MINIMAL SURFACES WITH NEGATIVE CURVATURE IN LARGE DIMENSIONAL SPHERES

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ABSTRACT. In this note, we answer positively a question of Yau by proving the existence of closed minimal surfaces with negative induced curvature in any sphere of large dimension. The proof follows the strategy of Song, applying it to closed Riemann surfaces with large automorphism groups, and obtaining almost hyperbolic minimal surfaces.

CONTENTS

1. INTRODUCTION

A classical result due to Robert Bryant [1] states the nonexistence of closed minimal surfaces with constant negative induced curvature in round spheres. In [16], Shing-Tung Yau asks the question of existence of closed minimal surfaces in spheres with negative induced curvature. In this paper, we give, as a corollary of our main result, a positive answer to this question:

Corollary 1.1. *There exists a closed minimal surface of negative curvature in any round sphere of large dimension.*

The proof follows the asymptotic strategy – when the dimension goes to infinity – developed by Antoine Song in [15]. In particular, we do not have any information on the

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dimension of the sphere for which it happens, although it is a well-known consequence of Gauß–Codazzi equations due to Blaine Lawson [10, Proposition 1.5] that closed negatively curved surfaces cannot exist in $\mathbf{S}^3$.

The novelty lies in the fact that we refine Song's construction and get the stronger result

Main Theorem. *For each integer n large enough, there exists a negatively curved closed minimal surface Σ_n in the round sphere of dimension n. Furthermore, we can choose Σ_n such that for all integer k*

$$\lim_{n\to\infty} \|\kappa_n + 8\|_{C^k} = 0,$$

where κ_n is the curvature of Σ_n.

Finally there is a closed Riemann surface X, such that for each n, there exists a finite group Γ_n in the isometry group of the n-sphere such that $X = \Gamma_n\backslash\Sigma_n$.

In other words, asymptotically, Bryant's result does not hold. We give an idea of the proof now: in his groundbreaking work, Song developed a new strategy to study harmonic maps into large dimensional spheres that are equivariant under unitary representations. Applying this general construction to the hyperbolic sphere with 3-cusps, Song obtained in [15, Theorem 0.3] a sequence of minimal surfaces in spheres that converges in the Benjamini–Schramm sense to the hyperbolic plane. The lack of smooth convergence in his theorem comes from the noncompactness of the cusped hyperbolic surface.

The main idea behind using a 3-holed sphere is that its Teichmüller space is reduced to a point. In this short note, we point out that one can replace the use of this 3-holed sphere by that of an orbifold whose Teichmüller space is a point, and then proceed to the proof as in [15] using the notion of induced representations that we spend some time recalling.

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2. Harmonic maps, minimal surfaces and results of Song

2.1. **Minimal surfaces.**

Let Σ be a closed surface and (M, h) a Riemannian manifold. The *area* of a smooth map $f : \Sigma \to M$ is defined by

$$\mathrm{A}(f) = \int_\Sigma \mathrm{d}\,\mathrm{vol}_{f^*h}\ .$$

The map f is called *minimal* if it is a critical point of the area, namely if for any smooth variation $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{A}(f_t) = 0\ .$$

We recall a definition of Robert Gulliver [7], see also Mario Micallef and Brian White for minimal maps [12]: let P be a discrete subset of Σ, a *branched minimal immersion* with *branch locus* P is a map $f : \Sigma \to M$ which is

- a minimal immersion away from P,
- for every p in P, the map f can be written in local coordinates centered at p as $f(z) = (z^{d+1}, g(z))$, where d is a positive integer and represents the order of branching, and moreover, g and $\mathrm{d}g$ are $\mathcal{O}(z^{d+1})$.

Remark 2.1. If p is a branched point of order d of a branched minimal immersion, the induced metric has a conical singularity of angle $2\pi(d+1)$ at p. Such a cone singularity yields a singularity of the sectional curvature of the induced metric.

2.2. **Harmonic maps from surfaces.** Let (S, g) be a closed Riemannian surface and (M, h) a Riemannian manifold. The *energy* of a smooth map $f : S \to M$ is

$$\mathrm{E}(f) = \frac{1}{2}\int_S \|\mathrm{d}f\|^2 \mathrm{d}\,\mathrm{vol}_g \ ,$$

where the norm $\|\mathrm{d}f\|$ is computed using the tensor product metric on $\mathsf{T}S \otimes f^*\mathsf{T}M$. The map f is called *harmonic* if it is a critical point of the energy, namely if for any smooth variation $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{E}(f_t) = 0 \ .$$

Since S has dimension 2, the energy $\mathrm{E}(f)$ only depends on the conformal class of g, and so harmonic maps can be defined on the underlying Riemann surface $X = (S, [g])$: a smooth map $f : X \to (M, h)$ is harmonic if and only if the $\mathbb{C}$-linear part ∂f of its differential is holomorphic.

The notion of harmonic maps is intimately linked with the theory of branched minimal immersions. The following result due to Robert Gulliver, Robert Osserman and Halsey Royden [7] is an extension of a classical observation of James Eells and Joseph Sampson [5] – see also Jonathan Sacks and Karen Uhlenbeck [14, Theorem 1.6].

Proposition 2.2. *A map f from X to (M, h) is a branched minimal immersion if and only if it is harmonic and conformal.*

The lack of conformality of a harmonic map $f : X \to (M, h)$ is encoded in its *Hopf differential*, which is defined by

$$\mathrm{Hopf}(f) = h^{\mathbb{C}}(\partial f, \partial f) \ ,$$

where $h^{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of h. Observe that since ∂f is holomorphic, $\mathrm{Hopf}(f)$ is a holomorphic quadratic differential on X, that is, an element of $H^0(K_X^2)$. The following Proposition is a direct consequence of the definition and Proposition 2.2:

Proposition 2.3. *A harmonic map f from X to (M, h) is a branched minimal immersion if and only if* $\mathrm{Hopf}(f) = 0$.

Let us observe that the proof of this result is local, so we do not need to assume that X is closed.

2.3. **The equivariant case.** In this paper, every group action on a Riemannian manifold will be by isometries.

Definition 2.4. Let Γ be a discrete group acting cocompactly on the hyperbolic plane $\mathbf{H}^2$ and ρ be a representation of Γ into $\mathsf{U}(n)$. The representation ρ has *finite energy* if there exists a ρ-equivariant map from $\mathbf{H}^2$ to the sphere $\mathbf{S}^{2n-1}$.

Observe that for n greater than 1 and Γ torsion free, any unitary representation has finite energy. However, if Γ has torsion, a necessary condition for ρ to be of finite energy is that any torsion element has a fixed point in the sphere.

Given a smooth equivariant map f, the pointwise norm $\|\mathrm{d}f\|$ is Γ-invariant, so it descends to a function on $\Gamma\backslash\mathbf{H}^2$. Thus, if Γ is cocompact, one can define the energy $\mathrm{E}(f)$ by integrating over a fundamental domain D for the action of Γ:

$$\mathrm{E}(f) = \frac{1}{2}\int_D \|\mathrm{d}f\|^2 \mathrm{d}\,\mathrm{vol}_g \ .$$

Subsequently, let

$$\mathrm{E}(\rho) := \inf \left\{ \mathrm{E}(f) \mid f : \mathbf{H}^2 \to \mathbf{S}^{2n-1} \text{ is smooth and } \rho\text{-equivariant} \right\} .$$

Note that we do not care whether Γ has torsion or not.

Song obtained in [15, Theorem 1.7] an extension of the classical result of Sacks–Uhlenbeck [13] in the spirit of [3, 4, 9]. We actually need a version of this result when torsion could be present:

Theorem 2.5 (SACKS–UHLENBECK, SONG)**.** *Let Γ be a discrete group acting cocompactly on the hyperbolic plane $\mathbf{H}^2$. Let ρ be a finite energy unitary representation of Γ in $\mathsf{U}(n)$. Then, there exists a ρ-equivariant energy minimizing harmonic map ψ from $\mathbf{H}^2$ to $\mathbf{S}^{2n-1}$.*

We just explain here how to modify the proof for group having torsion:

Proof of the extended of Sacks–Uhlenbeck Theorem 2.5. Consider the space

$$\mathcal{E}_\rho = \left\{ f : \mathbf{H}^2 \to \mathbf{S}^{2n-1} \mid f \text{ is } C^\infty \text{ and } \rho\text{-equivariant} \right\} .$$

By assumption $\mathcal{E}_\rho$ is nonempty so we can find a sequence $(f_j)_{j\in\mathbb{N}}$ such that

$$\lim \mathrm{E}(f_j) = \inf\{\mathrm{E}(f) \mid f \in \mathcal{E}_\rho\} .$$

Let D be the discrete set of branched points in $\mathbf{H}^2$. Let B be a ball in $\mathbf{H}^2$ whose iterates under Γ cover $\mathbf{H}^2$. Embed $\mathcal{E}_\rho$ in the Sobolev space $W^{1,2}(B, \mathbb{R}^{2n})$ (recall that the energy is defined on $W^{1,2}(B, \mathbb{R}^{2n})$). The sequence $(f_j)_{j\in\mathbb{N}}$ is now bounded in $W^{1,2}(B, \mathbb{R}^{2n})$ hence weakly converges by Banach–Alaoglu to ψ. By Rellich–Kondrachov theorem, $(f_j)_{j\in\mathbb{N}}$ also strongly converges in $L^2(B, \mathbb{R}^{2n})$ to ψ. From that we observe that ψ is ρ-equivariant and takes value in $\mathbf{S}^{2n-1}$ (almost everywhere). Finally ψ also minimizes locally the energy, away from D.

By lower semi-continuity of the energy, one obtains that $\mathrm{E}_{B\setminus D}(\psi) \leqslant \lim \mathrm{E}_{B\setminus D}(f_j)$, where E_U is the energy on an open set U. By [6, Section 8.4.3] one gets that ψ is a weak solution to the harmonic equation on $B \setminus D$. Finally, a classical result of Frédéric Hélein [8] implies that ψ is a strong solution on $B \setminus D$. Using the ρ-equivariance, we obtain a strong solution ψ on $\mathbf{H}^2 \setminus D$. Since ψ has locally finite energy on $\mathbf{H}^2$, it extends to a smooth harmonic map on $\mathbf{H}^2$ by a result of Sacks–Uhlenbeck [13, Theorem 3.6]. ☐

2.4. **Strong convergence.** Given a discrete group Γ, the *left regular representation* of Γ is the unitary representation

$$\lambda_\Gamma : \Gamma \longrightarrow \mathrm{End}(\ell^2(\Gamma, \mathbb{C}))$$

defined by $(\lambda_\Gamma(\gamma)(f))(x) = f(\gamma^{-1}x)$.

A sequence $(\rho_j)_{j\in\mathbb{N}}$ of representations with $\rho_j : \Gamma \to \mathsf{U}(N_j)$ *strongly converges* to a representation ρ if

$$\forall f \in \mathbb{C}[\Gamma] , \lim_{j\to\infty} \|\rho_j(f)\| = \|\rho(f)\| ,$$

where the norms are the operator norms, and we recall that the operator norm of a linear operator A is given by $\sup\{\|A(u)\| \mid u \in \mathbf{S}(\mathcal{H})\}$. We also say that two representations ρ_1 and ρ_2 are *weakly equivalent* if

$$\forall f \in \mathbb{C}[\Gamma] , \|\rho_1(f)\| = \|\rho_2(f)\| ,$$

By this very definition, if a sequence $(\rho_j)_{j\in\mathbb{N}}$ of representations with $\rho_j : \Gamma \to \mathsf{U}(N_j)$ *strongly converges* to a representation ρ_1, and furthermore ρ_1 and ρ_2 are weakly equivalent, then $(\rho_j)_{j\in\mathbb{N}}$ of also strongly converges to ρ_2.

We finally say a sequence $(\rho_j)_{j\in\mathbb{N}}$ *virtually strongly converges with respect to a subgroup* Γ_0 to the regular representation if there is a finite index subgroup Γ_0 such that the sequence restricted to Γ_0 strongly converges to λ_{Γ_0}

The following is provided by Lars Louder and Michael Magee [11]:

Theorem 2.6 (LOUDER–MAGEE). *Let Γ be the fundamental group of a closed connected oriented surface of genus at least 2. There exists a sequence $(\rho_j)_{j\in\mathbb{N}}$ of unitary representations of Γ in $\mathsf{U}(n_j)$ of finite image, that strongly converges to the regular representation.*

2.5. **A result of Antoine Song.** We need the Main Theorem of Song [15, Theorem 0.4].

Theorem 2.7 (SONG'S CONVERGENCE THEOREM). *Let X be a closed Riemann surface. Let $(\rho_j)_{j\in\mathbb{N}}$, where ρ_j is a representation of $\pi_1(X)$ in $\mathsf{U}(N_j)$. If $(\rho_j)_{j\in\mathbb{N}}$ strongly converges to the regular representation then,*

$$\lim_{j\to\infty} \mathrm{E}(\rho_j) = \frac{\pi}{4}|\chi(X)| .$$

Moreover if ψ_j from $\mathbf{H}^2$ to $\mathbf{S}^{2N_j-1}$ is a ρ_j-equivariant energy minimizing harmonic map, then

$$\lim_{j\to\infty} \psi_j^* \mathbf{g}_{\mathbf{S}^{2N_j-1}} = \frac{1}{8}\mathbf{g}_{\mathbf{H}^2} ,$$

where $\mathbf{g}_{\mathbf{S}^k}$ is the round metric on the sphere $\mathbf{S}^k$, $\mathbf{g}_{\mathbf{H}^2}$ is the hyperbolic metric on $\mathbf{H}^2$ and the convergence is in the C^∞ topology.

The following rigidity result [15, Corollary 2.4], strenghtened in [2], is crucial.

Theorem 2.8 (SONG'S RIGIDITY THEOREM). *Let X be a closed Riemann surface, $\mathcal{H}$ be a Hilbert space and ρ be a representation of $\pi_1(X)$ into $\mathsf{U}(\mathcal{H})$ which is weakly equivalent to the regular representation of $\pi_1(X)$. If there exists a ρ-equivariant map φ from $\mathbf{H}^2$ to the unit sphere in $\mathcal{H}$ whose energy equals $\frac{\pi}{4}|\chi(X)|$, then*

$$\varphi^* \mathbf{g}_{\mathbf{S}(\mathcal{H})} = \frac{1}{8}\mathbf{g}_{\mathbf{H}^2} ,$$

where $\mathbf{g}_{\mathbf{S}(\mathcal{H})}$ is the round metric on the unit sphere of $\mathcal{H}$.

Given Γ a discrete group (possibly with torsion) acting cocompactly on $\mathbf{H}^2$, one obtains an *orbifold Riemann surface* $X = \Gamma\backslash\mathbf{H}^2$. By Selberg Lemma, Γ admits a torsion free subgroup Γ_0 of finite index $[\Gamma : \Gamma_0]$. In particular, the Riemann surface $X_0 = \Gamma_0\backslash\mathbf{H}^2$ is smooth and is a branched cover of X of order $[\Gamma : \Gamma_0]$. The *orbifold Euler characteristic* $\chi_{\mathrm{orb}}(X)$ of X is defined by

$$\chi_{\mathrm{orb}}(X) := \frac{1}{[\Gamma : \Gamma_0]}\chi(X_0) .$$

We now prove the following consequence of Song's result in the torsion case, which is an extension of the second part:

Theorem 2.9. *Let Γ be a discrete group acting cocompactly on $\mathbf{H}^2$. Let $(\rho_j)_{j\in\mathbb{N}}$ be a sequence of representations, of Γ into $\mathsf{U}(N_j)$, assume that*

$$\lim_{j\to\infty} \mathrm{E}(\rho_j) = \frac{\pi}{4}|\chi_{\mathrm{orb}}(\Gamma\backslash\mathbf{H}^2)| . \tag{1}$$

Assume furthermore that $(\rho_j)_{j\in\mathbb{N}}$ virtually strongly converges with respect to a torsion free subgroup to the regular representation. Let u_j be a ρ_j-equivariant harmonic map from $\mathbf{H}^2$ to $\mathbf{S}^{2N_j-1}$. Assume that

$$\lim_{j\to\infty} \mathrm{E}(u_j) = \frac{\pi}{4}|\chi_{\mathrm{orb}}(\Gamma\backslash\mathbf{H}^2)| .$$

Then

$$\lim_{j\to\infty} u_j^*\mathbf{g}_{\mathbf{S}^{2N_j-1}} = \frac{1}{8}g_{\mathbf{H}^2}$$

where the convergence is in the C^∞ topology.

Proof. Let Γ_0 be the torsion free subroup of Γ such that $(\rho_j^0)_{j\in\mathbb{N}}$ strongly converges to the regular representation, where ρ_j^0 is the restriction of ρ_j to Γ_0.

For any j in $\mathbb{N}$, let u_j^0 be the harmonic map u_j seen as a ρ_j^0-equivariant map. Observe that

$$\mathrm{E}(u_j^0) = \frac{1}{2}\int_{\Gamma_0\backslash\mathbf{H}^2} \|\mathrm{d}u_j^0\|^2 \mathrm{d\,vol}_{\mathbf{H}^2} = \frac{[\Gamma:\Gamma_0]}{2}\int_{\Gamma\backslash\mathbf{H}^2} \|\mathrm{d}u_j\|^2 \mathrm{d\,vol}_{\mathbf{H}^2} = [\Gamma:\Gamma_0]\,\mathrm{E}(u_j) .$$

In particular, u_j^0 might not be energy minimizing, but we still have

$$\lim_{j\to\infty} \mathrm{E}(u_j^0) = [\Gamma:\Gamma_0]\left(\frac{\pi}{4}|\chi_{\mathrm{orb}}(\Gamma\backslash\mathbf{H}^2)|\right) = \frac{\pi}{4}|\chi(\Gamma_0\backslash\mathbf{H}^2)| .$$

We claim that this is enough to apply the conclusion of Theorem 2.7 to the closed Riemann surface $\Gamma_0\backslash\mathbf{H}^2$. In fact, by embedding each sphere totally geodesically into a fixed infinite dimensional sphere $\mathbf{S}(\mathcal{H})$ in a Hilbert space $\mathcal{H}$, we can consider the sequence $(u_j^0)_{j\in\mathbb{N}}$ as taking value in a fixed Riemannian manifold of infinite dimension. For any j, one can find a unitary transformation h_j in $\mathsf{U}(N_j)$ such that the 1-jet of $h_j \circ u_j^0$ at a fixed point x in $\mathbf{H}^2$ remains in a a fixed finite dimensional closed submanifold.

By the upper bound on the energy of $(u_j^0)_{j\in\mathbb{N}}$, one can apply Sacks–Uhlenbeck compactness result [13, Theorem 4.4] (which follows from ε-regularity, see [15, Theorem 1.8]): there is a discrete subset D of $\mathbf{H}^2$ such that $(u_j^0)_{j\in\mathbb{N}}$ converges C^∞ on any compact of $\mathbf{H}^2\setminus D$ to a harmonic map u_∞^0 of finite energy; by [13, Theorem 4.6][1], at any point of D where C^∞-convergence does not hold, one obtains a *bubble*, that is a nonconstant harmonic map β from $\mathbf{S}^2$ to $\mathbf{S}(\mathcal{H})$ such that

$$\mathrm{E}(u_\infty^0) + \mathrm{E}(\beta) \leqslant \lim_{j\to\infty} \mathrm{E}(u_j^0) . \tag{2}$$

By Theorem 2.7, since $(\rho_j^0)_{j\in\mathbb{N}}$ strongly converges to the regular representation of Γ_0, we have

$$\lim_{j\to\infty} \mathrm{E}(\rho_j^0) = \frac{\pi}{4}|\chi(\Gamma_0\backslash\mathbf{H}^2)| = \lim_{j\to\infty} \mathrm{E}(u_j^0) .$$

Since β has positive energy, equation (2) contradicts the above equality. It follows that u_∞^0 is a genuine harmonic map on $\mathbf{H}^2$ which, by [15, Proof of Proposition 4.4], is equivariant under a representation weakly equivalent to the regular representation of Γ_0 and satisfies $\mathrm{E}(u_\infty^0) = \frac{\pi}{4}|\chi(\Gamma_0\backslash\mathbf{H}^2)|$. The proof then follows from Theorem 2.8. □

[1] The original proof of Sack–Uhlenbeck holds for compact finite dimensional range. Nevertheless, its proof only relies on the ε-regularity which holds uniformly for all spheres (as pointed out in [15, Section 1.2]).

Remark 2.10. We will see in Theorem 3.2 that if the sequence $(\rho_j)_{j\in\mathbb{N}}$ above is obtained as the sequence of induced representations of a strongly convergence sequence, then assumption (1) is automatically satisfied and more can be said.

3. INDUCED REPRESENTATION

Let Γ be a discrete group acting cocomplactly on $\mathbf{H}^2$ and Γ_0 a finite index subgroup. In this section, we apply the construction of the *induced representation* (details can found in the book of Robert Zimmer [17]) to our setting. Given a Hilbert space $\mathcal{H}_0$ and a representation ρ_0 of Γ_0 into $\mathsf{U}(\mathcal{H}_0)$, this construction produces a new Hilbert space $\mathcal{H}_0^{\mathrm{ind}}$ together with a representation ρ_0^{ind} of Γ into $\mathsf{U}(\mathcal{H}_0^{\mathrm{ind}})$ satisfying the following

Theorem 3.1 (PROPERTIES OF THE INDUCED REPRESENTATION)**.** *Let Γ be a discrete group acting cocompactly on $\mathbf{H}^2$, Γ_0 be a finite index subgroup of Γ. Given a representation ρ_0 of Γ_0 into $\mathsf{U}(\mathcal{H}_0)$ for some Hilbert space $\mathcal{H}_0$, the induced representation ρ_0^{ind} of Γ into $\mathsf{U}(\mathcal{H}_0^{\mathrm{ind}})$ satisfies the following properties*

(1) *if $\mathcal{H}_0$ has finite dimension, then $\mathcal{H}_0^{\mathrm{ind}}$ has finite dimension.*
(2) *If ρ_0 has finite image, then ρ_0^{ind} has finite image.*
(3) *If ρ_0 has finite energy in the sense of Definition 2.4, then ρ_0^{ind} has finite energy and*

$$\mathrm{E}(\rho_0^{\mathrm{ind}}) \leqslant \frac{1}{[\Gamma:\Gamma_0]}\mathrm{E}(\rho_0)\ .$$

(4) *If $(\rho_j)_{j\in\mathbb{N}}$ is a sequence of unitary representations of Γ_0 that strongly converges to the regular representation, then the sequence $(\rho_j^{\mathrm{ind}})_{j\in\mathbb{N}}$ of induced representations of Γ virtually strongly converges with respect to Γ_0 to the regular representation.*[2]

As we will prove in Sections 3.1 and 3.2, all of these assertions follow from elementary considerations. The next theorem, however, uses the first part of Song's Theorem 2.7.

Theorem 3.2 (SONG'S THEOREM FOR INDUCED REPRESENTATIONS)**.** *Let Γ be a discrete group acting cocompactly on $\mathbf{H}^2$ and Γ_0 a torsion free subgroup of finite index. Let $(\rho_j^0)_{j\in\mathbb{N}}$ be a sequence of unitary representations of Γ_0 in $\mathsf{U}(n_j)$ strongly converging to the regular representation. Let $(\rho_j^{\mathrm{ind}})_{j\in\mathbb{N}}$ be the sequence of unitary representations of Γ in $\mathsf{U}(N_j)$ induced by $(\rho_j^0)_{j\in\mathbb{N}}$. Then*

$$\lim_{j\to\infty}\mathrm{E}(\rho_j^{\mathrm{ind}}) = \frac{\pi}{4}\chi_{\mathrm{orb}}(\Gamma\backslash\mathbf{H}^2)\ .$$

Moreover, if we choose for each j a ρ_j^{ind}-energy minimizing map u_j then

$$\lim_{j\to\infty} u_j^*\mathbf{g}_{\mathbf{S}^{2N_j-1}} = \frac{1}{8}\mathbf{g}_{\mathbf{H}^2}$$

where the convergence is in the C^∞ topology.

We postpone the proof of Theorem 3.2 until Section 3.3.

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[2]Actually, it is well known that in this situation $(\rho_j^{\mathrm{ind}})_{j\in\mathbb{N}}$ strongly converges to the regular representation. If the sequence strongly converges, then it virtually strongly converges with respect to any finite index subgroup. The direct proof presented here is for the sake of the reader.

3.1. **The induced representation.** We now describe more precisely the induced representation. Let Γ_0 a finite index subgroup of Γ. Assume that we have a representation ρ_0 from Γ_0 to $\mathsf{U}(\mathcal{H}_0)$ for $\mathcal{H}_0$ some Hilbert space. Define the vector space

$$\mathcal{H}_0^{\mathrm{ind}} := \left\{ f : \Gamma \to \mathcal{H}_0 \mid \forall \gamma_0 \in \Gamma_0,\ x \in \Gamma\ \ f(x\gamma_0) = \rho_0(\gamma_0) f(x) \right\} .$$

Note that $\mathcal{H}_0^{\mathrm{ind}}$ is a Hilbert space when equipped with the scalar product

$$\langle f, g \rangle_{\mathcal{H}_0^{\mathrm{ind}}} = \frac{1}{[\Gamma : \Gamma_0]} \sum_{\eta \in \Gamma_0 \backslash \Gamma} \langle f(\eta), g(\eta) \rangle_{\mathcal{H}_0} .$$

The *induced representation* ρ_0^{ind} from ρ_0 is the unitary representation of Γ into $\mathsf{U}(\mathcal{H}_0^{\mathrm{ind}})$ given by

$$\rho_0^{\mathrm{ind}}(\gamma) f \ : \ \begin{cases} \Gamma & \to \quad \mathcal{H}_0 , \\ \eta & \mapsto \quad f(\eta\gamma) . \end{cases}$$

Item (2) of Theorem 3.1 follows from

Lemma 3.3. *If ρ_0 has finite image then so has ρ_0^{ind}.*

Proof. It is enough to prove the result when ρ_0 is trivial. In that case, $\mathcal{H}_0^{\mathrm{ind}}$ is the space of maps from Γ to $\mathcal{H}_0$ which are left Γ_0-invariant. That is, the space of $\mathcal{H}_0$-valued maps on $S = \Gamma_0 \backslash \Gamma$. Then Γ acts on this finite set S. Let Γ_1 be the subgroup of Γ such that for any x in S, and γ_1 in Γ_1, we have $x\gamma_1 = x$. Note that Γ_0 is a subgroup of Γ_1, so Γ_1 has finite index in Γ_0. In particular, for all γ_1 in Γ_1, we have $\rho_0^{\mathrm{ind}}(\gamma_1\gamma) = \rho_0(\gamma)$. Thus Γ_1 is in the kernel of ρ_0^{ind} which therefore have finite image. ☐

The proofs of items (1) and (4) follow directly from

Lemma 3.4. *Let $\hat{\rho}_0^{\mathrm{ind}}$ be the restriction of ρ_0^{ind} to Γ_0. There is an isomorphism of Hilbert spaces*

$$\theta : \ \mathcal{H}_0^{\mathrm{ind}} \longrightarrow \bigoplus_{[\Gamma:\Gamma_0]} \mathcal{H}_0$$

that intertwines the Γ_0-representations $\hat{\rho}_0^{\mathrm{ind}}$ and $\bigoplus_{[\Gamma:\Gamma_0]} \rho_0$, that is, for any γ_0 in Γ_0 we have

$$\theta \circ \hat{\rho}_0^{\mathrm{ind}}(\gamma_0) \circ \theta^{-1} = \bigoplus_{[\Gamma:\Gamma_0]} \rho_0(\gamma_0) .$$

In particular, for any element f in $\mathbb{C}[\Gamma_0]$ we have

$$\|\hat{\rho}_0^{\mathrm{ind}}(f)\| = \|\rho_0(f)\| .$$

where the norm is the operator norm.

Proof. Choose elements $\eta_1, \ldots, \eta_k$ in Γ such that each class in $\Gamma_0 \backslash \Gamma$ is represented by a unique η_j. Then we have the following linear isomorphism

$$\theta : \begin{cases} \mathcal{H}_0^{\mathrm{ind}} & \to \quad \bigoplus_{[\Gamma:\Gamma_0]} \mathcal{H}_0 , \\ f & \mapsto \quad (f(\eta_1), ..., f(\eta_k)) . \end{cases}$$

By construction, θ is an isometry, where the Hilbert product on the left-hand side is defined so that the diagonal embedding of $\mathcal{H}_0$ is an isometry. Moreover, we have

$$\begin{aligned}\bigoplus_{[\Gamma:\Gamma_0]}\rho_0(\gamma_0)\theta(f) &= \bigoplus_{[\Gamma:\Gamma_0]}\rho_0(\gamma_0)(f(\eta_1),...,f(\eta_k))\\ &= (\rho_0(\gamma_0)f(\eta_1),...,\rho_0(\gamma_0)f(\eta_k))\\ &= (f(\eta_1\gamma_0),...,f(\eta_k\gamma_0))\\ &= \theta(\rho_0^{\mathrm{ind}}(\gamma_0)f)\ ,\end{aligned}$$

Thus, θ is an intertwiner.

The last statement follows from the fact that, for any endomorphism φ of a Hilbert space $\mathcal{H}$, the operator norm of $\varphi\oplus\varphi$ on $\mathcal{H}\oplus\mathcal{H}$, equipped with the direct sum scalar product, satisfies $\|\varphi\oplus\varphi\| = \|\varphi\|$. □

3.2. **Induced equivariant map.** We now prove item (3) of Theorem 3.1, namely

Proposition 3.5. *Let Γ be a discrete group acting cocompactly on $\mathbf{H}^2$ and Γ_0 be a torsion free finite index subgroup of Γ. Then the induced representation ρ_0^{ind} from any unitary representation ρ_0 has finite energy in the sense of Definition 2.4 and*

$$\mathrm{E}(\rho_0^{\mathrm{ind}}) \leqslant \frac{\mathrm{E}(\rho_0)}{[\Gamma:\Gamma_0]}\ .$$

Consider a ρ_0-equivariant map φ_0 from $\mathbf{H}^2$ to $\mathbf{S}(\mathcal{H}_0)$. Thus for any h in $\mathbf{H}^2$, $\varphi_0(h)$ is an element of $\mathcal{H}_0$ of norm 1. The *induced equivariant map* is the map φ_0^{ind} from $\mathbf{H}^2$ to $\mathcal{H}_0^{\mathrm{ind}}$, where $\varphi_0^{\mathrm{ind}}(h)$ is defined by

$$\begin{cases}\Gamma & \to \quad \mathcal{H}\\ \eta & \mapsto \quad \varphi_0(\eta h)\ .\end{cases}$$

Observe that since $\varphi_0(h)$ has norm 1, then

$$\|\varphi_0^{\mathrm{ind}}(h)\|^2_{\mathcal{H}_0^{\mathrm{ind}}} = \frac{1}{[\Gamma:\Gamma_0]}\sum_{\eta\in\Gamma_0\backslash\Gamma}\|\varphi_0(\eta h)\|^2_{\mathcal{H}_0} = 1\ ,$$

and so φ_0^{ind} takes value in $\mathbf{S}(\mathcal{H}_0^{\mathrm{ind}})$. The proof of Proposition 3.5 is a direct consequence of the following

Lemma 3.6. *The map φ_0^{ind} is ρ_0^{ind}-equivariant and its energy satisfies*

$$\mathrm{E}(\varphi_0^{\mathrm{ind}}) = \frac{\mathrm{E}(\varphi_0)}{[\Gamma:\Gamma_0]}\ .$$

Proof. For γ in Γ, the element $\varphi_0^{\mathrm{ind}}(\gamma h)$ is given by

$$\begin{cases}\Gamma & \to \quad \mathcal{H}_0\\ \eta & \mapsto \quad \varphi_0\left(\eta(\gamma h)\right)\ .\end{cases}$$

On the other hand $\rho_0^{\mathrm{ind}}(\gamma)(\varphi_0^{\mathrm{ind}}(h))$ is given

$$\begin{cases}\Gamma & \to \quad \mathcal{H}_0\\ \eta & \mapsto \quad \varphi_0((\eta\gamma)h)\ .\end{cases}$$

Thus $\rho_0^{\mathrm{ind}}(\gamma)\varphi_0^{\mathrm{ind}}(h) = \varphi_0^{\mathrm{ind}}(\gamma h)$ which means that φ_0^{ind} is ρ_0^{ind}-equivariant. Then, for any $h \in \mathbf{H}^2$, using the identification of Lemma 3.4, we have

$$\varphi_0^{\mathrm{ind}}(h) = \sum_{\eta\in\Gamma_0\backslash\Gamma} \varphi_0 \circ \eta(h) ,$$
$$\|\mathrm{d}_h\varphi_0^{\mathrm{ind}}\|^2 = \frac{1}{[\Gamma:\Gamma_0]} \sum_{\eta\in\Gamma_0\backslash\Gamma} \|\mathrm{d}_{\eta h}\varphi_0\|^2 ,$$

where in the last equation we use that η is an isometry of $\mathbf{H}^2$. From this it follows that

$$\begin{aligned}
\mathrm{E}(\varphi_0^{\mathrm{ind}}) &= \frac{1}{2}\int_{\Gamma\backslash\mathbf{H}^2} \|\mathrm{d}_h\varphi_0^{\mathrm{ind}}\|^2 \mathrm{d}\,\mathrm{vol}_{\mathbf{H}^2}(h)\\
&= \frac{1}{2[\Gamma:\Gamma_0]}\int_{\Gamma\backslash\mathbf{H}^2} \Big(\sum_{\eta\in\Gamma_0\backslash\Gamma} \|\mathrm{d}_{\eta h}\varphi_0\|^2\Big) \mathrm{d}\,\mathrm{vol}_{\mathbf{H}^2}(h)\\
&= \frac{1}{2[\Gamma:\Gamma_0]}\int_{\Gamma_0\backslash\mathbf{H}^2} \|\mathrm{d}_h\varphi_0\|^2 \mathrm{d}\,\mathrm{vol}_{\mathbf{H}^2}(h)\\
&= \frac{1}{[\Gamma:\Gamma_0]}\mathrm{E}(\varphi_0) ,
\end{aligned}$$

where the third equality follows from the fact that if D_Γ is a fundamental domain for the action of Γ on $\mathbf{H}^2$, its orbit under $\Gamma_0\backslash\Gamma$ is a fundamental domain for the action of Γ_0. The result follows. □

3.3. **Proof of Theorem 3.2.** Let $\hat\rho_j$ be the restriction of ρ_j^{ind} to Γ_0. By the fourth item of Theorem 3.1, $(\hat\rho_j)$ converges strongly to the regular representation of Γ_0. In particular the second hypothesis of Theorem 2.9 holds. Then, by applying Theorem 2.7 twice, we obtain

$$\lim_{j\to\infty} \mathrm{E}(\hat\rho_j) = \frac{\pi}{4}|\chi(\Gamma_0\backslash\mathbf{H}^2)| ,$$
$$\lim_{j\to\infty} \mathrm{E}(\rho_j^0) = \frac{\pi}{4}|\chi(\Gamma_0\backslash\mathbf{H}^2)| . \tag{3}$$

By item (3) of Theorem 3.1 the induced representations have finite energy. By Theorem 2.5, for each j there exists a ρ_j^{ind}-equivariant energy-minimizing map u_j. In particular, seeing u_j as a $\hat\rho_j$-equivariant map (denoted by $\hat u_j$), we get

$$\mathrm{E}(\hat u_j) = [\Gamma:\Gamma_0]\mathrm{E}(u_j) = [\Gamma:\Gamma_0]\mathrm{E}(\rho_j^{\mathrm{ind}}) .$$

Since $\hat u_j$ is $\hat\rho_j$-equivariant, we have $\mathrm{E}(\hat\rho_j) \leqslant \mathrm{E}(\hat u_j)$. Using the above equality, this gives

$$\frac{1}{[\Gamma:\Gamma_0]}\mathrm{E}(\hat\rho_j) \leqslant \mathrm{E}(\rho_j^{\mathrm{ind}}) .$$

On the other hand, item (3) of Theorem 3.1 gives

$$\mathrm{E}(\rho_j^{\mathrm{ind}}) \leqslant \frac{1}{[\Gamma:\Gamma_0]}\mathrm{E}(\rho_j) .$$

Taking the limit as j goes to infinity gives $\lim \mathrm{E}(\rho_j^{\mathrm{ind}}) = \frac{\pi}{4}|\chi_{\mathrm{orb}}(\Gamma\backslash\mathbf{H}^2)|$, thus proving the first assertion. The sequences $(u_j)_{j\in\mathbb{N}}$ and $(\rho_j)_{j\in\mathbb{N}}$ satisfy all the hypothesis of Theorem 2.9, thus we get the result. □

4. Proof of Main Theorem

4.1. **A Riemann surface with a large automorphism group.** We prove the existence of an orbifold Riemann surface whose Teichmüller space is a point. Given a closed Riemann surface X_0, we denote by $\mathrm{Aut}(X_0)$ its automorphism group, and by π the quotient map from X_0 to the orbifold $X := \mathrm{Aut}(X_0)\backslash X_0$. We prove

Proposition 4.1. *There exists a closed Riemann surface X_0 of genus greater than 1 with no nonzero $\mathrm{Aut}(X_0)$-invariant element in $H^0(K^2_{X_0})$.*

Let X_0 be as in Proposition 4.1, Γ_0 be its fundamental group, and Γ be the extension of Γ_0 such that $\Gamma\backslash\mathbf{H}^2 = \mathrm{Aut}(X_0)\backslash X_0$. Using Proposition 2.2, we have the following corollary:

Corollary 4.2. *Every Γ-equivariant harmonic mapping is a branched minimal immersion.*

Before proving Proposition 4.1, let us first prove a lemma.

Lemma 4.3. *Let q_2 be a meromorphic quadratic differential on $X := \mathrm{Aut}(X_0)\backslash X_0$ and D be the set of orbifold points of X. The pullback π^*q_2 to X_0 is holomorphic if and only if q_2 has at most simple poles at D.*

Proof. The computation is local: around a singular point in D the map π is given by $z \mapsto z^p$ for some positive integer p. In this coordinate system, we can write $q_2 = z^{-n}f(z)\mathrm{d}z^{\otimes 2}$ where f is a non vanishing holomorphic function. In particular

$$\pi^*q_2 = z^{-pn}f(z^p)(\mathrm{d}z^p)^{\otimes 2} = z^{2p-2-pn}p^2 f(z^p)\mathrm{d}z^{\otimes 2}\,.$$

So π^*q_2 is holomorphic if and only if $n \leqslant 1$, and the result follows. □

Proof of Proposition 4.1. Given a closed Riemann surface X_0 with automorphism group $\mathrm{Aut}(X_0)$, the quotient $X = \mathrm{Aut}(X_0)\backslash X_0$ is an orbifold with singular points $D = p_1, \ldots, p_k$. By the previous lemma, any $\mathrm{Aut}(X_0)$-invariant element of $H^0(K^2_{X_0})$ is the pullback via the quotient map $\pi : X_0 \to X$ of an element of $H^0(K^2_X(D))$.

In particular, if X is an orbifold structure on $\mathbf{P}^1$ with three singular points D, then the $\mathrm{Aut}(X_0)$-invariant elements of $H^0(K^2_{X_0})$ are the pullbacks of elements of $H^0(K^2_{\mathbf{P}^1}(D))$. Since $K^2_{\mathbf{P}^1}(D)$ has degree -1, $H^0(K^2_{\mathbf{P}^1}(D))$ is trivial. □

4.2. **Proof of the Main Theorem.** Let X_0 be the Riemann surface obtained in Proposition 4.1. Let Γ_0 be the fundamental group of X_0, and let Γ be a discrete group containing Γ_0 such that $\Gamma\backslash\mathbf{H}^2 = \mathrm{Aut}(X_0)\backslash X_0$.

Let $(\rho^0_j)_{j\in\mathbb{N}}$ be a sequence of representations of Γ_0 with finite image, strongly converging to the regular representation. Such a sequence exists by Theorem 2.6. Let ρ^{ind}_j denote the representation of Γ induced from ρ^0_j. Applying the extended Sacks–Uhlenbeck Theorem 2.5 to this situation, we obtain a ρ^{ind}_j-equivariant map u_j that is harmonic and energy minimizing.

By Corollary 4.2, each u_j is a branched minimal immersion. Furthermore, Theorem 3.2 gives

$$\lim_{j\to\infty} u_j^*\mathbf{g}_{\mathbf{S}^{2N_j-1}} = \frac{1}{8}\mathbf{g}_{\mathbf{H}^2}$$

where the convergence is C^∞. In particular, for j large enough, u_j is free of branch points: by Remark 2.1 branch points yields singularities of the curvature of the pullback metric. Thus, for j large enough, u_j is a negatively curved minimal immersion.

Observe also that ρ_j^{ind} has finite image (by item (2) of Theorem 3.1), hence, $\ker \rho_j^{\mathrm{ind}}$ has finite index in Γ. Let Γ_j be a torsion-free finite-index subgroup of $\ker \rho_j^{\mathrm{ind}}$, which therefore acts cocompactly on $\mathbf{H}^2$, and set

$$X_j = \Gamma_j \backslash \mathbf{H}^2 .$$

Then X_j is a closed Riemann surface and u_j is a minimal immersion of X_j in $\mathbf{S}^{2N_j-1}$. By construction, the induced metric $u_j^* \mathbf{g}_{\mathbf{S}^{2N_j-1}}$ has negative curvature for j large enough. The theorem is proved. □

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