Topological Superrigidity and Anosov Actions of Lattices

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Abstract

The main result proved here is a topological version of Zimmer's cocycle superrigidity theorem. A number of applications is also given concerning lattice actions, rigid geometric structures, and the cohomology of actions of higher rank semisimple groups with coefficients in vector-bundles.

0 Introduction

The main result obtained here is Theorem 1.2, which corresponds to a topological, and more generally C^r, version of Zimmer's Cocycle Superrigidity Theorem. Zimmer's theorem is concerned with cocycles over actions of a semisimple Lie group G of real rank at least 2 and its lattices by automorphisms of a finite measure space, while here we consider actions of G by automorphisms of principal bundles such that the actions of certain subgroups of G on the base are topologically transitive. Our results, in fact, extend the main theorem of [18].

The proof given here is not just an adaptation of the classical proofs. We shall sketch in the appendix how to obtain Margulis-Zimmer superrigidity using the ideas of our proof. We should point out, however, that there are no essentially new ideas in the approach given here. Its main advantages are that it is perhaps a shorter and in a sense more axiomatic way of presenting old, and beautiful, ideas.

In its general form, the main theorem gives information about the action only on an open dense G-invariant subset of the manifold. Using an idea due to Zimmer ([18]), which is explained in Section 6, Theorem 1.2 can be

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specialized so as to yield a conclusion that holds everywhere on the manifold. (Theorem 1.6.)

Part of the content of the main theorem can be rephrased in cohomological terms, which is done in Section 7. The main result in that direction, Theorem 7.1, gives conditions for the vanishing of certain cohomology classes associated to actions of G on vector bundles. One useful consequence of Theorem 7.1 is Corollary 7.4, which gives conditions for smooth G-actions on vector bundles to preserve a connection.

The main applications of Theorem 1.2 obtained here are Theorems 1.7 and 1.8. The former, discussed in Section 8, solves in a very special case a general problem posed by Zimmer of classifying smooth actions of lattices in G on compact or finite volume manifolds. The latter theorem, discussed in section 9, is concerned with the notion of rigid geometric A-structures, introduced by Gromov in [4]. We show that if a subgroup K of G preserves a rigid geometric A-structure and G and K satisfy the dynamical assumptions of the main theorem, then G itself must also leave invariant some rigid geometric structure, although it may only be defined on an open dense G-invariant set.

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1 Statements of the main results

The general setting for the theorems discussed here will be a C^s ($s \geq 0$) right principal H-bundle $\pi : P \to M$ over a manifold M, where H is a real algebraic group. Suppose, moreover, that a Lie group G acts on P by left principal bundle automorphisms, the action being C^s.

For a given subgroup $L \subset H$ and open subset $U \subset M$, we refer to a C^s L-subbundle Q of $P|_U$ as a C^s *L-reduction* of P over U. The L-reduction is said to be G-invariant if U is a G-invariant set and G acts by automorphisms of Q, that is, Q is a G-invariant subset of P.

An important definition is the following.

Definition 1.1 *(Cf. [17, 9.2.2].) Let $L \subset H$ be a real algebraic subgroup and suppose that P admits a G-invariant C^s L-reduction over some G-invariant open dense subset $U \subset M$. The conjugacy class of L in H will be called a C^s-*algebraic hull *of the G-action on P if L is minimal in the following sense: there is no proper real algebraic subgroup $L_1 \subset L$ such that P admits a G-invariant C^s L_1-reduction over some open dense G-invariant subset of M.*

Let G denote the connected component of a semisimple real algebraic group. Recall that the real-rank of G, $\mathrm{rank}_{\mathbb{R}}G$, is the dimension of a maximal split torus in G. A 1-parameter subgroup T of G is said to be $\mathbb{R}$-*semisimple* if for each linear representation ρ of G $\rho(a)$ is diagonalizable with real eigenvalues for all $a \in T$.

Our main goal is to prove the following theorem.

Theorem 1.2 (Topological superrigidity) *Suppose G has real rank at least 2 and that it acts by H-bundle automorphisms on some C^s principal H-bundle P over a manifold M such that the action is also C^s. Assume that*

(i) H is the C^s algebraic hull of the G-action;

(ii) every $\mathbb{R}$-semisimple 1-parameter subgroup of G acts topologically transitively on M and admits a dense set of recurrent points, i.e., points that are contained in their own ω-limit sets.

Assume furthermore that there is a subgroup $K \subset G$ with the following properties:

(iii) K acts topologically transitively on M,

(iv) K commutes with some $\mathbb{R}$-semisimple 1-parameter subgroup of G,

(v) the C^s algebraic hull of the K-action does not contain a nontrivial normal subgroup of H.

Then, there exists a continuous surjective homomorphism $\rho : G \to H$ and a C^s section σ of $P|_U$, for some open dense G-invariant subset U of M, such that for all $g \in G$ and $x \in U$,

$$g\sigma(x) = \sigma(gx)\rho(g).$$

The theorem can also be stated as follows. Given a homomorphism ρ of G into H and a G-action on M, we can build a G-action on the trivial bundle $M \times H$, which we call a ρ-action, such that $g(x, h) = (gx, \rho(g)h)$, $g \in G, (x, h) \in M \times H$. Then the conclusion of the theorem is that, at least on some open dense G-invariant subset of M, the original G-action on P is C^s conjugate to a ρ-action. (The conjugacy is given by $\Phi(x, h) = \sigma(x)h$.)

The theorem above is different from the topological superrigidity theorem of [18] in that we do not assume the existence of invariant measures or of a parabolic invariant structure. The latter is replaced with the much weaker hypothesis (v) on the hull of a smaller group, which can actually be just a 1-parameter subgroup.

In the C^0 case of the theorem M need only be a Baire topological space. If M is actually a finite dimensional topological manifold (or, more generally, a second countable, locally compact metrizable space), then the recurrence condition (ii) on 1-parameter groups follows from topological transitivity, as a simple argument shows. Moreover, if the G-action on M preserves an ergodic probability measure whose support is M, an application of Moore's ergodicity theorem and Poincaré recurrence gives the following corollary, in which topological transitivity and recurrence are replaced by ergodicity of G.

Theorem 1.3 *Suppose that a connected semisimple group G of real-rank at least 2 acts by H-bundle automorphisms on some C^s principal H-bundle P over a manifold M such that the action is also C^s. Assume that the action preserves an ergodic probability measure whose support is M and that H is the C^s algebraic hull of the G-action. Assume moreover that there is a noncompact subgroup $K \subset G$ such that K commutes with some $\mathbb{R}$-semisimple 1-parameter subgroup of G and the C^s algebraic hull of the K-action does not contain a nontrivial normal subgroup of H. Then, there exists a continuous surjective homomorphism $\rho : G \to H$ and a C^s section σ of $P|_U$, for some open dense G-invariant subset U of M, such that for all $g \in G$ and $x \in U$,*

$$g\sigma(x) = \sigma(gx)\rho(g).$$

The next result uses an observation due to Zimmer [18] that will be explained in section 6. Zimmer's result complements our main theorem by giving conditions for the section σ to be defined on the entire manifold. We first state a couple of definitions.

Let V be a smooth real algebraic variety equipped with a real algebraic left action of H. By a C^s *geometric structure* on P of type V we mean a C^s section of the associated V bundle P_V.

Definition 1.4 ([18]) *Suppose that G is the identity component of a real algebraic semisimple group. A geometric structure φ on P is called a* parabolic invariant *if it is invariant by some parabolic subgroup of G.*

Definition 1.5 ([18]) *The G-action on the H-bundle P is said to be* effective *relative to the geometric structure $\sigma : M \to P_V$ if, for some $x \in M$, the group of automorphisms of P_x fixing $(g_*\sigma)(x)$ for all $g \in G$ is trivial, where E is associated to φ as indicated earlier.*

Theorem 1.6 *In addition to the assumptions of the main theorem, suppose there exists a C^s parabolic invariant relative to which the G-action is effective. Then the C^s section σ obtained in the main theorem is defined over the whole manifold.*

We state now an application of the main theorem to smooth actions of a lattice group Γ in G. The main new point is that no invariant volume form or measure will be assumed. Even though our primary concern here is the action of Γ on some compact manifold M, it will be necessary to refer to an induced action by G, defined by a standard suspension construction. We recall that an action on M by a lattice Γ in a Lie group G induces in a canonical way an action of G on the manifold $N := (G \times M)/\Gamma$, which is the quotient of $G \times M$ by the action of Γ given by

$$(g, x) \cdot \gamma := (g\gamma, a(\gamma)^{-1}x).$$

G acts locally freely on N, so that the foliation of N by fibers $M_{[g]} := p^{-1}([g])$ is preserved by the action and is everywhere transverse to the G-orbits. We denote the transversal foliation by $\mathcal{M}$. Also remark that each fiber of $\mathcal{M}$ is diffeomorphic to M.

Since N may fail to be compact (when the lattice is not uniform), it will be necessary to assume the existence of a Riemannian metric on N with norm $\|\cdot\|$ for which $\|g_*\|$ is uniformly close to 1 for all g sufficiently close to e. In particular, as G is connected, $\|g_*\|$ is uniformly bounded for each $g \in G$. (This is clearly satisfied for the model actions, for example, the suspension of the affine action of $SL(n, \mathbb{Z})$ on the n-torus.)

Also with respect to $\|\cdot\|$, we say that $k \in G$ is an *Anosov* element if $T\mathcal{M}$ decomposes as a continuous direct sum of subbundles

$$T\mathcal{M} = E^- \oplus E^+$$

such that k (resp., k^{-1}) is uniformly contracting on E^- (resp., E^+), i.e., there is λ, $0 < \lambda < 1$, such that $\|k_*|_{E^-}\| \leq \lambda$ (resp., $\|(k^{-1})_*|_{E^+}\| \leq \lambda$). We call E^- (resp., E^+) the *stable* (resp., *unstable*) subbundle of k.

Theorem 1.7 *We assume that a lattice Γ of $G = SL(n, \mathbb{R})$, $n \geq 3$, acts smoothly on a compact manifold M of dimension n. Suppose, moreover, that for the induced G-action on $N = (G \times M)/\Gamma$, (i) every $\mathbb{R}$-semisimple 1-parameter subgroup of G acts topologically transitively on N and (ii) some regular element k of G is Anosov. Then M is a flat torus for some smooth Riemannian metric and the Γ-action is a standard affine action with respect to that metric.*

We point out that the standard actions on tori satisfy the conditions of the theorem. We do not know, however, how to restate condition (ii) in terms of Anosov elements in Γ, rather than in G.

Theorems concerning the rigidity of actions of lattices on tori have been obtained by several authors. We mention [5, 10, 8, 9, 14, 3] among others. To our knowledge Theorem 1.7 is the first global result of such kind which makes no reference to invariant measures, or to assumptions on the topology of the original manifold. We also point out that the Franks-Newhouse theorem, which implies that codimension-1 Anosov diffeomorphisms can only exist on tori, is not used here.

The result below is an application of the main theorem to rigid geometric structures. The notion of a rigid geometric A-structure was introduced in [4] and it generalizes the classical geometric structures of finite type of Cartan. We show in section 9 that under the dynamical assumptions of the main theorem, if a subgroup $K \subset G$ preserves a smooth rigid geometric structure, then G must also preserve a (possibly different) rigid geometric structure on a G-invariant open dense set. More precisely, we have the following theorem.

Theorem 1.8 *Let G be a connected semisimple Lie group of real rank at least 2 that acts on a smooth manifold M so that every $\mathbb{R}$-semisimple 1-parameter subgroup of G acts topologically transitively on M. Let K be a subgroup of G that commutes with some $\mathbb{R}$-semisimple 1-parameter subgroup of G and acts topologically transitively on M. Suppose moreover that K preserves a smooth rigid A-structure on M. Then G preserves a smooth rigid A-structure on some open dense G-invariant set $U \subset M$.*

An important example of rigid structure is an affine connection. We prove in section 7 that under the same conditions of the theorem above, if the K-action preserves a connection on some open dense subset of M, then so does G. This will be consequence of a general cohomology vanishing result for G-actions on vector bundles over M.

2 The topological Furstenberg lemma

Let P be, as before, an H-principal bundle over M and T a group that acts on P by bundle automorphisms. Let V be a smooth real algebraic variety and suppose that both T and H act on V algebraically, and that the actions commute. Let Φ be a $T \times H$-equivariant map from $P|_U$ into V, where U is an open dense set in M. The proposition below, which will be referred to as the Topological Furstenberg Lemma, gives conditions for Φ to take values in a single H-orbit in V. It will be of basic importance throughout the paper.

The usual set up of a C^s principal H-bundle P over a manifold M is now in place. V is a smooth real algebraic variety equipped with an algebraic action of H.

Proposition 2.1 *Let T be a 1-parameter group of homeomorphisms of P commuting with the right H-action and let S be a real algebraic group isomorphic to either the additive group $\mathbb{R}$ or the multiplicative group $\mathbb{R} - \{0\}$. We suppose that S acts algebraically on V, commuting with the action of H. Let Φ be an $H \times \mathbb{R}$-equivariant continuous map from $P|_U$ into V, where U is an open dense T-invariant subset of M. (More precisely, there is a homomorphism ρ from T into S such that*

$$\Phi(lph) = h^{-1}\rho(l)\Phi(p) = \rho(l)h^{-1}\Phi(p)$$

for all $l \in T$, $h \in H$ and $p \in P|_U$.) Then, after possibly having to restrict Φ to $P|_{U'}$ for some open dense subset U' of U, Φ takes values into a single H-orbit in each of the following two cases:

1. *T acts on M topologically transitively and S acts trivially on V.*
2. *T acts on M topologically transitively with a dense set of recurrent points, and S acts possibly nontrivially on V.*

Proof. We recall the following stratification theorem for algebraic actions due to Rosenlicht ([4]). If a real algebraic group B acts regularly on a smooth real algebraic variety V, then V decomposes as a disjoint union $V = V_1 \cup \cdots \cup V_m$ of B-invariant smooth subvarieties V_i such that the union $F_i := V_i \cup \cdots \cup V_m$ is Zariski closed in V for each $i \leq m$, V_i open and dense in F_i and the B-orbit of a point of V_i is closed in V_i. Furthermore, the quotient V_i/B has a natural structure of smooth real algebraic variety for each i and the quotient map $V_i \to V_i/B$ is a smooth fibration for each i. In particular, each B-orbit is embedded into V.

We let now V_i, $i = 1, \ldots, m$, be the smooth varieties given by the Rosenlicht stratification for the action of $H \times S$ on V. We first observe that Φ, restricted to some open and dense subset of the form $P|_{U'}$, $U' \subset U$, must take values into a single stratum V_i. In fact, for any $x \in U$ with dense T-orbit in M and any $p \in P$ in the fiber above x the image under Φ of the $T \times H$-orbit of p is entirely contained in some V_i, since these strata are $T \times H$-invariant. But V_i is open in its closure so, by continuity, $\Phi^{-1}(V_i)$

is open, $T \times H$-invariant, and contains a dense subset of $P|_U$. Therefore, $\Phi^{-1}(V_i)$ is of the form $P|_{U'}$ as claimed.

Consequently, the restriction of Φ to $P|_{U'}$ factors through the bundle projection to define a continuous T-equivariant map $\bar{\Phi}$ from U' into a smooth real algebraic variety, $W_i = V_i/H$. In case (1), this map is T-invariant, hence constant by topological transitivity, and the claim follows.

In order to consider case (2), we first remark that every recurrent point under the action of S on W_i is a fixed point. In fact, by the recurrence assumption and the fact that orbits of S are locally closed, the stabilizer subgroup of S for that point must be a nontrivial Zariski closed subgroup of S, hence it must be all of S since S is 1-dimensional; that is, every recurrent point must be a fixed point for S. The set of S-fixed points of W_i is closed, and by the assumption (2), a dense set of points in x is taken under $\bar{\Phi}$ to that set. Therefore, $\bar{\Phi}$ is S-invariant and the argument of part (1) applies, concluding the proof. □

Corollary 2.2 *Let V be a smooth real algebraic variety equipped with a regular action of a real algebraic group S isomorphic to $\mathbb{R}$ or $\mathbb{R} - \{0\}$ (as real algebraic groups). Let T be a 1-parameter group of homeomorphisms of a topological space M acting topologically transitively and having a dense set of recurrent points. Suppose that $\phi : U \to V$ is a continuous map defined on an open dense T-invariant subset $U \subset M$ and T-equivariant, i.e., there is a continuous homomorphism $\rho : T \to S$ such that*

$$\phi(lx) = \rho(l)\phi(x)$$

for each $l \in T$ and each $x \in U$. Then, Φ is constant and its value is a fixed point under S.

Proof. Set $H = S$ and $P = M \times S$. T acts on P on the left as $l(x, s') = (lx, s')$ and S acts on the right as $(x, s')s = (x, s's)$. Define $\Phi : U \times S \to V$ as $\Phi(x, s) := s^{-1}\phi(x)$. Then, since S is abelian, the equivariance condition of the proposition is satisfied and the claim follows. □

It will be helpful to keep in mind the following trivial remark. Suppose that manifolds N_1 and N_2 are equipped with C^s actions of groups B_1 and B_2, resp., and let $\rho : B_1 \to B_2$ be a homomorphism. Let $F : W \to N_2$ be a C^s map from an open subset W of N_1 that satisfies the following equivariance condition: For each $b \in B_1$ and $x \in W$ such that $bx \in W$, we have $F(bx) = \rho(g)F(x)$. Then there is a unique C^s map $\bar{F}$ from the

B_1-saturation of W into N_2 that restricts to F on W and satisfies: For all $x \in B_1 W$ and all $b \in B_1$, $\bar{F}(bx) = \rho(b)\bar{F}(x)$. In other words, in such a situation we may assume without loss of generality that W is B_1-invariant. $\bar{F}$ is, of course, given by $\bar{F}(bx) = \rho(b)F(x)$, for $b \in B_1$ and $x \in W$, and it is immediate to check that it is well-defined.

3 H-pairs

We denote by $\mathcal{E}^s(P|_U, V)$ the space of all geometric structures of type V and class C^s, defined over an open subset $U \subset M$. Recall that these are simply the H-equivariant C^s maps from $P|_U$ into V. A group B of C^s automorphisms of P leaving U invariant defines a left-action on $\mathcal{E}^s(P|_U, V)$ by $g \cdot \varphi := \varphi \circ g^{-1}$, for $g \in B$ and $\varphi \in \mathcal{E}^s(P|_U, V)$. Given $p \in P|_U$, we denote by e_p the evaluation map which associates to $\varphi \in \mathcal{E}^s(P|_U, V)$ its value $\varphi(p)$. Remark that $e_p \circ g = e_{g^{-1}p}$.

Definition 3.1 (H-pair for B) *Let W be a subset of $\mathcal{E}^s(P|_U, V)$, where U is an open dense B-invariant subset of M and W is assumed to be B-invariant, i.e. $b \cdot \varphi \in W$ for all $\varphi \in W$. We say that (W, V) defines a C^s* H-pair for B *if the following conditions hold:*

1. *For each $p \in P|_U$, the evaluation map $e_p : W \to V$ is injective and $W_p := e_p(W)$ is a real subvariety of V.*
2. *For each $p, p' \in P|_U$,*
$$\tau_{p,p'} := e_p \circ e_{p'}^{-1} : W_{p'} \to W_p$$
is an H-translation, *that is, one finds $h \in H$ such that $\tau_{p,p'}(v) = hv$ for all $v \in W_{p'}$.*
3. *H acts transitively and effectively on V.*

Heuristically, an H-pair can be thought of as follows. Starting with a principal H-bundle and an algebraic H-space V, one forms the associated bundle whose typical fiber is V. Then an H-pair is an *algebraic* subset of sections of this associated bundle. The reader is advised to think of the corresponding situation when one replaces the word 'algebraic' by 'linear,' and studies vector spaces of sections of a vector bundle as in [12].

It should be remarked that condition 3 implies that V is a homogeneous space of the form H/H_0, where H_0 does not contain a nontrivial normal subgroup of H.

The collection of all H-translations from W_p into V is naturally identified with H/F_p, where F_p is the group $F_p := \{h \in H \mid hv = v \text{ for all } v \in W_p\}$. Moreover, it follows from property 2 that for each $p \in P|_U$ and $b \in B$, there is $h \in H$ such that $e_p(b \cdot \varphi) = he_p(\varphi)$, for all $\varphi \in W$. (The H-translation corresponding to h is $\tau_{b^{-1}p,p}$.) Therefore, one associates to each $p \in P|_U$ a C^s homomorphism

$$\rho_p : B \to N_p/F_p, \quad \rho_p(b) = e_p \circ b \circ e_p^{-1},$$

where $N_p := \{h \in H \mid hv \in W_p \text{ for all } v \in W_p\}$. In other words,

$$e_p(b \cdot \varphi) = \rho_p(b)e_p(\varphi),$$

for $\varphi \in W$ and $b \in B$.

The H-translations $\tau_{p,q} : W_p \to W_q$ satisfy the following elementary properties, which we are going to use freely in the sequel and to which the reader is advised to refer when checking formal computations.

$$\begin{aligned}
\tau_{bp,q} &= \tau_{p,b^{-1}q} \\
\rho_p(b)\tau_{p,q} &= \tau_{p,q}\rho_q(b) \\
\rho_{ph}(b) &= h^{-1}\rho_p(b)h \\
\tau_{ph,qh'} &= h^{-1}\tau_{p,q}h' \\
\tau_{bp,b'q} &= \rho_p(b)^{-1}\tau_{p,q}\rho_q(b')
\end{aligned}$$

where $p, q \in P|_U$, $b, b' \in B$ and $h, h' \in H$.

We say that an H-pair (W_1, V) is *contained* in another H-pair (W_2, V) if $W_1 \subset \mathcal{E}^s(P|_{U_1}, V)$, $W_2 \subset \mathcal{E}^s(P|_{U_2}, V)$ are defined over open and dense sets U_1 and U_2 and for every $\varphi_1 \in W_1$ one finds $\varphi_2 \in W_2$ such that the two maps agree over $U_1 \cap U_2$.

It will be convenient in what follows to identify pairs (W_1, V) and (W_2, V) if each one is contained in the other. This indeed defines an equivalence relation and we write $[W, V]$ for the class represented by (W, V). We say that $[W_1, V]$ is contained in $[W_2, V]$, and write $[W_1, V] < [W_2, V]$, if a representative of the former is contained in a representative of the latter. Due to the fact that we consider open dense sets, $<$ defines a partial ordering.

We say that the H-pair $[W, V]$ for B is *maximal* if it is equal to any other H-pair in which it is contained.

Definition 3.2 (Invariant and hyperbolic pairs) *A C^s H-pair (W, V) for B defined over a dense B-invariant open set $U \subset M$ will be called* invariant *if $b \cdot \varphi = \varphi$ for all $\varphi \in W$ and $b \in B$. It is* A-hyperbolic*, for a subgroup*

A of B, if $\rho_p(a)$ is a $\mathbb{R}$-semisimple element of the real algebraic group N_p/F_p (defined earlier) for every $p \in P|_U$ and $a \in A$.

Lemma 3.3 *Every C^s H-pair for B is contained in a maximal C^s H-pair for B. Similarly, every invariant (resp., hyperbolic) C^s H-pair for B is contained in a maximal invariant (resp., maximal hyperbolic) C^s H-pair for B.*

Proof. We will construct for any given increasing sequence of C^s H-pairs for B,

$$[W^1, V] < [W^2, V] < \cdots < [W^i, V] < \cdots,$$

a C^s H-pair for B, $[W^\infty, V]$, which contains each one in the sequence. The claim will then follow from Zorn's lemma.

We choose a representative (W^i, V) for each pair, where W^i is a subset of $\mathcal{E}^s(P|_{U_i}, V)$, and U_i is an open dense B-invariant subset of M. By the Baire property, the intersection of all U_i is nonempty (in fact, dense) so that we can fix a $p_0 \in P$ that projects to a point in that intersection. For each $p \in P|_{U_i}$, we denote $W^i_p := e_p(W^i)$ and define

$$\psi^i_p := e_p \circ e^{-1}_{p_0} : W^i_{p_0} \to W^i_p,$$

which is an H-translation, by definition of an H-pair. Therefore, we may regard ψ^i_p as an element of $H/F^i_{p_0}$, where $F^i_{p_0}$ is the (real algebraic) subgroup of H that fixes $W^i_{p_0}$ pointwise. Thus, we obtain a C^s H-equivariant map

$$\Psi_i : P|_{U_i} \to H/F^i_{p_0}$$

defined by $\Psi_i(p) := \psi^i_p$. Remark that for any given $w \in W^i_{p_0}$ we recover $\varphi \in W^i$ by the equation $\varphi(p) = \Psi_i(p)w$.

The sequence of subvarieties $W^i_{p_0}$ of V is increasing, so the sequence of real algebraic subgroups $F^i_{p_0}$ is decreasing. By the descending chain condition for algebraic groups there must be a finite index i_0 such that $F^i_{p_0} = F^{i_0}_{p_0} =: F^\infty_{p_0}$ for all $i \geq i_0$. Consequently, for all $i, j \geq i_0$, the maps Ψ_i, Ψ_j agree on $P|_{U_i \cap U_j}$. (Remark that if, say, $j \geq i$, then the H-translation ψ^i_p is the restriction of ψ^j_p to W^i_p and is of the form $w \mapsto hw$ for some $h \in H$ and all $w \in W^j_{p_0}$, so that $\psi^i_p = \psi^j_p$ if $F^i_{p_0} = F^j_{p_0}$.)

The preceding shows that a map Ψ_∞ can be defined on $P|_{\cup_{i \geq i_0} U_i}$ extending all Ψ_i, for $i \geq i_0$ and is, in particular, a C^s map. Moreover, if $W^\infty_{p_0}$ denotes the Zariski closure of the union of all $W^i_{p_0}$, $i \geq i_0$, we have that $F^\infty_{p_0}$ must fix $W^\infty_{p_0}$ pointwise, hence $F^\infty_{p_0}$ is the full subgroup of H that does so.

We now define on the B-invariant dense open set $U_\infty := \bigcup_{i \geq i_0} U_i$ an H-pair (W^∞, V) for B as follows. Set $W^\infty \subset \mathcal{E}^s(P|_{U_\infty}, V)$ to be the collection of C^s maps φ_w, $w \in W^\infty_{p_0}$, defined by

$$\varphi_w(p) := \Psi_\infty(p)w.$$

It is now routine to check that properties 1, 2, and 3 in the definition of an H-pair are satisfied. In fact, property 1 results from H-translations being injective; for property 2, remark that $\tau_{p,p'} : W^\infty_{p'} \to W^\infty_p$ is the H-translation that corresponds to $\Psi_\infty(p)\Psi_\infty(p')^{-1}$; and property 3 holds since V and the H-action on it have not changed.

The homomorphisms ρ^∞_p associated to the just constructed H-pair can be described as follows. For each i and $p \in P|_{U_i}$, there is a C^s (hence C^∞) homomorphism $\rho^i_p : B \to N^i_p/F^i_p$ such that $e_p(g \cdot \varphi) = \rho^i_p(g)e_p(\varphi)$ for all $\varphi \in W^i$ and $g \in B$. It follows that for the same i, p, g,

$$\Psi_i(g^{-1}p) = \rho^i_p(g)\Psi_i(p).$$

Since Ψ_i and Ψ_j agree on $P|_{U_i \cap U_j}$, for $i, j \geq i_0$, we must have, for p in that set, that $\rho^j_p(g) = \rho^i_p(g)$ for all $g \in B$. We thus obtain a family of homomorphisms ρ^∞_p from B into N^∞_p/F^∞_p extending $p \mapsto \rho^i_p$ over $P|_{U_\infty}$ such that $\Psi_\infty(g^{-1}p) = \rho^\infty_p(g)\Psi_\infty(p)$. Remark that the homomorphisms depend C^s on p. Here, N^∞_p denotes the algebraic subgroup of H that fixes W^∞_p as a set (not necessarily pointwise). It contains F^∞_p as a normal subgroup. It now follows from the definition of $\varphi_w \in W^\infty$ that $e_p(g \cdot \varphi_w) = \rho^\infty_p(g)e_p(\varphi_w)$. (The action of B on W^∞ is canonically defined by restricting the natural action on the space of geometric structures over U.)

If each H-pair (W^i, V) is invariant, the same property is clearly inherited by (W^∞, V). If they are hyperbolic, let $g \in B, p \in P|_{U_\infty}$, and consider the decomposition $\rho^\infty_p(g) = l_u l_e l_h$ into unipotent, elliptic and $\mathbb{R}$-semisimple parts. Then for all $p \in U_\infty$, the union of W^i_p, $i \geq i_0$, is fixed pointwise by l_u and l_e. Since W^∞ is the Zariski closure of that union, l_u and l_e must be the unit element in N^∞_p/F^∞_p. $\square$

4 H-pairs for centralizers

The same set up of section 3 is still in force. Furthermore, we assume that a Lie group G acts C^s on P by principal bundle automorphisms and that B is a subgroup of G. If Z is a subgroup of the centralizer, $Z_G(B)$, of B

in G, we would like to know when an H-pair for B can be shown to be an H-pair for Z as well. The next lemma gives sufficient conditions for that to happen.

Lemma 4.1 *Suppose that (W, V) is a maximal invariant C^s H-pair for B, and that B acts topologically transitively on M. Then (W, V) is an H-pair for Z. More precisely, if W is defined over an open dense B-invariant set $U \subset M$, its elements can be extended above the saturation $U' = Z \cdot U$ so that (W, V), now defined above U', is a C^s H-pair for Z. The same conclusion holds if (W, V) is a maximal T-hyperbolic C^s H-pair for B for some $T \subset B$ such that T acts on M topologically transitively with a dense set of recurrent points.*

Proof. Fix $z \in Z$ and define $z \cdot W$ to be the set of all maps $z \cdot \varphi := \varphi \circ z^{-1}$, $\varphi \in W$. It is immediate to check that $(z \cdot W, V)$ is a C^s H-pair for B defined over $z(U)$.

Set $W^z := z \cdot W \cup W$ and $W^z_p := e_p(W^z) = W_{z^{-1}p} \cup W_p$. . We claim that (W^z, V) is a C^s H-pair for B, defined over $U^z := z(U) \cap U$. To show that, we first check that the evaluation maps $e_p : W^z \to V$ are injective, for p in some open dense B-invariant H-invariant subset of P.

For any $p, q_0 \in P|_{U^z}$, we recall that the H-translation τ_{p,q_0} on W_{q_0} may be regarded as an element of H/F_{q_0}, where F_{q_0} is the subgroup of H that leaves W_{q_0} pointwise fixed. Define now a map

$$\Psi^z_{q_0} : P|_{U^z} \to H/F_{q_0} \times H/F_{q_0} \quad \Psi^z_{q_0}(p) = (\tau_{p,q_0}, \tau_{z^{-1}p,q_0}).$$

Notice that $H \times B$ act on $H/F_{q_0} \times H/F_{q_0}$ diagonally, H acting on the left and B acting on the right via the homomorphism $\rho_{q_0} : B \to N_{q_0}/F_{q_0}$. The map is $H \times B$-equivariant:

$$\Psi^z_{q_0}((h, b) \cdot p) := \Psi^z_{q_0}(bph^{-1}) = h\Psi^z_{q_0}(p)\rho_{q_0}(b)^{-1} =: (h, b) \cdot \Psi^z_{q_0}(p)$$

where $p \in P|_{U^z}$ and $(h, b) \in H \times B$.

Applying the topological Furstenberg lemma, it follows that $\Psi^z_{q_0}$, restricted to $P|_{U'}$ for some open dense B-invariant subset $U' \subset U^z$, takes values into a single H-orbit. We can now conclude that the evaluation map $e_q : W^z \to W^z_q$ is injective for all $q \in P|U'$ in the following way. If $q \in P|_{U'}$, $\varphi_1 \in W$, and $z \cdot \varphi_2 \in z \cdot W$, then

$$\begin{aligned}
(\varphi_1(q), z \cdot \varphi_2(q)) &= (\tau_{q,q_0}(\varphi_1(q_0)), \tau_{z^{-1}q,q_0}(\varphi_2(q_0))) \\
&= (h\tau_{p,q_0}(\varphi_1(q_0)), h\tau_{z^{-1}p,q_0}(\varphi_2(q_0))) \\
&= (h\varphi_1(p), h\varphi_2(z^{-1}p)) \\
&= h(\varphi_1(p), z \cdot \varphi_2(p)).
\end{aligned}$$

Therefore, if $\varphi_1(q) = z \cdot \varphi_2(q)$, we conclude $\varphi_1(p) = z \cdot \varphi_2(p)$ for all $p \in P|_{U'}$.

It also follows from the equality $(\tau_{q,q_0}, \tau_{z^{-1}q,q_0}) = (h\tau_{p,q_0}, h\tau_{z^{-1}p,q_0})$ that $\tau_{q,p} : W_p^z \to W_q^z$ is an H-translation given by $\tau_{q,p}(w) = hw$. Such h is uniquely defined up to right translation by elements in the group

$$F_p^z = F_{z^{-1}p} \cap F_p = \{h \in H \mid hw = w \text{ for all } w \in W_p^z\}.$$

The third property of an H-pair is trivially satisfied since V and the H-action on it have not changed.

The previous discussion shows that (W^z, V) is an H-pair for B defined over $z(U) \cap U$ and by construction it contains (W, V). We can now use the maximality of (W, V) to conclude that

$$[W, V] = [z \cdot W, V] = [W^z, V].$$

In particular, each $\varphi \in W$ extends above a $\langle z \rangle$-invariant open dense subset of M, which is also B-invariant and we may assume that W is $\langle z \rangle$-invariant. Since z is an arbitrary element of Z, we conclude that W is Z-invariant. In particular, (W, V) is an H-pair for Z. $\square$

Lemma 4.2 *Let Z be a group of H-bundle automorphisms commuting with the B-action. Assume that the action of B on M is topologically transitive and there exists a C^s H-pair $(W, H/H_0)$ for B. Then, there exists a C^s H-pair $(W', H/F)$ for Z such that F is a subgroup of H_0. Moreover, denoting by π the natural projection from H/F onto H/H_0, there is $\varphi \in W'$ such that $\pi \circ \varphi \in W$.*

Proof. The H-pair $(W, H/H_0)$ for B is defined over some open dense B-invariant subset $U \subset M$ and we recall that H_0 does not contain a nontrivial normal subgroup of H. Fix a point $q_0 \in P|_U$. By translating q_0 in its fiber by some appropriate element in H, we may assume without loss of generality that W_{q_0} (the image of W in H/H_0 under the map e_{q_0}) contains the coset H_0, i.e. there is $\varphi_0 \in W$ such that $\varphi_0(q_0) = H_0$. Notice that if $p \in P|_U$, then

$$\varphi_0(p) = \tau_{p,q_0} H_0.$$

The group $F := F_{q_0}$ of elements of H that fix W_{q_0} pointwise is a subgroup of H_0 (since it fixes, in particular, the coset H_0), hence it does not contain a nontrivial normal subgroup of H.

Let $N := N_{q_0}$ be, as before, the subgroup of H that stabilizes W_{q_0}, which contains F as a normal subgroup, and set $A := N/F$.

We regard $P_1 := P|_U \times A$ as a principal bundle with group $H_1 = H \times A$ and right-action given by the product action. We define on H/F a right H_1-action given by

$$\tau \cdot (h, a) = h^{-1} \tau a,$$

for $\tau \in H/F$ and $(h, a) \in H_1$ and introduce the map

$$\psi : P_1 \to H/F, \quad \psi(p, a) := \tau_{p,q_0} a.$$

Notice that $\pi(\psi(p, e)) = \varphi_0(p)$, for $p \in P|_U$. A simple calculation also shows that ψ is H_1-equivariant, i.e. $\psi(ph, aa') = h^{-1}\psi(p, a)a'$, for $(p, a) \in P_1$ and $(h, a') \in H_1$. Moreover, with respect to the left B-action on P_1 given by

$$b \cdot (p, a) := (bp, \rho_{q_0}(b)a),$$

ψ is B-invariant. In fact,

$$\begin{aligned} \psi(b(p, a)) &= \psi(bp, \rho_{q_0}(b)a) \\ &= \tau_{bp,q_0} \rho_{q_0}(b)a \\ &= \tau_{p,b^{-1}q_0} \rho_{q_0}(b)a \\ &= \tau_{p,q_0} a \\ &= \psi(p, a). \end{aligned}$$

Therefore, $(\{\psi\}, H/F)$ defines an invariant C^s H_1-pair for B. By Lemma 3.3, there exists a maximal C^s invariant H_1-pair for B containing $(\{\psi\}, H/F)$, which we denote $(W_1, H/F)$.

Define a left Z-action on P_1 by

$$z \cdot (p, a) := (zp, a).$$

It is immediate to check that the Z-action commutes with both the B-action and the H_1-action on P_1. Therefore, by Lemma 4.1, $(W_1, H/F)$ is an H_1-pair for Z. Let now W' be the space of maps $\varphi : P \to H/F$ of the form $\varphi(p) := \varphi_1(p, e)$ for some $\varphi_1 \in W_1$, where e is the identity element in A. It also follows that $(W', H/F)$ defines a C^s H-pair for Z. Finally, as we already noted above, φ_0 is the image of some element of W' under the map from W' to W defined by post-composition with π. $\square$

5 Proof of the main theorem

Before beginning the proof of the main theorem, we collect a few facts concerning semisimple algebraic groups of real rank at least 2. The reader may refer to [2] or [12] for the facts stated here.

Let G denote a semisimple real algebraic group and T a maximal real split torus. It will now be assumed that G has real rank at least 2. Let $R = R(T, G)$ denote the set of roots of the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G and let T_α denote the codimension one torus defined as the identity component of the kernel of α in T. Then the centralizer $Z_G(T_\alpha)$ of T_α is a reductive $\mathbb{R}$-group whose semisimple part S_α is isomorphic to either $SL(2,\mathbb{R})$ or $PSL(2,\mathbb{R})$. S_α, moreover, intersects T in a one dimensional torus $\check{T}_\alpha$ such that T_α and $\check{T}_\alpha$ together generate T. The centralizer of $\check{T}_\alpha$ in G also is a reductive group whose semisimple part contains T_α.

Essentially the same argument as in [12, 1.2.2, p.39] gives the following claim.

Proposition 5.1 *Each $g \in G$ can be written as a product $g = g_1 g_2 \dots g_l$, where, for each i, $1 \le i \le l$, $g_i \in Z_G(T_\alpha)$ for some $\alpha \in R$.*

With these facts, we can now begin the proof of the main theorem. The C^s algebraic hull, H_0, of K determines a C^s K-invariant geometric structure $\varphi_0 : P|_{U_0} \to H/H_0$, where U_0 is some open dense K-invariant subset of M. It is immediate to check that $(\{\varphi_0\}, H/H_0)$ is a C^s H-pair for K defined over U_0.

We can now apply Lemma 4.2 to obtain a C^s H-pair for $\langle h_0 \rangle$, where h_0 is some $\mathbb{R}$-semisimple element of G commuting with K, which exists by assumption. That $\mathbb{R}$-semisimple element is contained in some real split torus T, and the same lemma implies the existence of a C^s H-pair for T, which is clearly also an H-pair for any of the T_α or $\check{T}_\alpha$.

Lemma 4.2 once again, applied now to T_α for any fixed α, gives a C^s H-pair for the centralizer of T_α. The result is a C^s H-pair which is *hyperbolic* for $\check{T}_\alpha$, since the latter group is contained in the semisimple part of that centralizer. Using Lemma 3.3 we obtain a maximal hyperbolic C^s H-pair (W, V) for $\check{T}_\alpha$.

We claim there exists a T-hyperbolic H-pair for T. To obtain such a pair, fix $\alpha \in R$ and a maximal $\check{T}_\alpha$-hyperbolic C^s H-pair for $\check{T}_\alpha$, which we denote $(W, H/L)$. We may assume that α and $(W, H/L)$ have been chosen so that L is minimal, that is, if $\beta \in R$ and $(W', H/L')$ is a maximal $\check{T}_\beta$-hyperbolic C^s H-pair for $\check{T}_\beta$ such that $L' \subset L$, then $L' = L$. By the descending chain condition for algebraic groups such α and $(W, H/L)$ indeed exist.

Lemma 4.1 implies that $(W, H/L)$ is also an H-pair for any element of T, since T centralizes $\check{T}_\alpha$. Let now Π be a hyperplane in T^* (the dual space of T) spanned by roots and complementary to α, and let $u \in T$ be a nonzero vector orthogonal to Π with respect to the Killing form. We apply Lemma 4.2 to obtain a C^s H-pair $(W', H/F)$ for the centralizer of u in G, where $F \subset L$. Notice, in particular, that $(W', H/F)$ is an H-pair for T. On the other hand, for any root $\beta \in \Pi$, S_β centralizes u, and since $\check{T}_\beta \subset S_\beta$, we conclude that $(W', H/F)$ is $\check{T}_\beta$-hyperbolic. Using the minimality of L we conclude that $F = L$. Moreover, by the second part of Lemma 4.2 and the fact that $F = L$, the intersection $W_0 := W \cap W'$ is not empty. Since both $(W, H/L)$ and $(W', H/L)$ are H-pairs for T, the intersection $(W_0, H/L)$ also defines an H-pair for T.

We claim that $(W_0, H/L)$ is the desired T-hyperbolic H-pair for T. First notice that since $(W, H/L)$ is $\check{T}_\alpha$-hyperbolic and $(W', H/L)$ is $\check{T}_\beta$-hyperbolic, $(W_0, H/L)$ is both $\check{T}_\alpha$-hyperbolic and $\check{T}_\beta$-hyperbolic for all roots $\beta \in \Pi$. But $\check{T}_\alpha$ and the groups $\check{T}_\beta$ for $\beta \in R \cap \Pi$ together span T. Therefore, as T is abelian, $(W_0, H/L)$ is T-hyperbolic.

Let (W, V) be a maximal C^s T-hyperbolic H-pair for T. We wish to prove that it is an H-pair for the centralizer $Z_G(T_\alpha)$ for each root α. To that end, let α be a root and let $\Pi \subset T^*$ be now a hyperplane containing α and spanned by roots. Let $u \in T$ be a nonzero element annihilated by all the elements of Π (in particular, $u \in T_\beta$ for all $\beta \in R \cap \Pi$) and consider a maximal $\langle u \rangle$-hyperbolic H-pair for T, (W', V), containing (W, V). By Lemma 4.1, (W', V) is an H-pair for $Z_G(u)$, hence it is $\check{T}_\beta$-hyperbolic for all $\beta \in \Pi \cap R$, since $\check{T}_\beta \subset S_\beta \subset Z_G(T_\beta) \subset Z_G(u)$. But u together with the groups $\check{T}_\beta$ generate T, so that (W', V) is T-hyperbolic. By maximality, we obtain that $W' = W$, so that (W, V) is an H-pair for $Z_G(u)$. But $u \in T_\alpha$, so that $Z_G(T_\alpha) \subset Z_G(u)$, hence (W, V) is an H-pair for $Z_G(T_\alpha)$, as claimed.

We claim, now, that the H-pair (W, V) obtained above is, in fact, an H-pair for the whole group G, defined over some G-invariant open dense set $U \subset M$. For that, all that is needed to show is that W is G-invariant, but this is now a consequence of Proposition 5.1.

The section of $P|_U$ asserted in the main theorem is obtained as follows. Fix $p_0 \in P|_U$ and let $W_{p_0}, N_{p_0}, F_{p_0}, \rho_{p_0} : G \to N_{p_0}/F_{p_0}$ be as before. Denote by Ψ_{p_0} the map

$$p \in P|_U \mapsto \Psi_{p_0}(p) := \tau_{p,p_0} \in H/F_{p_0}$$

and by $\bar{\Psi}_{p_0}$ the postcomposition of Ψ_{p_0} with the projection from H/F_{p_0} onto

H/N_{p_0}. Notice that Ψ_{p_0} satisfies

$$\Psi_{p_0}(gp) = \Psi_{p_0}(p)\rho_{p_0}(g)^{-1}$$

for all $g \in G$ and $p \in P|_U$, so that $\bar{\Psi}_{p_0}$ is G-invariant. Therefore, we obtain a G-invariant N_{p_0}-reduction of P over some G-invariant open dense subset of M. But H already is the C^s algebraic hull of G, so $H = N_{p_0}$ and since H is transitive on V, we conclude that $W_{p_0} = V$. Since, furthermore, H acts on V effectively, F_{p_0} must be trivial. Therefore, Ψ is an H-equivariant map from $P|_U$ onto H (i.e., $\Psi(ph) = h^{-1}\Psi(p)$) and is, in particular, a bijection from each fiber P_x, $x \in U$.

We can now define a C^s section σ of $P|_U$ by setting $\Psi(\sigma(x)) = e$, where e is the identity element in H and x is any element in U. The equation $g\sigma(x) = \sigma(gx)\rho_{p_0}(g)$ can be checked as follows, using the injectivity of Ψ on fibers:

$$\begin{aligned}
\Psi(g\sigma(x)) &= \Psi(\sigma(x))\rho_{p_0}(g)^{-1} \\
&= \rho_{p_0}(g)^{-1} \\
&= \rho_{p_0}(g)^{-1}\Psi(\sigma(gx)) \\
&= \Psi(\sigma(gx)\rho_{p_0}(g)).
\end{aligned}$$

6 Parabolic invariants

A C^s geometric structure $\varphi : P \to V$ can also be described as a C^s section of the bundle $P \times_H V := (P \times V)/\sim$ over M, where $\sim$ is the equivalence relation on the product such that $(p, v) \sim (ph, h^{-1}v)$. If we represent by pv an equivalence class, the section associated to φ is written as

$$x \in M \mapsto E(x) := p\varphi(p)$$

for any p in the fiber P_x of P above x. (This is well-defined due to the H-equivariance of φ.) The action of $g \in G$ on E is denoted

$$(g_*E)(x) := gE(g^{-1}x) = p\varphi(g^{-1}p).$$

Suppose that $F \to M$ is a smooth (C^∞) vector bundle with n-dimensional fibers and let P be an H-reduction of the bundle of frames of F, so that H is a subgroup of $GL(n, \mathbb{R})$. An element $p \in P$ is a linear isomorphism from $\mathbb{R}^n$ into the fiber F_x above $x = \pi(p)$. ($GL(n, \mathbb{R})$ operates on the right by postcomposition with π.) The reduction may arise due to the existence of some further geometric structure on F, such as a volume form, when $H = SL(n, \mathbb{R})$.

Let V be the Grassmannian of k-planes in $\mathbb{R}^n$ equipped with the natural action of $GL(n, \mathbb{R})$. A continuous H-equivariant map $\varphi : P \to V$ defines a continuous subbundle of F of fiber dimension k. If $E(x) \subset F_x$ denotes the subspace at $x \in M$, then for any $p \in P$ above x, $E(x) = p\varphi(p)$, i.e., the image of $\varphi(p)$ under the linear map p. Supposing now that F is a subbundle of TM, then any diffeomorphism g of M operates on E by the induced derivative map dg_x on T_xM. The action on E becomes

$$(g_*E)(x) := (dg)_{g^{-1}x} E(g^{-1}x) = p\varphi(g^{-1}p),$$

for any $p \in P_x$. If E is preserved by a subgroup B of G, one has for each $x \in M$ a map

$$gB \in G/B \mapsto (g_*E)(x) = p\Phi(p)(gB) \in \{k\text{-dimensional subspaces of } T_xM\},$$

where $\Phi(p)$, for each $p \in P$, is the map from G/B into the Grassmannian variety defined by $\Phi(p)(gB) = \varphi(g^{-1}p)$.

We give now an example of a parabolic invariant. Let $f \in G$ be contained in a split $\mathbb{R}$-torus $T \subset G$. Let U be the maximal unipotent subgroup contracted by f, so that for any $g \in U$ one has $f^n g f^{-n} \to e$ as n tends to $+\infty$, and denote by $Z_G(T)$ the centralizer group of T in G. Then $B = Z_G(T)U$ is a parabolic subgroup and each $g \in B$ has the property that the set

$$\{f^n g f^{-n} |\ n \in \mathbb{N}\}$$

is relatively compact in G. For $\chi \in \mathbb{R}$, and $x \in M$, where M is now a compact smooth manifold and G operates on M smoothly, we define a subspace $E(x)$ of F_x by the following property ([7, S.2.6]): A nonzero vector v belongs to $E(x)$ if and only if

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(df^n)_x v\| \leq \chi.$$

It is immediate to check that E is preserved by B. If E arises due to a gap in the Mather spectrum of f (cf. [15, 2.8, p.121]), then E is a Hölder continuous parabolic invariant.

The next theorem is due to Zimmer [18]. We provide below a proof adapted to our notations.

Theorem 6.1 *Let P be a C^s ($s \geq 0$) principal H-bundle defined over a manifold M and G a semisimple Lie group acting on P by principal bundle* *automorphisms all of whose $\mathbb{R}$-semisimple 1-parameter subgroups act topologically transitively on M, with a dense set of recurrent points. We suppose that H is real algebraic and is a C^s algebraic hull of the action. We also suppose that on some open dense G-invariant set $U \subset M$, $P|_U$ admits a C^s section σ such that $g\sigma(x) = \sigma(gx)\rho(g)$ for all $g \in G$ and $x \in U$, where $\rho : G \to H$ is a continuous surjective homomorphism. If $\varphi : P \to V$ is a C^s parabolic invariant, then there exists $w \in V$ whose H-orbit W is a compact real subvariety of V and for each $p \in P$ there exists $h_p \in H$ such that*

$$\varphi(g^{-1}p) = h_p\rho(g)w$$

for all $g \in G$. Consequently, if F denotes the normal subgroup of H that fixes W pointwise, and if P/F denotes the principal H/F-bundle obtained from P in a natural way, then the section $\bar{\sigma}$ of $(P/F)|_U$ induced from σ extends to a C^s section on P/F.

We now proceed to the proof of Theorem 6.1. Let P be as before a principal H-bundle over a manifold M and suppose that a Lie group G acts on P by bundle automorphisms. We suppose that there is a C^s section σ of $P|_U$, for an open dense G-invariant subset $U \subset M$, and a continuous homomorphism $\rho : G \to H$ such that $g\sigma(x) = \sigma(gx)\rho(g)$ for all $g \in G$ and $x \in U$. Also suppose there exists a C^s H-equivariant function $\varphi : P \to V$, where V is a real algebraic variety equipped with a real algebraic H-action, such that φ is invariant under a parabolic subgroup B of G. Recall that G/B is compact.

Denoting by $C(G/B, V)$ the space of continuous maps from G/B into V with the topology of uniform convergence, we obtain a continuous map

$$\Phi : P \to C(G/B, V),$$

such that $\Phi(p)([g]) := \varphi(g^{-1}p)$. Then we have the following lemma.

Lemma 6.2 *There is an element $w \in V$ such that*

$$\Phi(p)([g]) = h_p\rho(g)w,$$

for all $g \in G$ and $p \in P|_U$, where h_p is the unique $h \in H$ such that $p = \sigma(x)h_p^{-1}$, where x is the base point of p.

Proof. Define the map $\xi := \varphi \circ \sigma : U \to V$ and remark that for any $g \in B$

and $x \in U$, we have

$$\begin{aligned}\xi(bx) &= \varphi(\sigma(bx)) \\ &= \varphi(b\sigma(x)\rho(b)^{-1}) \\ &= \rho(b)\varphi(b\sigma(x)) \\ &= \rho(b)\varphi(\sigma(x)) \\ &= \rho(b)\xi(x).\end{aligned}$$

If T is a real algebraic 1-parameter subgroup of B associated to an $\mathbb{R}$-split group, we can apply the corollary of the Topological Furstenberg Lemma to conclude that ξ is constant, equal to some $w \in V$. Therefore, given any $g \in G$ and $p \in P|_U$, we write $p = \sigma(x)h_p^{-1}$ for some h_p, so that

$$\begin{aligned}\Phi(p)([g]) &= \varphi(g^{-1}\sigma(x)h_p^{-1}) \\ &= \varphi(\sigma(g^{-1}x)\rho(g)^{-1}h_p^{-1}) \\ &= h_p\rho(g)w\end{aligned}$$

which is the claim. □

Lemma 6.3 *We suppose now that the variety V is of the form H/H_0 and that ρ is a surjective homomorphism from G to H. Let $f : G/B \to V$ be defined by $f(gB) := \rho(g)w$ and suppose that a sequence h_jf converges uniformly to a continuous function $\theta : G/B \to V$, for $h_j \in H$. Then there is $h \in H$ such $\theta = hf$.*

Proof. This is a consequence of [17, 3.1.4] and the next lemma. □

Lemma 6.4 ([18]) *Let $W \subset P^N$ be a quasiprojective irreducible variety that is not contained in a proper projective irreducible subspace. Let X be a projective irreducible variety and $f : X \to \overline{W}$ a regular surjection. Suppose that h_j is a sequence in $GL(N+1,\mathbb{R})$, such that $h_j(W) \subset W$ and h_jf converges uniformly to a continuous function $\theta : X \to \overline{W}$. Then $\{h_j\}$ is bounded in $PGL(N+1,\mathbb{R})$.*

Proof. Fix a metric on P^N and choose $\epsilon > 0$ such that $d(\overline{W}, Y) \geq \epsilon$ for every proper projective subspace $Y \subset P^N$. Choose j_1 large enough so that $j \geq j_1$ implies

$$\sup_j d(h_jf(x), \theta(x)) \leq \frac{\epsilon}{2}.$$

Let $T_j = h_j/\|h_j\|$. Then if h_j is not bounded in $PGL(N+1,\mathbb{R})$, by passing to a subsequence we can assume that T_j converges in the space of $N+1$ by $N+1$ real matrices to a matrix T with $\|T\| = 1$ and $\det T = 0$. Let $X_1 = f^{-1}(P^N - [\ker T])$. Then $X_1 \subset X$ is (Zariski) open and dense. Observe that if $y \in P^N - [\ker T]$, then $h_j y = T_j y$ converges to Ty, in the image V of T in P^N. Therefore, if $x \in Z_1$, $h_j f(x)$ converges to $Tf(x) \in V$. Since $h_i f(x)$ converges to $\theta(x)$, we have $\theta(X_1) \subset V$. Therefore $\theta(X) \subset V$. However, this contradicts the fact that θ must be surjective. (Namely, if $y \in \overline{W}$, choose $x_j \in X$ such that $f(x_j) = h_j^{-1}(y)$. By passing to a subsequence, we can assume that x_j converges to some $x \in X$. Then $h_j f(x_j)$ converges to $\theta(x)$ since X is compact and $h_j f$ converges uniformly to θ. Thus $y = \theta(x)$.) $\square$

Using the previous result, the set U of Lemma 6.2 can be taken to be the entire manifold M. Theorem 6.1 now follows easily. Notice that the section claimed in the conclusion of Theorem 6.1 is deduced from φ in the same way that σ was obtained from Ψ at the end of section 5.

Theorem 1.6 is a consequence of the calculation given below, which shows that the group F that arises in Theorem 6.1 must be trivial if the action is effective relative to φ. We use the description of φ as a section $x \mapsto E(x) = p\varphi(p)$ of $P \times_H V$. For any given $x \in M$ and $p_0 \in P_x$, we can define for each h_0 in F an automorphism A of P_x by $A(p_0 h) := p_0 h_0 h$. A then induces a transformation on $(P \times_H V)_x$, still denoted A, and we have

$$\begin{aligned} A(g_* E)(x) &= Ap\varphi(g^{-1}p) \text{ for some } p = p_0 h \in P \\ &= p_0 h_0 h \varphi(g^{-1}p) \\ &= ph'\varphi(g^{-1}p) \text{ for some } h' \in F \\ &= p\varphi(g^{-1}p) \\ &= (g_* E)(x), \end{aligned}$$

for all $g \in G$. Therefore A, hence h_0, is the identity.

7 Cohomology vanishing

Let E be a C^s vector bundle over a C^s manifold M. Suppose that E is equipped with a C^s action by bundle automorphisms of a Lie group G. We denote by $\Gamma^s(E|_U)$ the space of all C^s sections of $E|_U$, where U is some open subset of M. If U is G-invariant, G acts on $\Gamma^s(E|_U)$ by defining $g_*\alpha$ for $\alpha \in \Gamma^s(E|_U)$ and $g \in G$ so that $(g_*\alpha)(x) := g\alpha(g^{-1}(x))$, for all $x \in U$.

We now define a cohomology group $\check{H}^1_s(G,E)$ associated to a C^s action of G on E as follows. A C^s *o.d. 1-cocycle* (for open and dense) is a map θ

from G into $\Gamma^s(E|_U)$, for some open dense G-invariant $U \subset M$, such that θ is C^s as a map from $G \times U$ into E and

$$\theta(g_1 g_2) = \theta(g_1) + g_{1*}\theta(g_2)$$

for all $g_1, g_2 \in G$. We call U the *domain* of θ. A C^s o.d. 1-cocyle θ is an *o.d. coboundary* if there is an $\alpha \in \Gamma^s(E|_V)$, for some open dense G-invariant $V \subset M$, such that over $U \cap V$

$$\theta(g) = \alpha - g_*\alpha$$

for all $g \in G$.

The C^s o.d. first cohomology group $\check{H}^1_s(G, E)$ is now defined as the quotient of the space of o.d. cocycles by the space of o.d. coboundaries. It is clear that for any given subgroup B of G there is a linear map

$$\check{H}^1_s(G, E) \to \check{H}^1_s(B, E)$$

obtained by restriction of the cocycles from G to B.

Theorem 7.1 *Let G be a connected semisimple Lie group of real rank at least 2 that acts on a C^s vector bundle E over M by automorphisms, the action being C^s. We assume that every $\mathbb{R}$-semisimple 1-parameter subgroup of G acts topologically transitively on M and admits a dense set of recurrent points. Let K be a subgroup of G that acts on M topologically transitively and commutes with some $\mathbb{R}$-semisimple 1-parameter subgroup of G. Then, the restriction map*

$$\check{H}^1_s(G, E) \to \check{H}^1_s(K, E)$$

is injective.

It will be convenient to have E as an associated vector bundle to a principal bundle P, which may be taken to be the bundle of frames of E. Thus, let P be a C^s principal H-bundle over M, where H is a real algebraic group, and write $E = (P \times V)/H$. Here V is a finite dimensional vector space upon which H operates via a linear representation $\eta : H \to GL(V)$ and the right H-action on the product is given by

$$(p, v)h := (ph, \eta(h)^{-1}v)$$

for $p \in P, v \in V, h \in H$. We may also suppose that G acts on P by bundle automorphisms and that the induced G-action on E, by operating on the first factor of the product, is the one of the theorem.

We denote by π_P and π_E the base point projections from P and E onto M, respectively, and define the bundle

$$\pi : P \times_M E = \{(p, \alpha) \in P \times E \mid \pi_P(p) = \pi_E(\alpha)\} \to M.$$

$P \times_M E$ is a principal bundle over M with group given by the semidirect product $A := H \ltimes_\eta V$, where η is the linear representation of H into $GL(V)$ that appears in the definition of E. A has multiplication defined by

$$(h, u)(h', u') = (hh', \eta(h)u' + u)$$

for $(h, u), (h', u') \in H \times V$. The right action of A on $P \times_M E$ is defined as

$$(p, \alpha)(h, u) = (ph, \alpha - pu),$$

where an element p in the fiber of P above $x \in M$ is being regarded as a linear isomorphism from V onto the fiber E_x of E at x, so that pu is an element of E_x for each $u \in V$.

Starting with an action of G by automorphisms of P, we can define for each C^s o.d. 1-cocycle θ an action of G on $(P \times_M E)|_U$, where U is the domain of θ, as follows. For each $g \in G$ and $(p, \alpha) \in (P \times_M E)|_U$, we set

$$g(p, \alpha) = (gp, g\alpha + \theta(g)(\pi_P(gp))).$$

It is a consequence of the cocycle identity satisfied by θ that the definition above indeed yields an action.

We identify H with the subgroup of A consisting of elements of the form $(h, 0)$, where $h \in H$ and 0 is the zero element of V.

Lemma 7.2 *Let θ be an o. d. 1-cocycle with domain U representing a class in $\check{H}^1_s(G, E)$. The bundle $(P \times_M E)|_U$ with the G-action derived from θ admits a C^s G-invariant reduction Q with structure group H over some open dense G-invariant set $V \subset M$ if and only if θ represents the 0 class in $\check{H}^1_s(G, E)$.*

Proof. We first assume that θ is a coboundary of the form $\theta(g) = \beta - g_*\beta$, for some C^s section β of $E|_V$. A G-invariant H-reduction is obtained by setting

$$Q := \{(p, \beta(\pi_P(p)) \mid \pi_P(p) \in V\}.$$

Conversely, assume that a C^s G-invariant H-reduction Q exists and that it is defined on an open and dense subset U of M. The projection map

$(p, \alpha) \mapsto p$ from Q into $P|_U$ is a C^s isomorphism of principal H-bundles and we denote its inverse by $p \mapsto (p, \gamma(p))$. From the equality

$$(ph, \gamma(ph)) = (p, \gamma(p))(h, 0) = (ph, \gamma(p))$$

we conclude that $\gamma(ph) = \gamma(p)$ for all $p \in Q$ and $h \in H$, so that γ is of the form $\beta \circ \pi_P$ for some C^s section β of $E|_U$. A simple calculation now shows that $\theta = \beta - g_* \beta$. □

We remark that if L is a subgroup of $H \ltimes_\eta V$, then L fixes a point in V if and only it is conjugate to a subgroup of H, as a simple calculation shows.

Lemma 7.3 *Let L be a subgroup of $H \ltimes_\eta V$ and let N be a closed normal subgroup of L such that L/N is a semisimple group with finitely many connected components and N fixes a point in V. Then L also fixes a point in V.*

Proof. The set $W \subset V$ of fixed points by N is a nonempty affine subspace of V. Since N is a normal subgroup of L, W is stabilized by L, so we have an affine action of $\bar{L} = L/N$ on W. If $\bar{L}$ fixes a point of W, then the same is true for L. Moreover, if $\bar{L}_0$ is the connected component of the identity of $\bar{L}$ and $\bar{L}_0$ fixes a point in W, then $\bar{L}$ also fixes a point, as one easily sees by averaging over the finite group $\bar{L}/\bar{L}_0$. Therefore, to show that L fixes a point in V it suffices to show that a connected semisimple Lie group acting on a linear space V by affine transformations must have a fixed point.

We have thus reduced to problem to showing that a connected semisimple subgroup S of $\bar{H} \ltimes V$, where $\bar{H}$ is any closed subgroup of $GL(V)$, is conjugate in $\bar{H} \ltimes V$ to a subgroup of $\bar{H}$. This, in turn, is a consequence of the similar claim for the Lie algebra of S. By a standard argument one shows that the assertion for the Lie algebra of S is a consequence of Whitehead's lemma concerning the vanishing of the first cohomology group for semisimple Lie algebras. ([16, p. 220].) □

The theorem can now be proved as follows. Let θ be a 1-cocycle for G with domain U whose restriction to K is a coboundary. From the discussion above we can assume that the algebraic hull H_K for the K-action on $(P \times_M E)|_U$ associated to θ is contained in H. Denote by H_G a representative of the algebraic hull for the G-action on $(P \times_M E)|_U$. We may assume that H_K is a subgroup of H_G and we denote by N the maximal normal subgroup of H_G that is contained in H_K. According to the main theorem, H_G/N is a homomorphic image of G, hence it is semisimple, and the lemmas above conclude the proof.

Corollary 7.4 *We assume the same conditions of the previous theorem and consider the natural action of G on E that comes from a C^∞ G-action on a C^∞ principal bundle P. If K preserves a C^s connection on E over some K-invariant open dense subset of M, then G also preserves a C^s connection on E over some G-invariant open and dense subset of M.*

Proof. If ∇ denotes an arbitrary C^s connection on E and $g \in G$, we denote by $g \cdot \nabla$ the natural push-forward of ∇ under g, which is also a connection on E. The difference

$$\theta(g) := \nabla - g \cdot \nabla$$

defines a 1-cocycle with coefficients in the vector bundle $E^* \otimes E^* \otimes E$, where E^* denotes the dual bundle to E. It is immediate that a G-invariant C^s connection exists exactly when θ defines a trivial element in $\check{H}^1_s(G, E^* \otimes E^* \otimes E)$. Therefore, the claim is a consequence of the previous theorem. $\square$

Corollary 7.5 *We assume the same conditions of the previous theorem. Let E be a vector bundle associated to P and suppose that K preserves a C^s volume form on E. Then G also preserves the same volume form.*

Proof. We first remark that if Ω is a volume form on $E|_V$ preserved by K, for an open dense K-invariant set $V \subset M$, and if G preserves some volume form Ω' on $E|_U$ for some open dense G-invariant set $U \subset M$, then $\Omega = f\Omega'$ for some continuous K-invariant function on $U \cap V$. In particular f must be constant since the action of K is topologically transitive. It follows that Ω is itself G-invariant. We can now prove the existence of Ω' by applying the theorem to the cocycle $\theta(g)$ defined by the equation

$$(g^{-1})^*\Omega = e^{\theta(g)}\Omega.$$

Then θ gives an element in $\check{H}^1_s(G, L)$, where L is the trivial bundle with fiber $\mathbb{R}$, so that the C^s sections of L are the C^s functions on M. $\square$

8 Lattice actions

We give here the proof of Theorem 1.7. The notations used in that theorem are now in force. Observe that if $k \in G$ is Anosov, then the same is true for any conjugate gkg^{-1}. Moreover, if l centralizes k and lies in a compact subgroup, then (some power of) kl is also Anosov. In particular, the Anosov regular element k referred to in the theorem can be assumed without loss of generality to lie in some Cartan subgroup A and, therefore, to be part of an $\mathbb{R}$-semisimple 1-parameter subgroup. It is also important to remark that, after conjugation by some element in G, the Cartan subgroup A containing k may be assumed without loss of generality to intersect Γ in a cocompact lattice in A. ([13].) Therefore, the A-orbit of $[e]$ in G/Γ is compact.

Before we begin the proof of the theorem, we make a general remark about smooth cohomology. Let N be, for the moment, a topological space equipped with an n-dimensional lamination $\mathcal{F}$. I.e., N admits an open cover $\{U_\alpha : \alpha \in I\}$ such that each U_α is homeomorphic to $\mathbb{R}^n \times T_\alpha$ via a homeomorphism $\varphi_\alpha : U_\alpha \to \mathbb{R}^n \times T_\alpha$, where T_α is a topological space, such that the changes of coordinates $\varphi_\alpha \circ \varphi_\beta^{-1}$ have the following form on their domain of definition:

$$(x, t) \in \mathbb{R}^n \times T_\beta \mapsto (F_1(x, t), F_2(t)) \in \mathbb{R}^n \times T_\alpha,$$

where F_2 is a homeomorphism and $F_1(\cdot, t)$ is, for each t, a smooth diffeomorphism all of whose derivatives depend continuously on t. We call a foliation or lamination with such continuously varying smooth structure on leaves an *HP-lamination* (for Hirsch and Pugh). One can also define a C^r HP-lamination in an obvious way.

An object such as a map, connection, tensor field, etc., defined on N will be called $\mathcal{F}$-regular for an HP-lamination $\mathcal{F}$ if it is smooth along the leaves of $\mathcal{F}$ and its derivatives along the leaves vary continuously on the transversal direction. A homeomorphism $\phi : N \to N$ is $\mathcal{F}$-regular if, by definition, it sends leaves homeomorphically onto leaves and the restrictions of ϕ and ϕ^{-1} to leaves are smooth maps. In a similar way one defines an $\mathcal{F}$-regular flow $\phi_t : N \to N$, $t \in \mathbb{R}$.

An example of an $\mathcal{F}$-regular vector bundle is the tangent bundle $T\mathcal{F}$ to the lamination $\mathcal{F}$. The prime situation to keep in mind is that in which ϕ is Anosov on $T\mathcal{M}$ and $\mathcal{F}$ is the stable Anosov foliation.

Denote by $\bar{\nabla}$ the Levi-Civita connection for a Riemannian metric on N (not yet assumed to be as in the theorem). We define for each $g \in G$ a $(2, 1)$-tensor field $\bar{B}_g$ on $T\mathcal{M}$ such that

$$\bar{B}_g(X, Y) := g_*^{-1}(\bar{\nabla}_{g_*X} g_*Y) - \bar{\nabla}_X Y$$

for vector fields X, Y tangent to $\mathcal{M}$. We also introduce for an HP-lamination $\mathcal{F}$ and $g \in G$ preserving $\mathcal{F}$ a tensor $\bar{B}_g^{\mathcal{F}}$, which is defined as above except that $\bar{\nabla}$ is now the Levi-Civita connection for the Riemannian metric induced on the leaves of $\mathcal{F}$.

For the next lemma, let $\mathcal{F}$ be an HP-lamination in a manifold N equipped with an $\mathcal{F}$-regular Riemannian metric on the leaves and let $\bar{\nabla}$ be its Levi-Civita connection, which is also $\mathcal{F}$-regular. Let E be an $\mathcal{F}$-regular vector bundle on N equipped with an $\mathcal{F}$-regular metric and a compatible connection ∇ that is also $\mathcal{F}$-regular. The vector bundles $E \otimes (T^*\mathcal{F})^{\otimes^m}$ are, then, $\mathcal{F}$-regular and are automatically equipped with an $\mathcal{F}$-regular metric and compatible $\mathcal{F}$-regular connection, which we also call ∇. Let ϕ be an $\mathcal{F}$-regular homeomorphism of N and Φ an $\mathcal{F}$-regular automorphism of E above ϕ such that the following properties are satisfied:

1. ϕ is uniformly contracting, i.e., there is λ, $0 < \lambda < 1$, such that $\|\phi_*\| \leq \lambda$;
2. each derivative of $\bar{B}^{\mathcal{F}}_{\phi}$ is bounded;
3. Φ^{-1} is uniformly contracting, i.e., there is Λ, $0 < \Lambda < 1$, such that for each $\eta \in E$, $\|\Phi^{-1}(\eta)\| \leq \Lambda\|\eta\|$;
4. each derivative of B is bounded, where B is the section of $T^*\mathcal{F} \otimes E^* \otimes E$ defined by $B(X, \eta) := \Phi\nabla_{\varphi_*^{-1}X}\Phi^{-1}(\eta) - \nabla_X \eta$.

We observe that Φ acts on the right on sections of $(T^*\mathcal{F})^{\otimes^m} \otimes E$ in the following way: if $T : N \to (T^*\mathcal{F})^{\otimes^m} \otimes E$ is a section, Φ^*T is the section defined by

$$(\Phi^*T)_x(X_1, \ldots, X_m) = \Phi^{-1}(T_{\varphi(x)}(\varphi_*X_1, \ldots, \varphi_*X_m)).$$

Due to 1 and 3, this action is also contracting.

Lemma 8.1 *Let E be, as above, an $\mathcal{F}$-regular vector bundle over N and denote by $\mathcal{E}$ the space of $\mathcal{F}$-regular sections of E. Let Φ be an $\mathcal{F}$-regular automorphism of E satisfying the conditions enumerated above. Then the first cohomology group for the $\mathbb{Z}$-action induced by Φ on $\mathcal{E}$ is trivial. More precisely, if $T : \mathbb{Z} \to \mathcal{E}$ is a 1-cocycle, there exists $S \in \mathcal{E}$ such that*

$$T(1) = S - \Phi^*S.$$

The same holds, mutatis mutandis, for an $\mathbb{R}$-action.

Proof. The proof is standard. To find S, we define

$$S^k := \sum_{i=0}^{k-1} (\Phi^i)^* T_1$$

where $T_1 := T(1)$, and remark that S^k converges uniformly to a continuous section S of E that solves the cohomological equation.

To prove that S is $\mathcal{F}$-regular it suffices to show that for each positive integer k the sequence $\{\nabla^k S^m : m \in \mathbb{N}\}$ of sections of $E \otimes (T^*\mathcal{F})^{\otimes^k}$ converges uniformly as m approaches ∞. Remark that it is enough to prove the claim for $k = 1$ since the general case follows by induction, by replacing $\nabla^{k-1}S^m$ for S^m.

Using that

$$(\Phi^{-1})^*(\nabla_{\varphi_*^{-1}X}(\Phi^* T_1)) = \nabla_X T_1 + B(X, T_1)$$

we can write

$$\begin{aligned}
\nabla_{X_x} S^m &= \sum_{r=0}^{m-1} \sum_{i=1}^{r} \Phi^{-r+i-1} B_{\phi^{r-i+1}(x)}(\phi_*^{r-i+1}X, (\Phi^{i-1})^* T_1) \\
&+ \sum_{r=0}^{m-1} \Phi^{-r} (\nabla_{\phi_*^r X} T_1)_{\phi^r(x)}.
\end{aligned}$$

Taking norms, we obtain

$$\|\nabla S^m - \nabla S^l\| \leq \frac{\Lambda^l}{(1-\lambda)(1-\Lambda)} \|B\| \|T_1\| + \frac{(\lambda\Lambda)^l}{1 - \lambda\Lambda} \|\nabla T_1\|$$

which shows that ∇S^m indeed converges. □

The previous lemma will be used later in the following situation.

Lemma 8.2 *Let $\mathcal{F}$ be an HP-foliation of a manifold N, invariant under an $\mathcal{F}$-regular flow ϕ_t, $t \in \mathbb{R}$,that satisfies the boundedness condition in property 2 and uniformly contracting on $\mathcal{F}$ according to the definition in 1. Suppose that $\check{E}$ and E_β are $\mathcal{F}$-regular ϕ_t-invariant vector bundles such that E_β is a subbundle of $\check{E}$. Suppose, moreover, that the action of $(\phi_{-1})_*$ on $E := (\check{E}/E_\beta)^* \otimes E_\beta$ is uniformly contracting, in the sense of 3. Then, there exists a ϕ_t-invariant $\mathcal{F}$-regular subbundle E_α of $\check{E}$ such that $\check{E} = E_\alpha \oplus E_\beta$. If the $\mathbb{R}$-action on N has a dense set of recurrent points, the subbundle E_α is unique.*

Proof. Finding E_α is a cohomological problem for which the previous lemma applies. We have to show that the exact sequence of ϕ_t-invariant $\mathcal{F}$-regular vector bundles

$$0 \longrightarrow E_\beta \xrightarrow{i} \check{E} \xrightarrow{\pi} \check{E}/E_\beta \longrightarrow 0$$

splits in an $\mathcal{F}$-regular and ϕ_t-invariant way.

Let $\tau : \check{E}/E_\beta \to \check{E}$ be an $\mathcal{F}$-regular splitting, not necessarily invariant, and define $T(t) := \tau - (\phi_t)_*\tau$. Then T takes values in E_β and is a cocycle for the $\mathbb{R}$-action on the bundle $E = (\check{E}/E_\beta)^* \otimes E_\beta$ of endomorphisms from $\check{E}/E_\beta$ into E_β. We can now apply the previous lemma to get a section S of E which is a coboundary of T. Then set $E_\alpha = \tau - S$. Uniqueness is a consequence of the property that the only ϕ_t-invariant continuous section of E is 0. $\square$

We proceed now with the proof of the theorem. Writing $E = T\mathcal{M}$, we let $F(E)$ denote the $GL(n,\mathbb{R})$-bundle of frames associated to E and $PF(E)$ the $PGL(n,\mathbb{R})$-bundle obtained as the quotient of the bundle of frames by the center of $GL(n,\mathbb{R})$. Equivalently, we could first consider the frame bundle associated to $\bigwedge^n(E^*)\otimes E$ and then pass to the G-invariant $SL(n,\mathbb{R})$-reduction P consisting of frames $\omega \otimes \sigma$ for which $\omega(\sigma) = 1$. We can then think of $PF(E)$ as the $PSL(n,\mathbb{R})$-bundle defined as P modulo the center of $SL(n,\mathbb{R})$.

Lemma 8.3 *We assume the hypothesis and notations of Theorem 1.7. Then the C^0 algebraic hull for the $SL(n,\mathbb{R})$-action on $PF(E)$ is $PGL(n,\mathbb{R})$ and the action is effective relative to the parabolic invariant corresponding to the stable subbundle of k.*

Proof. As indicated before, we may assume that k is in an $\mathbb{R}$-semisimple 1-parameter group, which we call L, and the action of L is Anosov on $T\mathcal{M}$.

We first remark that the L-invariant stable and unstable subbundles in $T\mathcal{M}$ cannot be G-invariant. In fact, let $\mathfrak{g}$ be the Lie algebra of G and let $X \in \mathfrak{g}$ be a nonzero vector in the Lie algebra of L. We also denote by X the vector field on N that generates the L-action. If G preserves, say, the stable bundle of L, then any conjugate $\mathrm{Ad}(g)X$, $g \in G$, will generate a flow that is Anosov on $T\mathcal{M}$ and is contracting on the stable subbundle of X. Denoting by $\mathfrak{a}$ the Cartan subalgebra containing X, then for any element s in the Weyl group $W(\mathfrak{g},\mathfrak{a})$ the flow of sX is also contracting on the stable bundle of L since sX is of the form $\mathrm{Ad}(g)X$ for some $g \in G$. On the other hand, any element of $\mathfrak{a}$ is a positive linear combination of elements in the orbit of X under $W(\mathfrak{g},\mathfrak{a})$. Consequently, for each $Y \in \mathfrak{a}$, including 0, the flow of Y is contracting on the stable bundle of L. But this is clearly a contradiction.

Therefore, the C^0 algebraic hull H_L for the L-action must be a proper subgroup of the C^0 algebraic hull for the G-action. According to the main theorem, there must be a normal subgroup N of H_G contained in H_L such that H_G/N is a nontrivial homomorphic image of G. Since G is

$SL(n,\mathbb{R})$, $H_G \subset PGL(n,\mathbb{R})$, and H_G/N is nontrivial, we conclude that $H_G = PGL(n,\mathbb{R})$ and N is trivial. $\square$

As a consequence of the lemma and Theorem 1.6, there is a C^0 section σ of $PF(E)$ and a homomorphism $\rho : G \to PGL(n,\mathbb{R})$ such that $g_*\sigma = \sigma\rho(g)$ for all $g \in G$. The homomorphism ρ must be equivalent to either the standard representation of $SL(n,\mathbb{R})$ or its inverse-transpose.

Lemma 8.4 *The section σ obtained above is smooth. Moreover, the Anosov element k may be assumed to have smooth stable and unstable foliations.*

Proof. If we show that σ is smooth on $M_{[e]}$, then it must be smooth on any other fiber of $\mathcal{M}$. Notice, in fact, that for each $g \in G$ and $y \in M_{[g]}$, we have $\sigma_y = (g_*\sigma)_y\rho(g)^{-1}$. (In reality, we are only interested in smoothness over $M_{[e]}$, which yields smoothness for the corresponding Γ-equivariant section over M.) Moreover, as we remarked earlier, the Cartan subgroup A that contains k may be assumed to be such that the A-orbit of $[e]$ in G/Γ is compact. Therefore, we may restrict the Anosov element k to a compact A-invariant submanifold of N containing $M_{[e]}$ so that the standard theory of (transversally) hyperbolic actions applies. In particular, the stable and unstable subbundles of k are integrable and produce HP-laminations and the boundedness assumptions needed for Lemma 8.2 are satisfied for those laminations. The same is true for any conjugate of k that still lies in A. In what follows, $T\mathcal{M}$ will denote the restriction of the bundle to that compact set.

Let R be the set of weights for the representation ρ and let X be an infinitesimal generator for the 1-parameter subgroup of A that contains k. After a small perturbation of X in A, we may assume that $\alpha(X) \neq \beta(X)$ for all distinct $\alpha, \beta \in R$ and that the corresponding flow is still Anosov. Let E_α be the continuous subbundle of $T\mathcal{M}$ that corresponds to the weight α, and let E^- be the stable bundle for the flow ϕ_t of X.

Let $\Omega^1 := |\bigwedge^n(T^*\mathcal{M})|$ denote the line bundle of 1-densities for $T\mathcal{M}$ and $\Omega^{1/n}$ the line bundle of $1/n$-densities, so that $(\Omega^{1/n})^{\otimes n} = \Omega^1$. Remark that, for each α, we can find a continuous norm on $E_\alpha \otimes \Omega^{1/n}$ which expands under ϕ_t with exact rate given by $e^{\alpha(X)t}$.

We now claim that E^- (resp., E^+) is the sum of E_α for $\alpha(X) < 0$ (resp., $\alpha(X) > 0$). First notice that any continuous L-invariant subbundle, such as E_α, decomposes continuously as a direct sum

$$E_\alpha = (E^- \cap E_\alpha) \oplus (E^+ \cap E_\alpha)$$

and we wish to show that $E^+ \cap E_\alpha = 0$ (resp., $E^- \cap E_\alpha = 0$). For this end, it will be enough to show that we can find sequences $x_k \in N$ and $t_k \to \infty$, such that for any nonzero elements $u_k \in E_\alpha(x_k)$

$$\lim_{k\to\infty} \frac{1}{t_k} \log \frac{\|D\phi_{t_k} u_k\|}{\|u_k\|} = \alpha(X) < 0.$$

This is now a consequence of the fact that the flow takes place on a set of finite volume. In fact, if ω is a nonvanishing volume density on our compact A-invariant submanifold given as the product of a continuous volume density on $T\mathcal{M}$ and an A-invariant volume density along the A-orbits, and if f_t is the function on the manifold defined by $\phi_t^*\omega = f_t^n\omega$, then for any t, f_t^n has average 1 over N. Fixing a sequence $t_k \to \infty$, we can then choose for each k a point x_k such that $f_{t_k}(x_k) = 1$, and the claim will be satisfied for this sequence since elements in $E_\alpha(x)$ grow with rate $f_t(x)e^{t\alpha(X)}$.

For each pair (α, β) of distinct weights of ρ, define $E_{\alpha,\beta} := E_\alpha \oplus E_\beta$. We claim that E_α and $E_{\alpha,\beta}$ are integrable and their respective foliations are HP. To show this fact, we use again the Weyl group $W(\mathfrak{g}, \mathfrak{a})$ and recall that for each s in it, sX commutes with X and is conjugate to X by some element of G. Therefore, the flow of sX has the form $g\phi_t g^{-1}$ and is also Anosov and commutes with the flow of X. Let $\mathcal{O}$ denote the orbit of X under the Weyl group. Using now the fact that the Weyl group for $SL(n, \mathbb{R})$ consists of all permutations of the weights, one can find subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{O}$ for which

i. the set of weights γ such that $\gamma(Y) < 0$ for all $Y \in \mathcal{U}_1$ is $\{\alpha\}$;

ii. the set of weights γ such that $\gamma(Y) < 0$ for all $Y \in \mathcal{U}_2$ is $\{\alpha, \beta\}$ and there is $Y_0 \in \mathcal{U}_2$ such that $\alpha(Y_0) < \beta(Y_0) < 0$.

Denoting by E_Y^- (resp., E_Y^+) the stable (resp., unstable) bundle for Y, we have

$$E_\alpha = \bigcap_{Y\in\mathcal{U}_1} E_Y^+ \quad \text{and} \quad E_{\alpha,\beta} = \bigcap_{Y\in\mathcal{U}_2} E_Y^+.$$

Therefore, as each E_Y^+ is the tangent bundle to an HP-lamination, the same is true for E_α and $E_{\alpha,\beta}$. We denote by $\mathcal{E}_\alpha$ and $\mathcal{E}_{\alpha,\beta}$ the laminations that integrate E_α and $E_{\alpha,\beta}$.

The final claim is that for each pair α, β, E_α is $\mathcal{E}_\beta$-regular. Once this is shown, we can apply a theorem of J. L. Journé ([6]) to conclude that E_α is smooth. Notice that, as a consequence, the stable bundle of X will also be smooth. Therefore, the same argument that concluded the existence of a continuous σ now gives a smooth σ.

The bundles E_β and $E_{\alpha,\beta}$ are $\mathcal{E}_\beta$-regular since they are both tangent bundles to HP-laminations that contain $\mathcal{E}_\beta$. It follows that the bundle

$$E := (E_{\alpha,\beta}/E_\beta)^* \otimes E_\beta$$

is also $\mathcal{E}_\beta$-regular. Now, elements of E grow under the flow of Y with the exact exponent $\beta(Y) - \alpha(Y) > 0$. It is, then, possible to apply Lemma 8.3 to conclude that the decomposition $E_\alpha \oplus E_\beta$ is $\mathcal{E}_\beta$-regular, which is what we wanted to show. $\square$

Lemma 8.5 *Under the hypothesis of the theorem, any G-invariant continuous tensor field on $T\mathcal{M}$ of type (r, s), $r \neq s$, defined on an open dense set, must vanish identically.*

Proof. Denote by $\mathcal{T}$ the linear space of tensors of type (r, s) on $\mathbb{R}^n$ and let R be a G-invariant tensor field of type (r, s). Then R defines a continuous, G-invariant, $GL(n, \mathbb{R})$-equivariant map from the frame bundle $F(E)$ into $\mathcal{T}$ (over some open dense subset of N). Since the G-action is topologically transitive, R actually maps into an orbit $H \cdot \tau_0$ of $H = GL(n, \mathbb{R})$ in $\mathcal{T}$. If H_0 denotes the isotropy group of τ_0 in H, we obtain in this way a continuous, G-invariant, H_0-reduction of $F(E)$ over some open dense subset of N. But we have already shown above that the C^0 algebraic hull of the G-action projects onto $PSL(n, \mathbb{R})$, so that H must contain $SL(n, \mathbb{R})$. Therefore, τ_0 is fixed by all of $SL(n, \mathbb{R})$. It is now immediate that $\tau_0 = 0$. $\square$

Lemma 8.6 *Under the hypothesis of the theorem, $\mathcal{M}$ admits a unique tangential G-invariant smooth connection ∇. The connection is torsion-free and its curvature tensor vanishes identically.*

Proof. Once we have proved the existence of ∇, the vanishing of its torsion and curvature will follow from the previous lemma. Also remark that the vector bundle $\check{E} := \bigwedge^n(T^*\mathcal{M}) \otimes T\mathcal{M}$ is equipped with a smooth G-invariant connection. In fact, if P denotes the frame bundle of $\check{E}$ and σ is the smooth section of $P/\{\pm I\}$ obtained in Lemma 8.4, then there exists a unique connection on $\check{E}$ with respect to which σ is parallel, and it can be easily shown to be smooth and G-invariant. Therefore, in order to obtain ∇, it suffices to obtain a smooth invariant connection on $\bigwedge^n(T^*\mathcal{M})$.

Let k be the Anosov element assumed in the statement of the theorem and $E^\pm$ its stable and unstable bundles. According to Lemma 8.4, $E^\pm$ are smooth. Also recall that k lies in an Anosov 1-parameter group generated by X in a Cartan subalgebra $\mathfrak{a}$. We denote by $\mathcal{E}^{\pm}$ the smooth foliations of $E^{\pm}$.

We first claim that the line bundle of top forms on E^{ϵ}, $\epsilon = +, -$, admits a continuous X-invariant connection, ∇^{ϵ}. In fact, observe that we can define a smooth partial connection of E^{ϵ} along $E^{-\epsilon}$ (a Bott connection) as follows:

$$D^{\epsilon}_X Y = \Pi^{\epsilon}[X, Y]$$

where Π^{ϵ} is the natural projection onto E^{ϵ}, X is a vector field tangent to $E^{-\epsilon}$ and Y is a vector field tangent to E^{ϵ}. D^{ϵ} induces a smooth X-invariant covariant derivative on the bundle of top forms on E^{ϵ} along $E^{-\epsilon}$. To obtain a covariant derivative on the same line bundle in the direction E^{ϵ}, we appeal to Lemma 8.2 and the characterization of invariant connections in cohomological terms discussed in section 7. We obtain in this way a continuous X-invariant connection ∇' on $\bigwedge^n(T^*\mathcal{M})$.

Fixing a smooth (not necessarily invariant) connection ∇° on $\bigwedge^n(T^*\mathcal{M})$, we define a continuous 1-form ω on $T\mathcal{M}$ by

$$\nabla'_v \xi - \nabla^{\circ}_v \xi = \omega(v)\xi,$$

where v is any element of $T_x\mathcal{M}$, $x \in N$, and ξ is a smooth section of $\bigwedge^n(T^*\mathcal{M})$. Then, by the way in which ∇' has been constructed, $\omega|_{E^{\epsilon}}$ is a $\mathcal{E}^{\epsilon}$-regular 1-form on E^{ϵ}, for $\epsilon = +, -$.

Due to Corollary 7.4, G preserves a connection $\bar{\nabla}$ on the bundle of top forms on $T\mathcal{M}$ over some open and dense G-invariant subset of N. But the difference $\nabla' - \bar{\nabla}$ is (up to sign) a 1-form on $T\mathcal{M}$ invariant under the flow of the Anosov element X, so that it must vanish. Therefore, ∇' also is G-invariant.

It remains to show that ∇' is smooth or, equivalently, that ω is smooth. Remark that its restriction to $\mathcal{E}^{\epsilon}$, $\epsilon = +, -$, is already known to be smooth (along $\mathcal{E}^{\epsilon}$).

We employ now the notations from Lemma 8.4. Let Y be an image of X under some element of the Weyl group $W(\mathfrak{a}, \mathfrak{g})$. Then Y is also Anosov and its stable (resp., unstable) foliation denoted $\mathcal{E}^-_Y$ (resp., $\mathcal{E}^+_Y$) is smooth. Just as for X, the restriction of ω to these foliations must be tangentially smooth, so that the restriction of ω to any $\mathcal{E}_{\alpha,\beta}$, for any pair (α, β) of distinct weights of ρ (as in the proof of Lemma 8.4) is also tangentially smooth. Therefore, since

$$E_{\alpha,\beta} = \bigcap_{Y \in \mathcal{U}_2} E^+_Y$$

we conclude that $\omega(Z_\alpha)$ is smooth along $\mathcal{E}_{\alpha,\beta}$ for each α and β. Therefore, the same argument in Lemma 8.4 that uses a theorem of Journé, implies that $\omega(Z_\alpha)$ is smooth for each α, so that ω is smooth. $\square$

We now conclude the proof of the theorem by appealing to a result of Y. Benoist and the second author [1] that classifies affine Anosov diffeomorphisms. First, remark that some $\gamma \in \Gamma$ is an Anosov diffeomorphism of M. To see that, recall that the vector field X can be taken to lie in a Cartan subgroup A whose orbit through $[e]$ is compact. I.e., $A/(A \cap \Gamma)$ is a torus. Since a small perturbation of X inside the Lie algebra $\mathfrak{a}$ of A is also the generator of an Anosov flow, we can choose $Y \in \mathfrak{a}$ such that $[e]$ is a periodic point for the flow of Y on G/Γ and the flow of Y on $T\mathcal{M}|_{\mathcal{O}}$ is Anosov, where $\mathcal{O}$ is here the preimage in N of the orbit of $[e]$. In particular, the 1-parameter group generated by Y contains a $\gamma_0 \in \Gamma$ different from e. The element γ_0 is therefore Anosov on M and it preserves the flat connection ∇ on M, obtained from the identification of M and $M_{[e]}$. Therefore γ_0 is an affine Anosov diffeomorphism with smooth Anosov foliations preserving a smooth, flat, torsion-free connection ∇. It now follows from [1] that with respect to the invariant affine structure given by ∇, M is a flat torus.

9 Rigid geometric structures

Let $G^k(n,\mathbb{R})$ be the real algebraic group of k-jets at $0 \in \mathbb{R}^n$ of smooth local diffeomorphisms of $\mathbb{R}^n$ fixing the origin. Let M be a smooth manifold and let $F^k(M)$ denote the bundle of k-frames, whose fiber above $x \in M$ consists of k-jets at 0 of smooth local diffeomorphism from $\mathbb{R}^n$ into M sending 0 to x. $F^k(M)$ is a $G^k(n,\mathbb{R})$-principal bundle over M and we denote the base point projection by π^k. Given $\sigma \in F^k(M)$ and $g \in G^k(n,\mathbb{R})$, we denote the natural right-action of g on σ by σg. The natural projection from $G^k(n,\mathbb{R})$ to $G^l(n,\mathbb{R})$ as well as that from $F^k(M)$ to $F^l(M)$ will be denoted by π^k_l, for $k \geq l$.

Let V be a smooth m-dimensional real algebraic variety and let $J^k(V) := J^k_0(\mathbb{R}^n, V)$ be the space of all k-jets at $0 \in \mathbb{R}^n$ of germs of smooth maps from $\mathbb{R}^n$ into V. $J^k(V)$ has the structure of a smooth real algebraic variety. If $\alpha \in J^k(V)$, then $\alpha = j_k f(0)$, the k-jet at 0 of a smooth map f from a neighborhood of $0 \in \mathbb{R}^n$ into V.

Let $V(M)$ denote the associated V-bundle over M for a given real algebraic left-action of $G^k(n,\mathbb{R})$ on V. A geometric A-structure on M of order k and type V is defined in [4] as a smooth section s of $V(M)$. Equivalently, it can be defined as a smooth $G^k(n,\mathbb{R})$-equivariant map $\mathcal{G} : F^k(M) \to V$.

Starting with a real algebraic action ρ of $G^k(n,\mathbb{R})$ on V, it is possible to define on $J^l(V)$ a real algebraic action ρ_l of $G^{k+l}(n,\mathbb{R})$ in a canonical way, so that if $\mathcal{G} : F^k(M) \to V$ is a $G^k(n,\mathbb{R})$-equivariant smooth function, one can construct its *prolongation*, also canonically defined, which is a smooth geometric structure $\mathcal{G}_l : F^{k+l}(M) \to J^l(V)$ of order $k+l$ and type ρ_l. (See [11, IV-14].)

Let $\eta_m \in J^m(V)$ be a sequence such that η_m maps onto η_{m-1} under the natural projection and denote by H^m the isotropy group of η_m in $G^{r+m}(n,\mathbb{R})$. Taking now somewhat of a shortcut to the main definition of [4], we state

Definition 9.1 *A geometric structure $\mathcal{G}$ of type V and order k is* rigid *if there exists N_0 big enough, depending only on ρ, for which the following holds: for each sequence η_m in the image of $\mathcal{G}_m$ in $J^m(V)$ such that η_{m+1} maps onto η_m under the natural projection, the isotropy group H^{m+1} is isomorphic to H^m for each $m \geq N_0$.*

Suppose now that G acts smoothly on a smooth n-dimensional manifold M so that the action is topologically transitive. Therefore, over some open dense G-invariant set in M, the image of $\mathcal{G}_m$ lies in a single orbit $G^{r+m}(n,\mathbb{R})\eta_m$.

Let $P^k(U_k)$ denote a G-invariant H_k-reduction of $F^k(M)|_{U_k}$, where $U_k \subset M$ is open dense and G-invariant and $H_k \subset G^k(n,\mathbb{R}^n)$ is a representative of the smooth algebraic hull of the induced action of G by automorphisms of $F^k(M)$. By the general properties of algebraic hulls, we can choose the reduction $P^k(U_k)$ (after possibly having to translate on the right by some element of $G^k(n,\mathbb{R})$, and conjugating H_k by the same element) so that the natural projection from $P^{k+1}(U_k \cap U_{k+1})$ into $F^k(M)|_{U_k \cap U_{k+1}}$ actually maps onto $P^k(U_k \cap U_{k+1})$ and H_{k+1} maps onto H_k.

Definition 9.2 *We say that the G-action is* hull-rigid *if the H_k, defined above, eventually stabilize. More precisely, for some k_0 and for all positive integers r,s such that $k_0 \leq s \leq r$, the projections $\pi^r_s : H_r \to H_s$ are isomorphisms.*

Lemma 9.3 *Under the conditions of Theorem 1.8, the G-action is hull-rigid.*

Proof. We denote by P_k a G-invariant H_k-reduction of $F^k(M)$ such that H_k is the smooth algebraic hull for the G-action. For simplicity, we omit reference to the open dense subset U_k where the reduction is defined. Similarly, we define K-invariant smooth reductions Q_k of P_k with group $L_k \subset H_k$. (Also defined on some open dense K-invariant subset of M.) After conjugating by appropriate elements of $G^k(n, \mathbb{R})$ we may assume that π_l^k maps P_k onto P_l and Q_k onto Q_l, for $k > l$ and l big enough.

Denote by N_k the maximal normal real algebraic subgroup of H_k contained in L_k and remark that π_l^k maps N_k onto N_l. By the topological superrigidity theorem, H_k/N_k is a homomorphic image of G, which we denote $\rho_k(G)$.

If the K-action preserves a smooth rigid geometric structure, then by the next lemma L_k stabilizes, so that N_k must have bounded dimension. Therefore, as $\rho_k(G)$ also has bounded dimension, we conclude that the same holds for H_k. On the other hand, $\pi_s^r : H_r \to H_s$ is surjective for $s \leq r$, and its kernel is either trivial or infinite. (Remark that the kernel is contained in the nilpotent radical of $G^r(n, \mathbb{R})$.) Therefore, for sufficiently big r, s, $\pi_s^r : H_r \to H_s$ are isomorphisms. □

Lemma 9.4 *Suppose that a group G acts smoothly and topologically transitively on a smooth manifold M. Then the action is hull-rigid if and only if it preserves some rigid A-structure on some open dense G-invariant subset of M.*

Proof. If the action preserves a rigid A-structure, the sequence H_k of algebraic hulls must be contained in subgroups of $G^k(n, \mathbb{R})$ that eventually stabilize. Hence the H_k also eventually stabilize.

For the converse, suppose that the G-action is hull-rigid and consider a sequence $P^k \subset F^k(M)$, $k \geq 1$, of smooth G-invariant reductions defined over open dense G-invariant subsets of M with group H_k. We may suppose that π_l^k projects P^k onto P^l. Then, by the definition of a hull-rigid action, we have for big enough k that $\pi_k^{k+1} : P^{k+1} \to P^k$ is an isomomorphism. We simplify the notation by writing $\pi := \pi_k^{k+1}$, $P := P^{k+1}$ and $Q := P^k$.

Each $\sigma \in P$ determines a horizontal n-plane in $T_{\pi(\sigma)}Q$, where n is the dimension of M, and such a plane determines a frame for $T_{\pi(\sigma)}Q$. Therefore, π defines a G-invariant, smoothly varying frame at each point of Q. More precisely, over some open dense G-invariant subset $U \subset M$ there is a smooth section σ of the frame bundle $F^1(Q)|_U$ such that each $g \in G$ maps $\sigma(q)$ onto $\sigma(g_{(k)}q)$, $q \in Q|_U$.

We claim that σ defines a G-invariant rigid structure. This amounts to the following elementary fact: If a manifold is equipped with a smooth full frame field and a diffeomorphism f fixes a point in the manifold and preserves the frame at the fixed point up to order l, then f has the same

l-jet as the identity map. $\square$

10 Appendix

We sketch below a mostly self-contained proof of Margulis-Zimmer super-rigidity in the following case.

Theorem 10.1 *Suppose G is a simple real Lie group of rank at least 2 that acts ergodically by measure preserving transformations on a standard Borel space M with finite measure. Assume furthermore that this action can be lifted to a left action on a measurable H-bundle P by bundle automorphisms such that H is a simple noncompact real Lie group and is the measurable hull of the action of G. Then there exists a representation ρ of G into H and a measurable section σ of P such that*

$$g \cdot \sigma(x) = \sigma(gx)\rho(g).$$

In the statement above we have preferred the geometric language of action and principal bundle to the equivalent language of cocycles, though of course measurable bundle is just a somewhat unconventional way of talking about the trivial bundle.

We want to use the following dictionary:

mesurable $\leftrightarrow$ continuous ergodic $\leftrightarrow$ topologically transitive

Everything essentially translates easily except the topological Furstenberg Lemma, which requires a little care. We need for that a classical fact that is used in the proof of various versions of the Borel density theorem, namely, if L is a 1-dimensional noncompact algebraic group, then every L-invariant measure on an algebraic variety V on which L acts algebraically is supported on the set fixed points. A nice proof of this fact is contained in [4], and relies on the following consequence of Rosenlicht's stratification theorem: the action becomes proper on a Zariski open dense subset of the product of a sufficiently large number of copies of V.

Using this dictionary, and adding the hypothesis that T (as in section 2) preserves a measure μ, our topological Furstenberg Lemma translates now into an avatar of the classical Furstenberg Lemma. Let us see how it is done using the notations of our proof. The map $\bar{\Phi}$ will also take its values in the set of fixed points of T but for a somewhat different reason: it will take its values in the support of the T-invariant measure $\bar{\Phi}_*\mu$. The rest of the proof works *mutatis mutandis*.

With this point settled, we can carry out a translation of our main theorem (plus the assumption that G preserves a finite ergodic measure) into a mesurable statement by using the dictinary. This yields the conclusion of the measurable superrigidity theorem, but with more restrictive hypothesis.

To improve it and get rid of the two extra hypothesis, namely that every $\mathbb{R}$-semisimple element acts ergodically, and that the hull of an $\mathbb{R}$-semisimple element is smaller than H, we need two observations.

The first is a theorem of which we do not know a topological analogue: Moore's ergodicity theorem, which exactly states that if a noncompact simple algebraic group acts ergodically preserving a finite measure, then every noncompact closed subgroup of G (in particular any $\mathbb{R}$-semisimple element) is ergodic. This is classically the first use of ergodic theory in the proof of Margulis superrigidity.

To eliminate the second hypothesis which correspond to the use of Oseledec's theorem in [12] and of amenable cocycles in [17], we just need the following lemma, of which a topological analogue is certainly wrong.

Lemma 10.2 *Let P be a principal H-bundle over a standard Borel space M, such that H is simple noncompact. Let T be a 1-parameter group acting ergodically on M preserving a finite measure m. Then the hull of T is smaller than H.*

Proof. Let $V = H/Q$ be a compact algebraic variety on which H acts transitively, and which is not a point. Of course we can take Q to be a parabolic in H, or we can take a minimal algebraic variety invariant by H in the projective space of an irreducible representation; indeed, using the Rosenlicht stratification theorem, the action of H is transitive on such a minimal variety.

Now the classical Kakutani-Markov theorem yields a T-invariant probability measure μ on the total space P_V of the associated V-bundle that projects onto the G-invariant measure on M. Let μ_x be the measure supported on the fiber of P_V above the point x, obtained by the desintegration of μ. By ergodicity, the Zariski closure J_x in H of the stabilizer of μ_x is, for almost every x, conjugate to a certain group J which will contain the hull for the action of T. H itself does not preserve a measure on V. This follows from the fact that such a measure should be supported on the set of fixed points $F(h)$ of an arbitrary $\mathbb{R}$-semisimple element and that the intersection of all the $F(h)$ is empty since it is globally invariant under H. Therefore, J

is strictly smaller than H and the proof is complete. □

The lemma above is to be compared with Zimmer's result stating that the hull of an amenable group is amenable. The point we wish to emphasize is that we do not prove it for a parabolic in G as in [17], only for a single 1-parameter group, in which case the result follows at once from the Kakutani-Markov fixed point theorem.

Ultimately, since H is simple, every normal subgroup of H contained in the hull of T will be the identity.

To summarize, the proof sketched above is self-contained apart from the use of Moore's ergodicity theorem, the Kakutani-Markov theorem, Rosenlicht stratification theorem and the fundamental fact alluded to above concerning invariant measures by algebraic groups. We have used nothing about the structure theory of H, and the structure theory we have used of G is summarised in Proposition 5.1 and the following page. By the use of H-pairs, a fancy name for "algebraic sets of sections of algebraic bundles," we have avoided the need for parabolic invariants as in the proof in [17], and we have adapted the ideas using vector spaces of sections of vector bundles of [12], avoiding use of proximal maps. For our topological superrigidity theorem, we added hypothesis when no topological analogue of a mesurable result was available, namely Moore's theorem and the Kakutani-Markov theorem.

Although we have not checked, we expect the same proof to work when H is a p-adic group.

Finally, from the point of view of Zimmer's program, the fundamental question is whether or not an element satisfying our hypothesis (iii), (iv) and (v) exists for general, not hyperbolic, dynamical systems, at least for the lifted actions on the jet bundles.

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