\infty\pi_1(S) \to \mathbf{F}_\Theta$ and a unique left-continuous ρ-equivariant boundary map $\xi_-\colon \partial_\infty\pi_1(S) \to \mathbf{F}_\Theta$, coinciding with the map ξ on dense subset.

Let $\mathcal{T}_{\pi_1(S)}$ be the set of triples of pairwise distinct points of $\partial_\infty\pi_1(S)$. For $t = (x,y,z)$ in $\mathcal{T}_{\pi_1(S)}$, let us define

$$\tau(t) = (\xi_+(x), \xi_+(y), \xi_+(z)) \ .$$

Lemma 5.5. *The $\pi_1(S)$-invariant function f defined by*

$$f(x,y,z) = d_{\tau(t)}(\xi_+(y), \xi_-(y)) \ .$$

is bounded: there is a constant D such that, for all (x,y,z) in $\mathcal{T}_{\pi_1(S)}$, $f(x,y,z) \leqslant D$.

Proof. Let Q be the set of quadruples (a,b,c,d) in $\mathbf{F}_\Theta$ such that there exists a diamond V with extremities a and d and containing both b and c. Using Lemma 5.2, we see that the map

$$(x,y,z) \mapsto (\xi_+(x), \xi_+(y), \xi_-(y), \xi_+(z))\ ,$$

from $\mathcal{T}_{\pi_1(S)}$ to Q is bounded. As the real valued function on Q sending a quadruple (a,b,c,d) to $d_{(a,b,d)}(b,c)$ is continuous, we get the result. □

Lemma 5.6. *The map ξ_+ is continuous.*

Proof. Since ξ_+ is right-continuous we only have to prove that it is left-continuous. Let x and y be in $\partial_\infty\pi_1(S)$, and let $\{x_m\}_{m\in\mathbb{N}}$ be a sequence in $\partial_\infty\pi_1(S)$, such that (x_m, x, y) is cyclically oriented with respect to the orientation on $\partial_\infty\pi_1(S)$, and that $\{x_m\}_{m\in\mathbb{N}}$ converges to x. Let $t_m = (\xi_+(x_m), \xi_+(x), \xi_+(y))$.

Recall that by Corollary 3.15, $\{\xi_+(x_m)\}_{m\in\mathbb{N}}$ converges to $\xi_-(x)$.

We now apply Proposition 4.12 to the following setting:

$$t_m^- = \xi_+(x_m)\ ,\ t_m^0 = \xi_+(x)\ ,\ u_m = \xi_-(x)\ ,\ t_m^+ = \xi_+(y)\ .$$

Since

$$\{d_{t_m}(t_m^0, u_m)\}_{m\in\mathbb{N}} = \{d_{t_m}(\xi_+(x), \xi_-(x))\}_{m\in\mathbb{N}}$$

is bounded by Lemma 5.5 and $\{\mathrm{K}(t_m)\}_{m\in\mathbb{N}}$ is bounded by Proposition 5.4, we get that

$$\lim_{m\to\infty} \xi_+(x_m) = \xi_+(x)\ .$$

This proves that ξ_+ is left-continuous. □

As a consequence we obtain

Proposition 5.7. *Let $\rho\colon \pi_1(S) \to \mathsf{G}$ be a positive representation and ξ the positive ρ-invariant boundary map from a $\pi_1(S)$-invariant dense subset of $\partial_\infty\pi_1(S)$ to $\mathbf{F}_\Theta$. Then ξ extends to a ρ-equivariant positive continuous map from $\partial_\infty\pi_1(S)$ to $\mathbf{F}_\Theta$.*

The extended map ξ from $\partial_\infty\pi_1(S)$ to $\mathbf{F}_\Theta$ will be called *the positive boundary map* of ρ.

5.5. **The Anosov property.** We are now in position to prove Theorem A from the introduction. More precisely we show

Proposition 5.8. *Let ρ from $\pi_1(S)$ to G be a positive representation and ξ from $\partial_\infty\pi_1(S)$ to $\mathbf{F}_\Theta$ be the ρ-equivariant continuous positive boundary map. Then ρ is Θ-Anosov and its boundary map is ξ.*

Let us start with a general lemma

Lemma 5.9. *Let $\{b^0_m\}_{m\in\mathbb{N}}$ and $\{b^1_m\}_{m\in\mathbb{N}}$ be two sequences in $\mathbf{F}_\Theta$ converging to c. Let d_0 and d_1 be in $\mathbf{F}_\Theta$ such that (d_0, c, d_0) is a positive triple and assume that, for all m in $\mathbb{N}$,*

$$(d_0, b^0_m, b^1_m, d_1)$$

is a positive quadruple. Let V_m be the unique diamond with extremities b^0_m and b^1_m contained in the diamond $V_c(d_0, d_1)$. Then

$$\lim_{m\to\infty} V_m = \{c\} .$$

Proof. Let a_0 be in $V^*_c(d_0, d_1)$ and a_1 be in $V^*_c(a_0, d_1)$ so that (a_0, d_0, c, d_1, a_1) is a positive quintuple and, for all big enough m, $(a_0, d_0, b^0_m, b^1_m, d_1, a_1)$ is a positive configuration.

Let z_m belong to V_m, we want to prove that

$$\lim_{m\to\infty} z_m = c .$$

Let p in $\overline{V}_c(d_0, d_1)$ be an accumulation point of the sequence $\{z_m\}_{m\in\mathbb{N}}$. Up to extracting a subsequence we may assume

$$\lim_{m\to\infty} z_m = p ,$$

By Corollary 3.9, p belongs to $V_c(a_0, a_1)$ and in particular it belongs to $\Omega_{a_0} \cap \Omega_{a_1}$. From the fact that z_m belongs to $V_{d_0}(a_0, b^1_m)$ we get that p belongs to $\overline{V}_{d_0}(a_0, c)$; similarly p belongs to $\overline{V}_{d_1}(a_1, c)$. Therefore

$$p \in \overline{V}_{d_0}(a_0, c) \cap \overline{V}_{d_1}(a_1, c) \cap \Omega_{a_0} \cap \Omega_{a_1} .$$

Let $V = V_{d_0}(a_0, c)$, and recall that by Lemma 2.8,

$$V_{d_1}(a_1, c) \subset V^* .$$

Finally remark that

$$\overline{V} \cap \Omega = \overline{\mathsf{N}}_{a_1} \cdot c \, , \overline{V}^* \cap \Omega = \overline{\mathsf{N}}_{a_1}^{-1} \cdot c \, ,$$

for the (positive) semigroup N_{a_1} in U_{a_1}. Since

$$\overline{\mathsf{N}}_{a_1} \cap \overline{\mathsf{N}}_{a_1}^{-1} = \{\mathrm{id}\} ,$$

one has $p = c$, which is what we wanted to prove. □

Proof of Proposition 5.8. The chosen hyperbolization of S defines a $\pi_1(S)$-invariant cross-ratio on $\partial_\infty\pi_1(S) \cong \mathbf{P}^1(\mathbb{R})$. Let us also fix an orientation on $\partial_\infty\pi_1(S)$. For any cyclically oriented triple $t = (x, y, z)$, let us consider the harmonic (with respect to the cross-ratio) quadruple (x, y, z, w), and let then

$$Y_t := V_{\xi(z)}(\xi(y), \xi(w)) .$$

By construction Y_t is an open neighborhood of $\xi(z)$. Moreover if (x, y_1, y_0, z) is a cyclically oriented quadruple,

$$Y_{(x,y_0,z)} \subset Y_{(x,y_1,z)} . \tag{21}$$

Finally, since ξ is continuous , by Lemma 5.9

$$\lim_{y\to z} Y_{(x,y,z)} = \{\xi(z)\} . \tag{22}$$

We now deduce the Anosov property from Assertion (22).

Recall that the chosen uniformization of the surface enables us to identify the space of triples in the boundary at infinity with the unit tangent bundle $\mathsf{U}\mathbf{H}^2$ of the universal cover of S. Let $\{\phi_s\}_{s\in\mathbb{R}}$ be the geodesic flow on $\mathsf{U}\mathbf{H}^2$. Let $\mathcal{F}$ be the trivial bundle $\mathbf{F}_\Theta \times \mathsf{U}\mathbf{H}^2$. The actions of $\pi_1(S)$ on $\mathsf{U}\mathbf{H}^2$ and on $\mathbf{F}_\Theta$ —through ρ— give rise to an action of $\pi_1(S)$ on $\mathcal{F}$.

Let $\mathcal{U}$ be the subbundle of $\mathcal{F}$ with open fibers given by

$$\mathcal{U} = \{(x, v) \in \mathcal{F} \mid v \in \mathsf{U}\mathbf{H}^2,\ \ x \in Y_v\} .$$

The bundle $\mathcal{U}$ is invariant by the $\pi_1(S)$-action, moreover it has a canonical section σ_0 given by

$$\sigma_0(x, y, z) = \xi(z) .$$

Let us lift the flow $\{\phi_s\}_{s\in\mathbb{R}}$ to a flow $\{\Phi_s\}_{s\in\mathbb{R}}$ on $\mathcal{F}$ acting trivially on the first factor. By assertion (21), for all positive s

$$\Phi_{-s}(\mathcal{U}) \subset \mathcal{U} .$$

Moreover the section σ_0 is invariant by $\{\Phi_s\}_{s\in\mathbb{R}}$.

The diamond metric g_t and the diamond distance d_t on each Y_t give a metric on each fiber of $\mathcal{U}$ which depends continuously on the base and is equivariant under the action of $\pi_1(S)$.

For any R, let $\mathcal{U}(R)$ be the neigbourhood of the image of the section σ_0, given by

$$\mathcal{U}(R) = \{(x, v) \in \mathcal{U} \mid v \in \mathsf{U}\mathbf{H}^2,\ \ d_v(x, \sigma_0(v)) \leqslant R\} .$$

It now follows from assertion (22) and Proposition 4.11, that for every u in $\mathsf{U}\mathbf{H}^2$, there is s_u such that, for all (x, u) in $\mathcal{U}(R)$

$$\text{for all } s \geqslant s_u,\ \ g_{\Phi_s(x,u)} \circ \mathsf{T}_{(x,u)}\Phi_{-s} \leqslant \frac{1}{2} g_{(x,u)} \quad . \tag{23}$$

Let now s be the real valued function on $\mathsf{U}\mathbf{H}^2$ defined by

$$s(u) = \inf\{s_u \mid s_u \text{ satisfies assertion (23) on } \mathcal{U}(R)\} .$$

The function $u \mapsto s(u)$ is upper semicontinuous and invariant under the action of $\pi_1(S)$. Thus by compactness of $\mathsf{U}\mathbf{H}^2/\pi_1(S)$ the function has an upper bound s_0. Then for all s greater than s_0

$$\Phi_s^* g \leqslant \frac{1}{2} g\,,$$

on $\mathcal{U}(R)$. In other words, the action of $\{\Phi_{-s}\}_{s\in\mathbb{R}}$ is contracting on $\mathcal{V} = \mathcal{U}(R)$ and σ_0 is an invariant section.

Thus ρ is Θ-Anosov according to the definition given in the beginning of the section and ξ is its limit curve. □

Now Corollary B in the introduction follows directly from the openness of the set of Θ-Anosov representations. More precisely

Proof of Corollary B. A positive representation ρ_0 with limit map ξ_0, is Θ-Anosov. Thus there is an open neighborhood U of ρ_0 containing only Θ-Anosov representations. For any ρ in this neighbourhood, let ξ_ρ be the limit map. Note that this map is equivariant and transverse. The map $\rho \mapsto \xi_\rho$ which sends an Anosov representation to its limit curve is continuous. Since ξ_{ρ_0} is positive it sends pairwise distinct triples in $\partial_\infty\pi_1(S)$ to positive triples. Moreover $\pi_1(S)$ acts cocompactly on the set of triples of pairwise distinct points of $\partial_\infty\pi_1(S)$, thus, for ρ close enough to ρ_0, the continuous curve ξ_ρ sends pairwise distinct triples to positive triples. Hence by Proposition 3.16, ξ_ρ is positive. □

Remark 5.10. The definition of Θ-positive representations can be made in more generality for non-elementary word hyperbolic group Γ whose boundary admits a cyclic ordering. This holds if Γ is a surface group, but also if Γ is virtually free. For example, an appropriate extension of the arguments in this section shows that a representation of Γ is Θ-Anosov if it admits a ρ-equivariant positive boundary map $\xi\colon \partial_\infty\Gamma \to \mathbf{F}_\Theta$.

6. Closedness

In this section we consider the space $\mathrm{Hom}^\Theta(\pi_1(S), \mathsf{G})$ of homomorphisms from $\pi_1(S)$ to G, which contains a contains a Θ-loxodromic element, as well as the set $\mathrm{Hom}^\Theta(\pi_1(S), \mathsf{G})$ of homomorphisms from $\pi_1(S)$ to G, who do not factor, even after taking a finite index subgroup through a parabolic subgroup of G.

We show that the set of Θ-positive representations $\mathrm{Hom}_{\Theta-\mathrm{pos}}(\pi_1(S), \mathsf{G})$ is an open and closed subset of $\mathrm{Hom}^\Theta(\pi_1(S), \mathsf{G})$, hence a union of connected components.

We first have

Proposition 6.1. *Every Θ-positive representation is an element of the set* $\mathrm{Hom}^{\Theta}(\pi_1(S), \mathsf{G})$. *Moreover, every Θ-positive representation has a compact centralizer and does not factor through a proper parabolic subgroup of* G.

Proof. The first part is a consequence of Proposition 3.18 Let us first note that since the centralizer of a positive triple is compact, the centralizer of a positive representation is compact as well. Let $\rho\colon \pi_1(S) \to \mathsf{G}$ be a positive representation with ρ-equivariant positive boundary map $\xi\colon \partial_\infty\pi_1(S) \to \mathbf{F}_\Theta$. Then ρ is Θ-Anosov with boundary map ξ. This remains true when restricting the representation to a finite index subgroup. To argue by contradiction we can thus assume that without loss of generality $\rho(\pi_1(S))$ is contained in a proper parabolic subgroup of G. We consider the semi-simplification ρ^{ss} of ρ, whose image is contained in a Levi factor of the parabolic subgroup.

By [21, Proposition 1.8] the semi-simplification ρ^{ss} is Θ-Anosov, denote ξ^{ss} the ρ^{ss}-equivariant boundary map. Since ρ^{ss} belongs to the closure of the G-orbit of ρ; there exists thus a sequence $\{g_m\}_{m\in\mathbb{N}}$ in G such that ρ^{ss} is the limit of $g_m\rho g_m^{-1}$. Since the boundary map ξ^{ss} is transverse, Lemma 3.5 implies that the boundary map ξ^{ss} is also positive as well. But this is a contradiction because the centralizer of ρ^{ss} in G contains the center of the Levi factor of the parabolic subgroup which is non-compact. □

By a classical result of Borel and Tits [6, Corollaire 3.3] (proved also by Morozov [34] in characteristic zero), the set $\mathrm{Hom}^{*}(\pi_1(S), \mathsf{G})$ is contained in the set of reductive homomorphisms, *i.e.* representations $\rho\colon \pi_1(S) \to \mathsf{G}$, whose Zariski closure is reductive. Thus a direct consequence of Theorem 6.1 is

Corollary 6.2. *Let $\rho\colon \pi_1(S) \to \mathsf{G}$ be a Θ-positive representation, then the Zariski closure of $\rho(\pi_1(S))$ is reductive.*

We also show the following result as a consequence of a result of Benoist–Labourie [2].

Proposition 6.3. *Assume that the Zariski closure H of the image of ρ is such that the exponential of every element in the open Weyl chamber of H is loxodromic with respect to $\mathcal{F}$. Then ρ belongs to* $\mathrm{Hom}^{\Theta}(\pi_1(S), \mathsf{G})$.

In particular, every representation with Zariski dense image belongs to $\mathrm{Hom}^{\Theta}(\pi_1(S), \mathsf{G})$.

Proof. Indeed by [2, Theorem A.1.1], an element h of the image of ρ has an hyperbolic part whic belongs to the Weyl Chamber. Hence h is loxodromic. □

We expect that the list of possible Zariski closures of Θ-positive representations is indeed restrictive. Classifications of the Zariski closures for maximal representations were given in [11, 12, 26, 27] and for Hitchin representations in [22, 36].

Since the set of Θ-positive representations is open in $\mathrm{Hom}(\pi_1(S), \mathsf{G})$ by Corollary B, it is also open in $\mathrm{Hom}^*(\pi_1(S), \mathsf{G})$.

We will now show

Theorem 6.4. *The set of Θ-positive homomorphisms is closed in the set* $\mathrm{Hom}^\Theta(\pi_1(S), \mathsf{G})$.

We will first prove the following proposition of independent interest:

Proposition 6.5. *Let $\{\rho_m\}_{m\in\mathbb{N}}$ be a sequence of Θ-positive representations converging to a representation ρ_∞. Let ξ_m be the limit curve of ρ_m. Assume that we can find x_0 and y_0 in $\partial_\infty\pi_1(S)$ such that $\{(\xi_m(x_0), \xi_m(y_0))\}_{m\in\mathbb{N}}$ converges to a transverse pair, then ρ_∞ is positive.*

6.1. **Proof of proposition 6.5.** We fix a countable set A in $\partial_\infty\pi_1(S)$, invariant by $\pi_1(S)$ and containing x_0 and y_0. We may now assume, by the Cantor diagonal argument, that $\{\xi_m|_A\}_{m\in\mathbb{N}}$ converges simply to a map ξ_∞ from A to $\mathbf{F}_\Theta$. By hypothesis $\xi_\infty(x_0)$ and $\xi_\infty(y_0)$ are transverse.

For any pair of distinct points (x, y) in A^2, denote by $]x, y[$ the interval in the oriented circle $\partial_\infty\pi_1(S)$ with origin x and extremity y, let c be in $A\cap]x, y[$ and set

$$W_\infty(x, y) := \lim_{n\to\infty} \overline{V}^*_{\xi_n(c)}(\xi_n(x), \xi_n(y)) ,$$

the convergence being for the Hausdorff topology, and using again the Cantor diagonal extraction, we can and will assume that all those sequences converge of all (x, y) in A^2 with $x \neq y$. Observe that $W_\infty(x, y)$ only depends on x, y, and the interval $]x, y[$. Furthermore the following equivariance property holds: $\rho_\infty(\gamma)W_\infty(x, y) = W_\infty(\gamma \cdot x, \gamma. \cdot y)$.

Lemma 6.6. *Assume that $\xi_\infty(x)$ and $\xi_\infty(y)$ are transverse then $W_\infty(x, y)$ is a closure of a diamond with extremities $\xi_\infty(x)$ and $\xi_\infty(y)$ and is Zariski dense.*

Proof. Since $\xi_\infty(x)$ and $\xi_\infty(y)$ are transverse, $W_\infty(x, y)$ is the closure of a diamond (see Proposition 3.8). It thus contains an open set, and in particular is Zariski dense. □

Lemma 6.7. *For every pairs of distinct points (x, y) and (z, t) in A, one has*

$$\overline{W}^Z_\infty(x, y) = \overline{W}^Z_\infty(z, t)$$

where $\overline{M}^Z$ denotes the Zariski closure of a set M. In particular, for all distinct x and y, $W_\infty(x,y)$ is Zariski dense.

Observe that only the last assertion depends on the assumption that $\xi_\infty(x_0)$ and $\xi_\infty(y_0)$ are transverse.

Proof. We shall use freely the following fact. If γ is an algebraic automorphism of a variety V, if B is a Zariski closed subset such that $\gamma(B) \subset B$ then $\gamma(B) = B$.

We first prove that if $[u,v] \subset [w,s]$, then we have

$$\overline{W}^Z_\infty(u,v) = \overline{W}^Z_\infty(w,s) \ . \tag{24}$$

We can always find an element γ of $\pi_1(S)$ such that

$$\gamma[w,s] \subset [u,v] \ .$$

Thus

$$\rho_\infty(\gamma)\Big(\overline{W}^Z_\infty(w,s)\Big) \subset \overline{W}^Z_\infty(u,v) \subset \overline{W}^Z_\infty(w,s) \ .$$

By the initial observation we get that

$$\overline{W}^Z_\infty(w,s) \subset \overline{W}^Z_\infty(u,v) \subset \overline{W}^Z_\infty(w,s) \ ,$$

and thus the assertion (24) follows. Take now γ in $\pi_1(S)$ such that

$$\gamma[x,y] \subset [x,y] \ , \ \ \gamma[x,y] \cup [z,t] \neq \partial_\infty \pi_1(S) \ .$$

We can then find distinct points u and v such that

$$(\gamma[x,y] \cup [z,t]) \subset [u,v] \ .$$

Thus, applying thrice assertion (24), we have

$$\overline{W}_\infty(x,y) = \overline{W}^Z_\infty(\gamma\cdot x, \gamma\cdot y) = \overline{W}^Z_\infty(u,v) = \overline{W}^Z_\infty(z,t) \ .$$

The last assertion follows from the fact that $\xi_\infty(x_0)$ and $\xi_\infty(y_0)$ are transverse and thus $W_\infty(x_0,y_0)$ is Zariski dense by Lemma 6.6. □

We are now in the position to show that ρ_∞ is Θ-positive. This will be a consequence of the following proposition:

Proposition 6.8. *For any pair of distinct points (x,y), the pair $(\xi_\infty(x), \xi_\infty(y))$ is transverse. Moreover, ξ_∞ is a positive map.*

Proof. Let (x,y,z) be a triple of pairwise distinct points in $\partial_\infty\pi_1(S)$. Let us denote for simplicity $x_n = \xi_n(x)$, $y_n = \xi_n(y)$ and $z_n = \xi_n(z)$ for n in $\mathbb{N} \cup \{\infty\}$. We choose diamonds by letting

$$V^0_n = V^*_{z_n}(x_n,y_n) \ , \ \ V^1_n = V^*_{y_n}(x_n,z_n) \ , \ \ V^2_n = V^*_{y_n}(z_n,y_n) \ .$$

Since $W_\infty(x,y)$, $W_\infty(y,z)$, and $W_\infty(z,x)$ are Zariski dense, we can pick three points a, b, and c so that

(1) $a \in W_\infty(x,y)$, $b \in W_\infty(y,z)$, $c \in W_\infty(z,x)$,
(2) a,b,c are pairwise transverse,
(3) any point in $\{a,b,c\}$ is transverse to any point in $\{x_\infty, y_\infty, z_\infty\}$.

Let now pick sequences $\{a_m\}_{m\in\mathbb{N}}$, $\{b_m\}_{m\in\mathbb{N}}$, and $\{c_m\}_{m\in\mathbb{N}}$, with $a_m \in V^0_m$, $b_m \in V^1_m$, and $c_m \in V^2_m$, and converging to a, b, and c respectively.

We will now apply the necklace property several times. By Proposition 3.2, (a_m, b_m, c_m) is a positive triple and since a,b,c are pairwise transverse it follows that (a,b,c) is a positive triple.

Then, since x_m belongs to $V^*_{b_m}(a_m, c_m)$, it follows that x_∞ belongs to $\overline{V^*_b}(a,c)$. Since x_∞ is transverse to both a and c, x_∞ belongs to $V^*_b(a,c)$. Symmetrically y_∞ belongs to $V^*_c(a,b)$, z_∞ belongs to $V^*_a(c,b)$. Applying Proposition 3.2 again, $(x_\infty, y_\infty, z_\infty)$ is a positive triple.

The fact that, for any cyclically oriented quadruple (x,y,z,w), the quadruple $(\xi_\infty(x), \xi_\infty(y), \xi_\infty(z), \xi_\infty(w))$ is positive, now follows from Proposition 3.1.(3). Hence the positivity of ξ_∞ by definition. $\square$

6.2. **Proof of Theorem 6.4.** We consider a sequence $\{\rho_m\}_{m\in\mathbb{N}}$ of Θ-positive representations converging to a representation ρ_∞. Let $\{\xi_m\}_{m\in\mathbb{N}}$ be the corresponding sequence of positive limit maps. Our assumption is that image of ρ_∞ contain a Θ-loxodromic element $\rho_\infty(\gamma_0)$.

We fix a countable set A in $\partial_\infty\pi_1(S)$, invariant by $\pi_1(S)$. and containing γ_0^+ and γ_0^-. We may now assume applying the Cantor diagonal argument, that $\{\xi|_A\}_{m\in\mathbb{N}}$ converges simply to a map ξ_∞ from A to $\mathbf{F}_\Theta$.

Observe now that if y^+ is the attractive fixed point of $\rho_\infty(\gamma_0) = \lim_{n\to\infty} \rho_\infty(\gamma_0)$, then y^+ is the limit of the attracting fixed point of $\{\rho_m(\gamma_0)\}_{m\in\mathbb{N}}$. It follows that $y^+ = \xi_\infty(\gamma^+)$. The same holds for the repelling fixed point y^- of $\rho_\infty(\gamma_0)$ and we have $y^- = \xi_\infty(\gamma^-)$. Since y^+ and y^- are transverse, we can apply Proposition 6.5 to obtain that ρ_∞ is positive. .

6.3. **Proof of Theorem C.** By Theorem 6.4 the set of Θ-positive representations is closed in $\mathrm{Hom}^\Theta(\pi_1(S), \mathsf{G})$. Using furthermore Corollary B this set is open. Thus the set of Θ-positive representations is a union of connected components of $\mathrm{Hom}^\Theta(\pi_1(S), \mathsf{G})$. Since finally, we can obtain positive representatsion by factoring in a positive $\mathsf{PSL}_2(\mathbb{R})$ we deduce the Theorem.

7. Positive representations and Cayley components

Let us recall that for a real split Lie group G, the Hitchin component was originally defined by Hitchin as the image of the Hitchin section Φ which assigns to a tuple of holomorphic differentials on a

Riemann surface Σ a G-Higgs bundle on Σ. Let us denote the image of Φ by $\mathcal{P}(\Sigma, \mathsf{G})$. Through the non-Abelian Hodge correspondence the set $\mathcal{P}(\Sigma, \mathsf{G})$ corresponds to a subset of the representation variety $\mathrm{Rep}^+(\pi_1(S), \mathsf{G})$, which we denote by the same symbol. Hitchin showed that $\mathcal{P}(\Sigma, \mathsf{G})$ is open and closed (hence a union of connected components) and the map Φ gives a parametrization of $\mathcal{P}(\Sigma, \mathsf{G})$. In the case of maximal representations a similar but more complicated parametrization of the space of maximal representations was obtained in [8], [18], and [5]. For any simple Lie groups admitting a Θ-positive structure, the authors of [7] define in a similar way subsets $\mathcal{P}_e(\Sigma, \mathsf{G})$ of the moduli space of G-Higgs bundles by giving explicit parametrizations, see also [14] and [1] for indefinite orthogonal groups. They prove that $\mathcal{P}_e(\Sigma, \mathsf{G})$ is open and closed in $\mathrm{Rep}^+(\pi_1(S), \mathsf{G})$. They further prove that all representations in $\mathcal{P}_e(\Sigma, \mathsf{G})$ have compact centralizer and thus do not factor through a proper parabolic subgroup. They further show that the set $\mathcal{P}_e(\Sigma, \mathsf{G})$ contains an open subset of Θ-positive representations.

Theorem C implies that any connected components of $\mathcal{P}_e^{\Theta}(\Sigma, \mathsf{G}) = \mathcal{P}_e(\Sigma, \mathsf{G}) \cap \mathrm{Rep}^{\Theta}(\pi_1(S), \mathsf{G})/\mathsf{G}$ that contains a Θ-positive representation consists entirely of Θ-positive representations. Due to the extension of our main result in [3] we further have that any connected components of $\mathcal{P}_e(\Sigma, \mathsf{G})$ that contains at least one Θ-positive representation consists entirely of Θ-positive representations. For many G this implies that $\mathcal{P}_e(\Sigma, \mathsf{G}) \subset \mathrm{Hom}_{\Theta-\mathrm{pos}}(\Gamma, \mathsf{G})$.

For this let us introduce the standard components of $\mathcal{P}_e(\Sigma, \mathsf{G})$. Consider an embedding of $\mathsf{SL}_2(\mathbb{R})$, such that the induced map from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}_\Theta$ is a positive circle, then the corresponding Fuchsian representation is positive. These Fuchsian representations can now in addition be twisted by a representation of $\pi_1(S)$ into the centralizer of this $\mathsf{SL}_2(\mathbb{R})$ in G. This is called a twisted positive Fuchsian representation. We call a component of $\mathcal{P}_e(\Sigma, \mathsf{G})$ standard if it contains a twisted positive Fuchsian representation.

When G is a classical group and not locally isomorphic to $\mathsf{Sp}_4(\mathbb{R})$, or $\mathsf{SO}(p, p+1)$, every component of $\mathcal{P}_e(\Sigma, \mathsf{G})$ is standard [7].

For $\mathsf{Sp}_4(\mathbb{R})$ with the Θ-positive structure with $\Theta \neq \Delta$, positive representations correspond precisely to maximal representations [10, 12]. In particular, the exceptional connected components of maximal representations in $\mathcal{P}_e(\Sigma, \mathsf{G})$ which do not contain any twisted positive Fuchsian representation [20], are positive. Similarly for the exceptional Hermitian Lie group of tube type, $\mathcal{P}_e(\Sigma, \mathsf{G}) = \mathrm{Hom}_{\Theta-\mathrm{pos}}(\Gamma, \mathsf{G})$ is the set of maximal representations.

For $\mathsf{SO}(p, p+1)$ with the Θ-positive structure with $\Theta \neq \Delta$, there also exist exceptional connected components in $\mathcal{P}_e(\Sigma, \mathsf{G})$ which do not contain any twisted positive Fuchsian representation. To deduce that they are positive we can use an embedding argument. Embedding $\mathsf{SO}(p, p+1) \to \mathsf{SO}(p, p+2)$, these components are sent to standard components for $\mathsf{SO}(p, p+2)$ [7], and thus they consist also entirely of Θ-positive representations seen in $\mathsf{SO}(p, p+2)$. Since any Θ-positive representation in $\mathsf{SO}(p, p+2)$, whose image is contained in $\mathsf{SO}(p, p+1)$ is also Θ-positive as a representation into $\mathsf{SO}(p, p+1)$, we conclude that these exceptional components consist entirely of Θ-positive representations.

However, for exceptional groups G whose restricted root system have a Dynkin diagram of type F_4, we do not know if all connected components of $\mathcal{P}_e(\Sigma, \mathsf{G})$ contain positive representations. If this where try, this would imply $\mathcal{P}_e(\Sigma, \mathsf{G}) \subset \mathrm{Hom}_{\Theta-\mathrm{pos}}(\Gamma, \mathsf{G})$. We further expect to have $\mathcal{P}_e(\Sigma, \mathsf{G}) = \mathrm{Hom}_{\Theta-\mathrm{pos}}(\Gamma, \mathsf{G})$. For this to hold one would need to show that any positive representation lies in $\mathcal{P}_e(\Sigma, \mathsf{G})$, or what might be the better approach, that none of the connected components of $\mathrm{Hom}^+(\Gamma, \mathsf{G}) \backslash \mathcal{P}_e(\Sigma, \mathsf{G})$ contain a Θ-positive representation.

References

1. Marta Aparicio-Arroyo, Steven Bradlow, Brian Collier, Oscar García-Prada, Peter B. Gothen, and André Oliveira, *SO(p, q)-Higgs bundles and higher Teichmüller components*, Invent. Math. **218** (2019), no. 1, 197–299.
2. Yves Benoist and François Labourie, *Sur les difféomorphismes d'Anosov affines à feuilletages stable et instable différentiables*, Inventiones Mathematicae **111** (1993), no. 2, 285–308.
3. Jonas Beyrer, Olivier Guichard, François Labourie, Beatrice Pozzetti, and Anna Wienhard, *Positivity, cross-ratios, photons, and the Collar Lemma*, arXiv:2409.06294, 2024.
4. Jonas Beyrer and Beatrice Pozzetti, *Positive surface group representations in PO(p,q)*, arXiv:2106.14725.
5. Olivier Biquard, Oscar García-Prada, and Roberto Rubio, *Higgs bundles, the Toledo invariant and the Cayley correspondence*, J. Topol. **10** (2017), no. 3, 795–826.
6. Armand Borel and Jacques Tits, *Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I*, Invent. Math. **12** (1971), 95–104.
7. Steve Bradlow, Brian Collier, Oscar García-Prada, Peter B. Gothen, and André Oliveira, *A general Cayley correspondence and higher Teichmüller spaces*, arXiv:2101.09377.
8. Steven Bradlow, Oscar García-Prada, and Peter B. Gothen, *Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces*, Geometriae Dedicata **122** (2006), 185–213.
9. Martin J. Bridgeman, Richard Canary, François Labourie, and AndrÃ©s Sambarino, *The pressure metric for Anosov representations*, Geometric And Functional Analysis **25** (2015), no. 4, 1089–1179.

10. Marc Burger, Alessandra Iozzi, François Labourie, and Anna Wienhard, *Maximal representations of surface groups: symplectic Anosov structures*, Pure Appl. Math. Q. **1** (2005), no. 3, 543–590.
11. Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Tight homomorphisms and Hermitian symmetric spaces*, Geom. Funct. Anal. **19** (2009), no. 3, 678–721.
12. ______, *Surface group representations with maximal Toledo invariant*, Ann. of Math. (2) **172** (2010), no. 1, 517–566.
13. Suhyoung Choi and William M. Goldman, *Convex real projective structures on closed surfaces are closed*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 657–661.
14. Brian Collier, $\mathrm{SO}(n, n+1)$*-surface group representations and Higgs bundles*, Ann. Sci. ENS (4) **53** (2020), no. 6, 1561–1616.
15. Thomas Delzant, Olivier Guichard, François Labourie, and Shahar Mozes, *Displacing representations and orbit maps*, Geometry, rigidity, and group actions, Univ. Chicago Press, Chicago, IL, 2011, pp. 494–514.
16. Subhadip Dey and Michael Kapovich, *Klein-maskit combination theorem for anosov subgroups: Amalgams*, arXiv:2301.02354, to appear in J. Reine Angew. Math.
17. Vladimir V. Fock and Alexander B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103, 1–211.
18. Oscar García-Prada, Peter B. Gothen, and Ignasi Mundet i Riera, *Higgs bundles and surface group representations in the real symplectic group*, Journal of Topology **6** (2013), no. 1, 64–118.
19. William M. Goldman, *Topological components of spaces of representations*, Inventiones Mathematicae **93** (1988), no. 3, 557–607.
20. Peter B. Gothen, *Components of spaces of representations and stable triples*, Topology **40** (2001), no. 4, 823–850.
21. François Guéritaud, Olivier Guichard, Fanny Kassel, and Anna Wienhard, *Anosov representations and proper actions*, Geom. Topol. **21** (2017), 485–584.
22. Olivier Guichard, *The Zariski closure of Hitchin and positive representations*, Unpublished work.
23. Olivier Guichard and Anna Wienhard, *Generalizing Lusztig's total positivity*, arXiv:2208.10114.
24. Olivier Guichard and Anna Wienhard, *Anosov representations: domains of discontinuity and applications*, Inventiones Mathematicae **190** (2012), no. 2, 357–438.
25. ______, *Positivity and higher Teichmüller theory*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2018, pp. 289–310.
26. Oskar Hamlet, *Tight holomorphic maps, a classification*, J. Lie Theory **23** (2013), no. 3, 639–654.
27. ______, *Tight maps and holomorphicity*, Transform. Groups **19** (2014), no. 4, 999–1026.
28. Nigel J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473.
29. Bertram Kostant, *Root systems for Levi factors and Borel-de Siebenthal theory*, Symmetry and spaces. In Honor of Gerry Schwarz on the occasion of his 60th birthday, Basel: Birkhäuser, 2010, pp. 129–152.
30. François Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.

31. ______, *Cross Ratios, Anosov Representations and the Energy Functional on Teichmüller Space.*, Ann. Sci. Éc. Norm. Supér. (4) (2008), 1–38.
32. François Labourie and Jérémy Toulisse, *Quasicircles and quasiperiodic surfaces in pseudo-hyperbolic spaces*, Invent. Math. **233** (2023), no. 1, 81–168.
33. George Lusztig, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568.
34. Vladimir V. Morozov, *Proof of the regularity theorem*, Usp. Mat. Nauk **11** (1956), no. 5(71), 191–194 (Russian).
35. Andrei S. Rapinchuk, Valerii V. Benyash-Krivetz, and Vladimir I. Chernousov, *Representation varieties of the fundamental groups of compact orientable surfaces*, Israel J. Math. **93** (1996), 29–71.
36. AndrÃ©s Sambarino, *Infinitesimal Zariski closures of positive representations*, arXiv:2012.10276.
37. Anna Wienhard, *An invitation to higher Teichmüller theory*, Proceedings of the international congress of mathematicians, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume II. Invited lectures, Hackensack, NJ: World Scientific; Rio de Janeiro: Sociedade Brasileira de Matemática (SBM), 2018, pp. 1013–1039.

Université de Strasbourg et CNRS, IRMA, UMR 7501, 7 rue Descartes, F-67000 Strasbourg
Email address: olivier.guichard@math.unistra.fr

Université Côte d'Azur, CNRS, LJAD, France
Email address: francois.labourie@univ-cotedazur.fr

Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig Germany
Email address: wienhard@mis.mpg.de