DYNAMICS ON MODULI SPACES OF GEOMETRIC STRUCTURES

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This spring's program (January 12, 2015 until May 22, 2015) concentrates on dynamical systems arising from the classification of locally homogeneous *geometric structures* on manifolds.

What is a geometric structure?

Geometry concerns spatial relationships and quantitative measurements, whereas *Topology* concerns the loose organization of points. Every geometric space has an underlying topological structure. Given a topological manifold Σ, and some geometry modeled on a homogeneous space X, can the local geometry of X (invariant under G) be put on the topology of Σ? If so, in how many ways? How does one understand the different ways of locally imparting the G-invariant geometry of X into Σ? The resulting *moduli space* (roughly speaking, the space of geometric structures on Σ) often has a rich geometry and symmetry of its own, and may be best understood, not as space but rather as a dynamical system. Here is a familiar example: *The sphere S^2 has no Euclidean geometry structure.* In other words, there is no metrically accurate world atlas. Therefore the moduli space of Euclidean structures on S^2 is empty. In contrast, the 2-torus T^2 has many Euclidean structures. The corresponding moduli space naturally identifies with the quotient of the upper halfplane H^2 by $\mathsf{PGL}(2,\mathbb{Z})$, the group of integral homographies, as depicted in Figure **??**. This quotient enjoys a rich and well-studied *hyperbolic geometry* of its own, which had been described in the mid-nineteenth century.

Some historical background

The subject's roots indeed go back to the nineteenth century. Following Sophus Lie and Felix Klein's work on continuous groups of symmetry, the *Erlangen Program* focused on the idea that a classical geometry (such as Euclidean geometry or projective geometry) is just the study of the G-invariant objects on a homogeneous space X. For instance, Euclidean geometry occurs when $X = \mathbb{E}^n$ and G is the group $\mathsf{Isom}(\mathbb{E}^n)$ of Euclidean isometries. In classical differential-geometric terms, this

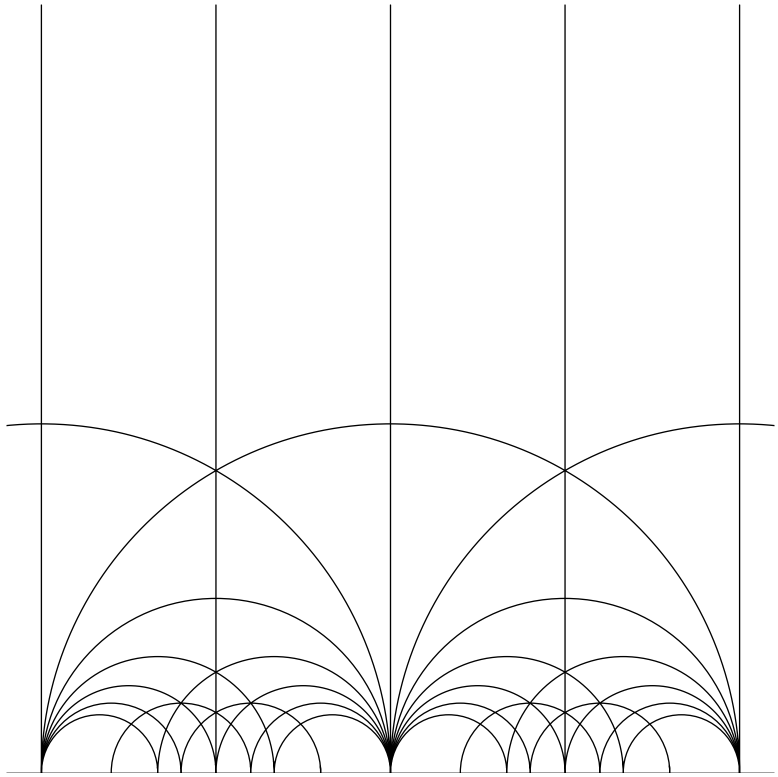

FIGURE 1. $\mathsf{PGL}(2,\mathbb{Z})$-invariant tiling of the upper half-plane H^2.

structure agrees with the notion of a *flat Riemannian metric.* Projective geometry concerns $X = \mathbb{R}\mathsf{P}^n$ and $G = \mathsf{PGL}(n+1,\mathbb{R})$ with a similar differential-geometric description.

The theory of crystallographic groups and their classification by Bieberbach is another historical source. This is equivalent to the classification of flat Riemannian manifolds, and in turn, to the classification of discrete groups of Euclidean symmetries.

Yet another source arose from integration of analytic differential equations, which related to conformal mappings of plane domains, as studied by Schwarz, Klein and Poincaré, and many others.

This was part of a larger development of the theory of connections by Ricci, Levi-Civita and É. Cartan, which generalized classical surface theory. Einstein's theory of relativity used these ideas and also was a major contribution.

Some of the most important examples arise from geometric strctures on surfaces. In higher dimensions, the moduli spaces are often finite sets, since the fundamental group is *overdetermined* in this case.

For instance, by the Mostow rigidity theorem a manifold of dimension greater than 2 admits at most one hyperbolic structure. In dimension two, the moduli spaces often admit *symplectic structures,* and natural Hamiltonian flows constructed out of the topology of the surface and the invariants of G provide ways of navigating around the moduli space.

Geometric structures and their models

Start with a geometry in the sense of Lie and Klein, that is, a homogeneous space X upon which a Lie group G act transitively. In other words, we restrict to "geometries" in which neighborhoods of all points "look the same." We model a manifold M locally on X as follows. Choose an atlas of *coordinate charts* on M, mapping coordinate patches $U \subset M$ by homeomorphisms $U \xrightarrow{\psi} \psi(U) \subset X$. We require that on overlapping coordinate patches, the *coordinate change* locally lies in G. Therefore the G-invariant geometry on X is transplanted locally to M. For example, a Euclidean structure defines notions of distance, angles, lines, area locally satisfying Euclidean rules. Similarly, a projective structure on M defines notions of lines, etc. which satisfy rules of projective geometry, such as Pappus's theorem.

The notion of local coordinates in X on Σ and the above *notion of a* (G, X)*-structure* was first explicitly defined by Charles Ehresmann in the 1930's. This notion was rejuvenated in the 1970's when Thurston formulated his Geometrization Program for 3-manifolds in the context of (G, X)-structures. In this theory, hyperbolic geometry plays the prominent role.

One convenient way to globalize the coordinate atlas of a geometric structure involves the *universal covering space* $\widetilde{M}$ of the geometric manifold M. If M is already simply connected, a geometric structure boils down to an immersion of M into the model space X. In general, one can describe the geometric structure in terms of a *developing map* $\widetilde{M} \xrightarrow{\mathsf{dev}} X$ and a compatible *monodromy* (or *holonomy*) representation $\pi_1(M) \xrightarrow{\rho} G$. The developing map globalizes the coordinate charts and the monodromy representation globalizes the coordinate changes.

While some developing maps are bijective and identify M with a quotient X/Γ, others may just identify M with quotients of proper domains $\Omega \subset X$. Others may not even be covering spaces of domains, and may wildly wrap $\tilde{M}$ onto all of X in an extemely complicated way. In the nicest cases the holonomy image Γ may be a discrete subgroup of G, but in the wildest cases it may even be dense. The developing map/holonomy representation itself displays potentially complicated dynamical behavior.

Let us now describe two important sources of examples: (1) Tilings and discrete symmetries, as epitomized by Euclidean crystallographic groups and flat Riemannian manifolds; (2) Monodromy of differential equations, as epitomized by projective structures on Riemann surfaces (that is, $\mathbb{C}\mathsf{P}^1$-manifolds) and classical Kleinian groups and uniformization.

Tilings and discrete symmetries. Regular tilings in Euclidean space led to the notion of a *crystallographic group,* which in modern parlance, is just a *lattice* Γ (a discrete subgroup of finite covolume) in the group $\mathsf{Isom}(\mathbb{E}^n)$.

Gradually the point of view changed, as the role of the transformations between the tiles became more prominent: *The shape of the tile is less relevant than the motions of the tiler.*

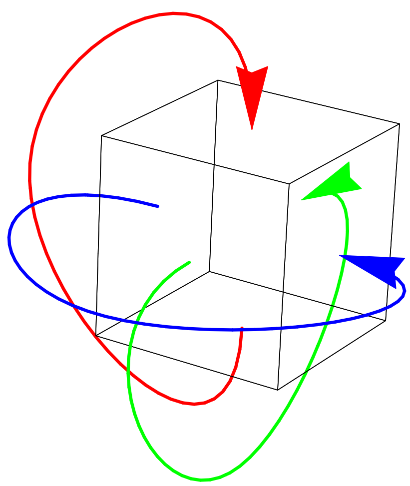

FIGURE 2. Identifications of a crystal to make a Euclidean manifold

The Bieberbach theorems gave an effective classification of crystallographic groups Γ, as finite extensions of *lattices* $\Lambda \subset \mathbb{R}^n$ of translations. Geometry arises through the quotient $M = \mathbb{E}/\Gamma$, which is often a manifold with a Euclidean structure. That M is actually a quotient of the model space X — that is, dev is a homeomorphism — relates to the metric nature of the structure, and we will see more complicated phenomena later. Nevertheless, this example underscores the intimate relationship between geometric structures and discrete subgroups of Lie groups.

An interesting development is the understanding of the analog of the Bieberbach theorems in *indefinite metric,* and in particular the construction and classification of geodesically complete spacetimes when there are two space dimensions and one time dimension. Figure **??** depicts one type of example, a so-called *Margulis spacetime,* whose fundamental group is a free group of rank two.

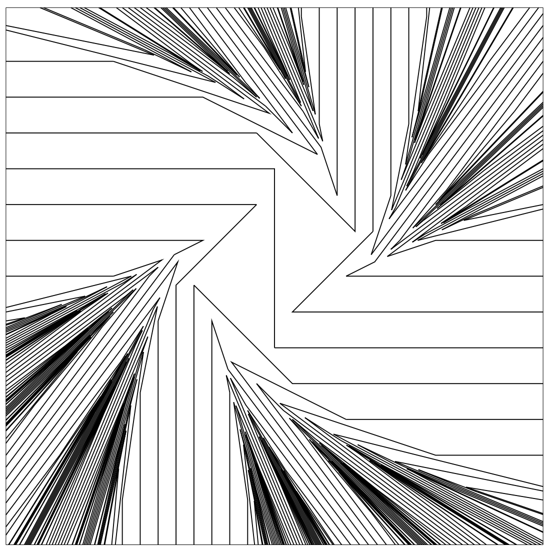

FIGURE 3. Cross-section of a tiling arising from a properly discontinuous group of affine Lorentz isometries in dimension three

Monodromy of differential equations. Another precursor of the theory of geometric structures on surfaces is the study of differential equations on complex domains. Even on $\mathbb{R}$, solutions of periodic linear differential equations on $\mathbb{R}$ may not be periodic. The solutions f of

$$f'(z) = a(z)f(z),$$

when $a(z+T) = a(z)$ (where T is the period), are not necessarily periodic, but satisfy

$$f(z+T) = \lambda f(z)$$

for some λ. The solution f looks like a developing map for a geometric structure, where λ generates the monodromy.

Similarly, on the unit disc Δ, Hill's equation

$$w''(z) + q(z)w(z) = 0$$

leads to Riemann surface M with a projective structure (a $\mathbb{C}\mathsf{P}^1$-structure). If $w_1(z), w_2(z)$ is a basis of the space of solutions on Δ, then the *projective solution*

$$f(z) := w_1(z)/w_2(z)$$

defines a map $\Delta \to \mathbb{C}\mathsf{P}^1$. If furthermore $q(z)$ is periodic with respect to a Fuchsian group Γ, then f defines a developing map for a $\mathbb{C}\mathsf{P}^1$-structure on the quotient Δ/Γ.

These classical examples are both basic and extremely rich. For $q = 0$, the developing map is the embedding of the disc Δ into the projective line $\mathbb{C}\mathsf{P}^1$. If q is sufficiently small, the developing map embeds Δ as a domain bounded by a fractal curve (a *quasicircle*) (see Figures **??** and **??**). However at this stage the developing map remains injective. As q increases, the developing map does not embed $\widetilde{M}$, nor admits a boundary. The asymptotic behavior of the monodromy as $q \longrightarrow \infty$ is an active subject of research related to mathematical physics.

Moving from ordinary differential equations to partial differential equations produces a wealth of new examples of fundamental importance, such as Yang–Mills equations, harmonic maps and Hitchin systems.

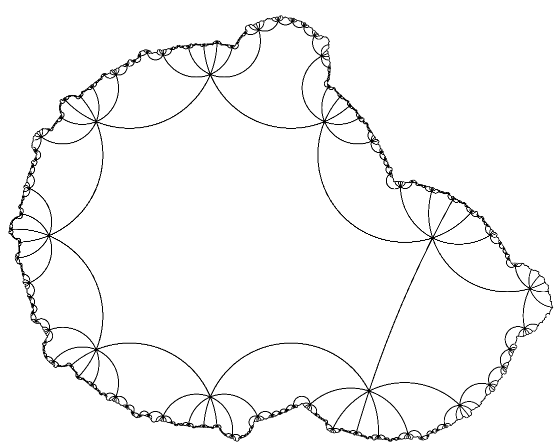

FIGURE 4. A quasi-Fuchsian $\mathbb{C}\mathsf{P}^1$-structure

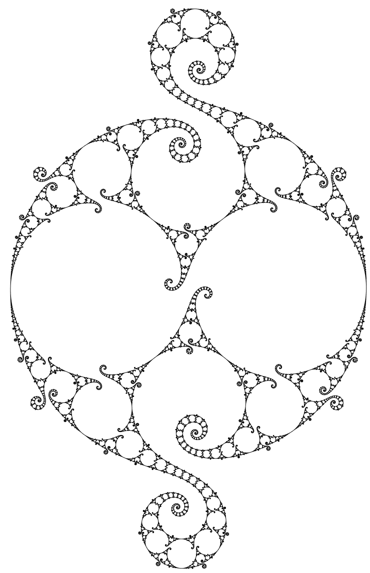

FIGURE 5. A quasicircle limit set

Moduli spaces

Isomorphism classes of (G, X)-structures on Σ locally modeled on X form a space, called the *moduli space,* which enjoys its own interesting geometry and dynamics.

Here are two examples: (1) Moduli of Euclidean tori; (2) Projective triangle tilings. In the first case, the moduli space of unmarked structures is the quotient space $\mathsf{H}^2/\mathsf{PGL}(2, \mathbb{Z})$ and in the second case the moduli space of (either marked or unmarked structures) is the half-open interval $[0, \infty)$, parametrized by a cross-ratio invariant.

Moduli of Euclidean tori. Euclidean structures on T^2 form a space enjoying hyperbolic geometry. If M is a Euclidean manifold homeomorphic to T^2, then the geometric structure identifies M as a quotient $\mathbb{E}^2$ by a lattice $\Lambda \subset \mathbb{R}^2$. In this context, a marking of M is just a basis of $\pi_1(M)$ which identifies with Λ.

The *moduli space* of unit-area marked Euclidean structures identifies with the upper halfplane in $\mathbb{C}$ as follows. A *marked Euclidean structure* is then just the choice of a parallelogram of area 1 with one horizontal side. Corresponding to the other side of the parallelogram is a complex number τ with positive imaginary part, which is a point in the Poincaré upper halfplane H^2.

Changing the marking amounts to applying an element of $\mathsf{GL}(2,\mathbb{Z})$ to the parallelogram, which corresponds to applying the associated integral linear fractional transformation of $\mathsf{PGL}(2,\mathbb{Z})$ to $\tau \in \mathsf{H}^2$. Therefore the corresponding *moduli space* of unit area *unmarked* Euclidean tori identifies with the quotient $\mathsf{H}^2/\mathsf{PGL}(2,\mathbb{Z})$, as depicted in Figure **??**. Note that the group associated with changing the markings preserves the hyperbolic geometry of the upper halfplane H^2.

Projective triangle tilings. Now let us move from tiling the Euclidean plane $\mathbb{E}^2$ by parallelograms to tiling domains in the projective plane by triangles. As $\mathbb{E}^2 \subset \mathbb{R}\mathsf{P}^2$ is a domain, and isometries of $\mathbb{E}^2$ extend to projective transformations of $\mathbb{R}\mathsf{P}^2$, every Euclidean structure is a *projective structure.* Now the familiar tiling of the Euclidean plane by equilateral triangles deforms projectively in a nontrivial way. Figure **??** depicts a projective deformation of this tiling. Here the developing map is not onto, but remains injective. Similarly, the Klein-Beltrami model embeds the hyperbolic plane H^2 in the projective plane $\mathbb{R}\mathsf{P}^2$, where the isometries of H^2 extend as projective transformations. Figure **??** depicts a projective deformation of a triangle group in H^2. The new domain Ω is tiled by triangles. Furthermore Ω is bounded by a C^1 convex curve which is nowhere C^2.

The symmetry group of each of these tesselations of domains Ω contains a finite-index subgroup Γ_0 such that Ω/Γ_0 is a surface with an $\mathbb{R}\mathsf{P}^2$-structure. These provide examples where the developing map is injective but not surjective. In contrast, Figure **??** depicts an $\mathbb{R}\mathsf{P}^2$-structure on T^2 whose developing map is neither injective nor surjective.

Dynamics

To make the moduli space more tractable, it is often useful to introduce some extra topological structure, called a *marking.* Changing the marking leads to the action of a group which defines a dynamical system.

We saw that the moduli space of *unmarked* unit-area Euclidean structures on T^2 is the quotient of H^2 by $\mathsf{PGL}(2,\mathbb{Z})$, whereas H^2 is the moduli space of *marked* Euclidean structures. Although the $\mathsf{PGL}(2,\mathbb{Z})$-action is not free, the quotient has a nice structure. However, sometimes the moduli space of *unmarked structures* is not a *space in the classical sense:* it may not admit nonconstant continuous functions. Therefore studying the space of *marked structures,* together with the group action corresponding to *changing the marking,* is more natural.

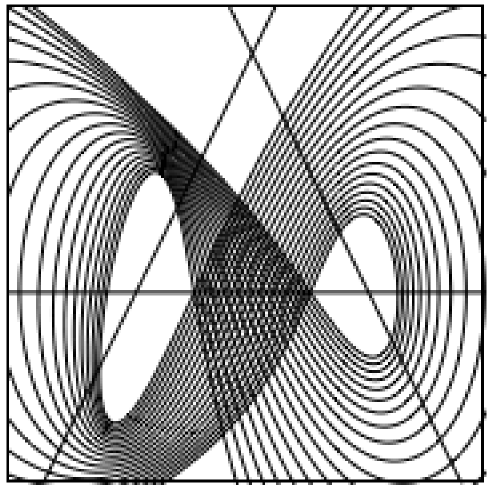

FIGURE 6. Exotic developing map for $\mathbb{R}\mathsf{P}^2$-structure on T^2

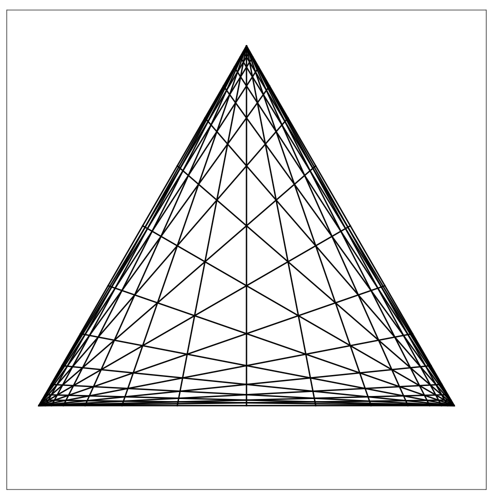

FIGURE 7. Projective deformation of tiling by Euclidean equilateral triangles

In two dimensions, this is described by the action of the *mapping class group* of Σ.

Here is an example of chaotic dynamics. A *marked complete affine structure* on T^2 is an identification of T^2 as a quotient of the affine plane $\mathbb{R}^2$ by a discrete group Γ of affine transformations. The moduli of marked complete affine structures on T^2 identifies with $\mathbb{R}^2$, where standard Euclidean structures on T^2 all correspond to the origin. Corresponding to changing marking is the standard linear action of of

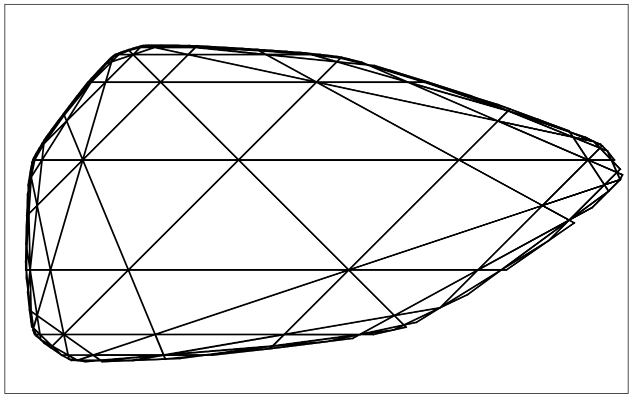

FIGURE 8. Projective deformation of hyperbolic triangle group

$\mathsf{GL}(2,\mathbb{Z})$. In this case, the quotient space is non-Hausdorff and doesn't even support nonconstant continuous functions.

In this way, classification of geometric structures naturally leads to interesting dynamical systems. Here is an example related to a venerable subject in number theory.

The dynamics of Markoff triples. Sometimes a marked geometric structure on Σ identifies with its holonomy representation. Thus the moduli space of marked structures identifies with a subset of an algebraic set: the coordinates are matrix entries and the defining equations arise from the defining relations in $\pi_1(M)$.

A simple and fundamental example is the space of equivalence classes of pairs of matrices in $\mathsf{SL}(2,\mathbb{C})$, corresponding to representations of the free group on two generators. In this case Σ is the once-punctured torus. Since the nineteenth century, we know that such a pair of matrices is described (up to equivalence) by the traces x, y of the two generators and the trace z of their product. Thus the moduli space identifies with $\mathbb{C}^3$. Further imposing the natural boundary condition around the puncture leads to the cubic equation

$$x^2 + y^2 + z^2 - xyz = t,$$

where $t \in \mathbb{C}$ corresponds to the trace of holonomy around the puncture. This moduli space has a rich group of symmetries generated by polynomial automorphisms such as

$$(x, y, z) \longmapsto (x, y, xy - z)$$

and permutations, corresponding to changes of markings. When $t = 0$, this is the classical Markoff equation, arising from the classification of binary quadratic forms. For other values of t, the dynamics ranges from proper dynamics (with a *Hausdorff* quotient space) to dynamically interesting chaos.

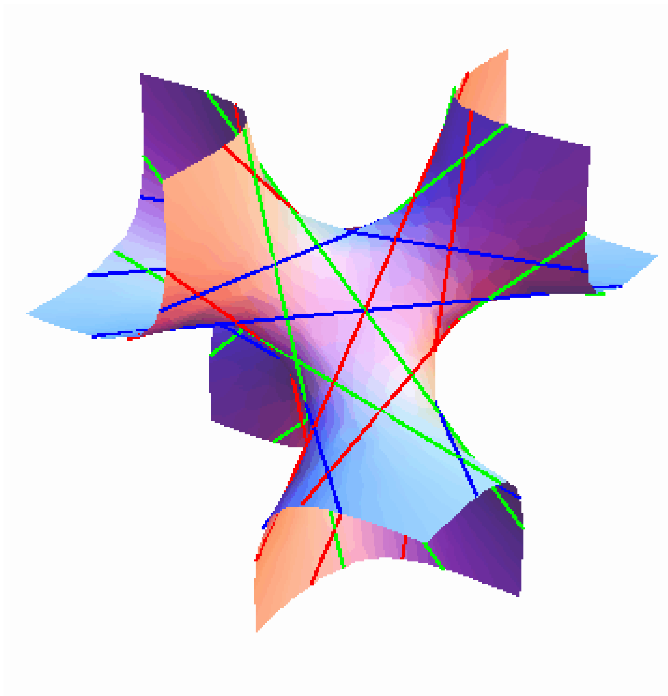

FIGURE 9. A cubic surface parametrizing geometric structures on a surface

CONCLUSION

Rooted in classical origins, our research area is thriving. The study of geometric structures involves many fields and diverse techniques: ergodic theory, geometric analysis, geometric group theory, Lie theory and combinatorics. As a natural extension of classical Riemann surface theory and Lie theory, it relates to the interests of many mathematicians and theoretical physicists. Many more connections are expected, notably with the companion program "Geometric and Arithmetic Aspects of Homogeneous Dynamics." Both the NSF-funded GEAR Research Network and the European Research Council have supplemented the MSRI budget to spread intellectual benefits of our MSRI program to a broader group of mathematicians. This program has been instrumental in expanding, clarifying and consolidating the general field.